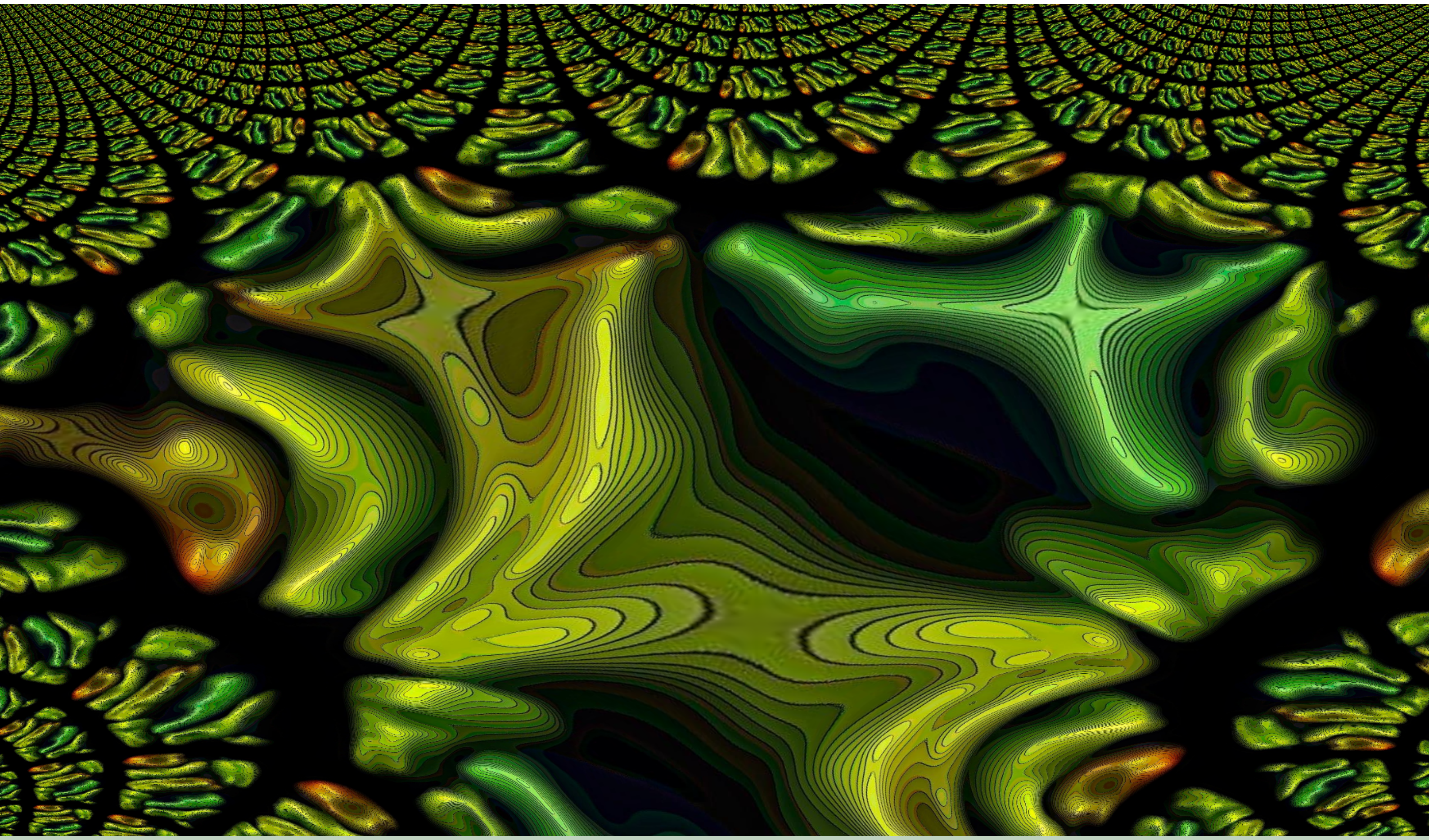

A Gentle Introduction to Ordinary Differential Equations

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PREFACE

A differential equation is an equation in mathematics that relates a function to its derivatives. The study of differential equations will always be important because it plays a central role in describing phenomena that change over time. When we wish to predict the future based on some knowledge of a current observable event, differential equations can help us to understand how that particular phenomenon evolves as a function of time. Differential equations play a prominent role in many disciplines including engineering, physics, economics, and biology.

Our goal in writing this text was to provide students at State College of Florida with both an introduction to and a survey of methods, applications, and theories of this beautiful and powerful mathematical tool. As a first course in differential equations, the book is intended for science and engineering majors who have completed two semesters of the calculus sequence, but not necessarily multivariable calculus. (Topics from multivariable calculus are introduced as needed.)

The many exciting and unanswered questions found in the theory of differential equations make it a popular field of study for graduate students. At the introductory level, however, it may seem like a collection of tricks that must be mastered. The beauty lies in the opportunity to challenge one's ability to analyze a problem and evaluate the known facts while forming a solution. Often the question in differential equations is not *how* to solve a problem but *how best* to solve a problem.

We encourage students to work their way through the examples with pencil and paper before attempting the exercises on their own. The examples outline the necessary procedures for each section but only with practice can one expect to learn the nuances of each approach in order to weigh the advantages of one technique over another when presented with a particular situation. Some time for reflection will be needed at the end of the course to contrast and compare the variety of methods available for solving differential equations.

CHAPTER 1 INTRODUCTION

Complete disorder is impossible.

Theodore Motzkin

IN THIS CHAPTER we begin our study of differential equations.

SECTION 1.1 introduces basic concepts and definitions related to differential equations.

SECTION 1.2 presents some applications that require differential equations in the construction of their mathematical models.

SECTION 1.3 analyzes solution curves without solving the corresponding differential equation.

1.1 BASIC CONCEPTS AND DEFINITIONS

The derivative dy/dx of a function $y = f(x)$ is itself another function. For example, the exponential function $y = e^{3x^2}$ is differentiable for all real numbers x and has first derivative $dy/dx = 6xe^{3x^2}$. We can replace e^{3x^2} on the right-hand side of the previous equation by the symbol y so that the equation for the derivative becomes

$$\frac{dy}{dx} = 6xy. \tag{1.1.1}$$

Now imagine you handed (1.1.1) to your differential equations instructor and asked them what function was represented by the symbol y. Without any knowledge of how the equation was constructed, your instructor could easily recover the original exponential function.

Equations such as (1.1.1) that contain one or more derivatives of an unknown function are referred to as *differential equations*. Of course, there are many uses for these types of equations outside of the classroom, too. Assumptions made about real-life systems frequently involve the rate of change of one or more of the variables being studied. This means that in order to construct a *mathematical model* to provide a mathematical description of the system, a differential equation or a system of differential equations may be required.

Much of calculus is devoted to learning mathematical techniques that are applied in later courses in mathematics and the sciences. However, you wouldn't have time to learn much calculus if you insisted on seeing a specific application of every topic covered in the course. Similarly, much of this book is devoted to methods that can be applied in later courses. Only a relatively small part of the book is devoted to the derivation of specific differential equations from mathematical models, or to relating the differential equations that we study to specific applications. In this section, we examine an application that you have probably encountered in a previous math course and then discuss some basic definitions and terminology; in the next section, we will discuss a few more applications.

It is rare for a mathematical model of an applied problem to capture every nuance of the situation being studied. This is because simplifying assumptions are usually required to obtain a mathematical problem that can be solved. If the results predicted by the model do not agree with physical observations, the underlying assumptions of the model must be revised until a satisfactory agreement is obtained.

To summarize, a good mathematical model is a balance between two important properties.

- It is simple enough for the mathematical problem to be solved.
- It is complex enough to represent the actual situation well enough for the solution to the mathematical problem to predict outcomes within a useful degree of accuracy.

Population Growth and Decay

Let us consider a mathematical model that represents population growth and decay. The number of members of a population (people in a given country, bacteria in a laboratory culture, wildflowers in a forest, etc.) at any given time t must be an integer. However, for this mathematical model using differential equations to describe the growth and decay of populations, we will use the simplifying assumption that the number of members of the population can be regarded as a differentiable function $P = P(t)$. (In particular, recall that a differentiable function must be continuous.) We can achieve a good model for population growth by assuming that the differential equation takes the form

$$P' = aP, \tag{1.1.2}$$

where a is a constant. This is referred to as the *Malthusian model* due to the work of Thomas Robert Malthus, which he published in 1798 as *An Essay on the Principle of Population*. The model assumes that the numbers of births and deaths per unit time are proportional to the size of the population. (If the constant of proportionality for the birth rate is b and the constant of proportionality for the death rate is d, then $P' = bP - dP$. Simplified, a is the birth rate minus the death rate.)

You learned in calculus that for any nonzero real number c,

$$P = ce^{at} \tag{1.1.3}$$

satisfies (1.1.2). This means that a single differential equation can possess an infinite number of solutions corresponding to an unlimited number of choices for the parameter c. A *particular solution* is one that is free of parameters. To find the particular solution, we would need to know the population P_0 at an initial time, say $t = 0$. Setting $t = 0$ in (1.1.3) yields $P(0) = c$, so if we relabel c as P_0 the particular solution would be

$$P(t) = P_0e^{at}.$$

Notice that

$$\lim_{t\to\infty} P(t) = \begin{cases} \infty & \text{if } a > 0, \\ 0 & \text{if } a < 0; \end{cases}$$

that is, this model predicts that the population will approach infinity if the birth rate exceeds the death rate and that it will approach zero if the death rate exceeds the birth rate.

To better understand the limitations of the Malthusian model, suppose we model the population of a country starting from a time $t = 0$ when the birth rate exceeds the death rate (so $a > 0$). If we know that the country's resources in terms of space, food supply, and other necessities of life can support the existing population, then the prediction $P = P_0e^{at}$ will be reasonably accurate as long as it remains within the limits that the country's future resources can support. However, the model must inevitably lose validity when the prediction exceeds these limits. (If nothing else, eventually there won't be enough space for the predicted population!)

This flaw in the Malthusian model suggests the need for a revised model that accounts for limitations of space and resources. Indeed, more complex models for population growth have been designed that better agree with physical observations of human populations. However, at the time of its publication, the Malthusian model turned out to be a reasonably accurate prediction of the United States population during the first half of the nineteenth century. The Malthusian model is still used today to predict the growth of small populations over short intervals of time.

The equation created in the opening discussion (1.1.1) and the Malthusian model of population growth (1.1.2) are both differential equations with solutions involving exponential functions. Naturally, there are differential equations that are much more complicated! Just as a student in an algebra course learns to solve equations such as $x^2 + 3x + 1 = 0$ to determine the unknown number x, a student in a differential equations course learns to solve equations such as $y'' + 3y' + 1 = 0$ to determine the unknown function y. Let us begin the journey with some useful definitions and terminology.

The *order* of a differential equation is the order of the highest derivative that it contains. A differential equation is an *ordinary differential equation* if it involves an unknown function of only one variable, or a *partial differential equation* if it involves partial derivatives of a function of more than one variable. For now we'll consider only ordinary differential equations, and we'll just call them differential equations. In this text, all variables and constants are real unless stated otherwise.

The simplest differential equations are first order equations of the form

$$\frac{dy}{dx} = f(x) \quad \text{which can also be written as,} \quad y' = f(x),$$

where f can be solved explicitly as a function of x. (The notation on the left is referred to as *Leibniz notation*, and the one on the right is referred to as *prime notation, Lagrange notation*.) We already know from calculus how to find many functions that satisfy this kind of equation. For example, if

$$y' = x^3,$$

then

$$y = \int x^3\, dx = \frac{x^4}{4} + c,$$

where c is an arbitrary constant. For higher order differential equations where $n > 1$ we can find functions y that satisfy equations of the form

$$y^{(n)} = f(x) \tag{1.1.4}$$

by repeated integration. Again, this is a calculus problem. (Recall that $y^{(n)}$ denotes the n^{th} derivative of y.)

Except for illustrative purposes in this section, there's no need to revisit differential equations like (1.1.4). Instead, we'll usually consider differential equations that can be written in the *normal form*

$$y^{(n)} = F(x, y, y', \dots, y^{(n-1)}). \tag{1.1.5}$$

Here are some examples:

$$\begin{aligned}
\frac{dy}{dx} - x^2 &= 0 && \text{(first order)},\\
\frac{dy}{dx} + 2xy^2 &= -2 && \text{(first order)},\\
\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y &= 2x && \text{(second order)},\\
xy''' + y^2 &= \sin x && \text{(third order)},\\
y^{(n)} + xy' + 3y &= x && (n^{th} \text{ order}).
\end{aligned}$$

Although none of these equations is written as in (1.1.5), all of them *can* be written in this form:

$$\begin{aligned}
y' &= x^2,\\
y' &= -2 - 2xy^2,\\
y'' &= 2x - 2y' - y,\\
y''' &= \frac{\sin x - y^2}{x},\\
y^{(n)} &= x - xy' - 3y.
\end{aligned}$$

Solutions of Differential Equations

A *solution* of a differential equation is a function that satisfies the differential equation on some interval. When we think of the solution to a differential equation, we must always simultaneously consider the interval on which it exists! The interval is called the *domain of the solution* and can be an open interval with bounds such as (a, b), a closed interval with bounds such as $[a, b]$, an open interval that is unbounded such as (a, ∞), and so on. For simplicity, we will refer to the domain of the solution as an open interval. More precisely, y is a solution of (1.1.5) on (a, b) if y is n times differentiable on (a, b) and

$$y^{(n)}(x) = F(x, y(x), y'(x), \dots, y^{(n-1)}(x))$$

for all x in the interval (a, b). (We will abuse the notation a bit and allow (a, b) to represent intervals such as (∞, b), (a, ∞), and $(-\infty, \infty)$.)

Functions that satisfy a differential equation at isolated points are not interesting. For example, $y = x^2$ satisfies

$$xy' + x^2 = 3x$$

if and only if $x = 0$ or $x = 1$, but it's not a solution of this differential equation because it does not satisfy the equation on an open interval. Also uninteresting is any solution that is identically zero on an interval; such a solution is said to be a *trivial solution*. For example, the equation $y'' - 2y' + y = 0$ has the trivial solution $y = 0$ on the interval (∞, ∞).

The graph of a solution y of a differential equation is a *solution curve*. Notice that y must be continuous on the domain of the solution since it is known to be differentiable there. This means there may be a difference between the graph of the *function* y and the graph of the *solution* y. Again, the solution to a differential equation must always be accompanied by the domain of the solution.

Example 1.1.1 Verify that

$$y = \frac{x^2}{3} + \frac{1}{x} \tag{1.1.6}$$

is a solution of

$$xy' + y = x^2 \tag{1.1.7}$$

on $(0, \infty)$.

Solution Notice that y is not defined when $x = 0$. So although the domain of the function y is the set of all real numbers other than 0, the domain of the solution y is restricted to a single open interval that does not contain $x = 0$.

Now, substituting (1.1.6) and its derivative

$$y' = \frac{2x}{3} - \frac{1}{x^2}$$

into (1.1.7) yields

$$xy'(x) + y(x) = x\left(\frac{2x}{3} - \frac{1}{x^2}\right) + \left(\frac{x^2}{3} + \frac{1}{x}\right)$$

for all $x \neq 0$. This simplifies to x^2, which is the right-hand side of (1.1.7). Therefore y is a solution of (1.1.7) on $(0, \infty)$. Alternatively, we could have taken the domain of the solution to be $(-\infty, 0)$. In either case, we use the largest open interval possible for the domain of the solution. ■

Figure 1.1 shows the graph of (1.1.6). The portion of the graph on $(0, \infty)$ is a solution curve of (1.1.7), as is the part of the graph on $(-\infty, 0)$.

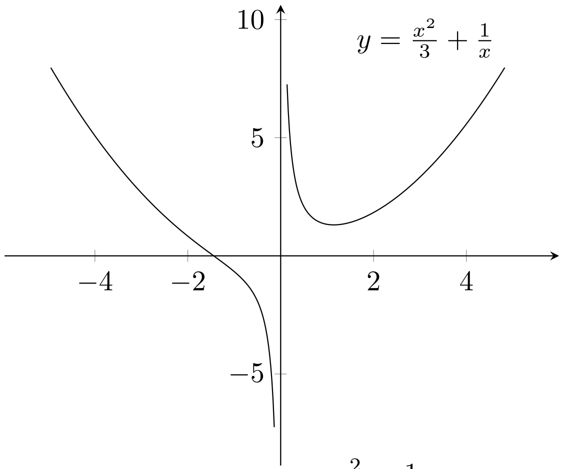


Figure 1.1 $y = \dfrac{x^2}{3} + \dfrac{1}{x}$

Example 1.1.2 Show that if c_1 and c_2 are constants then

$$y = (c_1 + c_2x)e^{-x} + 2x - 4 \tag{1.1.8}$$

is a solution of

$$y'' + 2y' + y = 2x \tag{1.1.9}$$

on $(-\infty, \infty)$.

Solution Differentiating (1.1.8) twice yields

$$y' = -(c_1 + c_2x)e^{-x} + c_2e^{-x} + 2$$

and

$$y'' = (c_1 + c_2x)e^{-x} - 2c_2e^{-x}.$$

Substituting these into (1.1.9) gives us

$$
\begin{aligned}
y'' + 2y' + y &= (c_1 + c_2x)e^{-x} - 2c_2e^{-x} \\
&\quad +2\left[-(c_1 + c_2x)e^{-x} + c_2e^{-x} + 2\right] \\
&\quad +(c_1 + c_2x)e^{-x} + 2x - 4 \\
&= (1 - 2 + 1)(c_1 + c_2x)e^{-x} + (-2 + 2)c_2e^{-x} \\
&\quad +4 + 2x - 4 \\
&= 2x
\end{aligned}
$$

for all values of x. Therefore y is a solution of (1.1.9) on $(-\infty, \infty)$. ■

Example 1.1.3 Find all solutions of

$$y^{(n)} = e^{2x}. \tag{1.1.10}$$

Solution Integrating (1.1.10) yields

$$y^{(n-1)} = \frac{e^{2x}}{2} + k_1,$$

where k_1 is a constant. If $n \geqslant 2$, integrating again yields

$$y^{(n-2)} = \frac{e^{2x}}{4} + k_1x + k_2.$$

If $n \geqslant 3$, repeatedly integrating yields

$$y = \frac{e^{2x}}{2^n} + k_1\frac{x^{n-1}}{(n-1)!} + k_2\frac{x^{n-2}}{(n-2)!} + \cdots + k_n, \tag{1.1.11}$$

where $k_1, k_2, \dots, k_n$ are constants. This shows that every solution of (1.1.10) has the form (1.1.11) for some choice of the constants $k_1, k_2, \dots, k_n$. On the other hand, differentiating the function y in (1.1.11) n times shows that if $k_1, k_2, \dots, k_n$ are arbitrary constants, then y satisfies (1.1.10). ■

Since the constants $k_1, k_2, \dots, k_n$ in (1.1.11) are arbitrary, so are the constants

$$\frac{k_1}{(n-1)!}, \frac{k_2}{(n-2)!}, \cdots, k_n.$$

Therefore Example 1.1.3 shows that all solutions of (1.1.10) can be written as

$$y = \frac{e^{2x}}{2^n} + c_1 + c_2x + \cdots + c_nx^{n-1},$$

where we renamed the arbitrary constants in (1.1.11) to obtain a simpler form. Keep in mind that two individuals correctly solving a differential equation may arrive at dissimilar expressions for their answers. This can be due to constants that have been relabeled, algebraic simplification, or application of trigonometric identities.

Initial Value Problems

In Example 1.1.3 we saw that the differential equation $y^{(n)} = e^{2x}$ has an infinite family of solutions that depend upon the n arbitrary parameters $c_1, c_2, \dots, c_n$. In the absence of additional conditions, there's no reason to prefer one solution of a differential equation over another. However, we'll often be interested in finding a solution of a differential equation that satisfies one or more specific conditions. The next example illustrates this using a process learned in calculus.

Example 1.1.4 Find a solution of

$$y' = x^3$$

such that $y(1) = 2$.

Solution From calculus, we know that the solutions of $y' = x^3$ are

$$y = \frac{x^4}{4} + c. \tag{1.1.12}$$

To determine a value of c such that $y(1) = 2$, we substitute $x = 1$ and $y = 2$ to obtain

$$2 = \frac{1}{4} + c, \quad \text{so} \quad c = \frac{7}{4}.$$

Therefore the required solution is

$$y = \frac{x^4}{4} + \frac{7}{4}.$$

■

Figure 1.2 shows the graph of this solution. Recall from algebra that a nonzero value of c in (1.1.12) results in a vertical translation of c units. Imposing the condition $y(1) = 2$ is equivalent to requiring the graph of y to pass through the point $(1, 2)$, which determines the specific vertical translation required.

We can rewrite the problem considered in Example 1.1.4 more succinctly as

$$y' = x^3, \quad y(1) = 2.$$

This type of problem is referred to as an *initial value problem*, and the requirement $y(1) = 2$ is an example of an *initial condition*. Initial value problems can also be posed for higher order differential equations. For example,

$$y'' - 2y' + 3y = e^x, \quad y(0) = 1, \quad y'(0) = 2 \tag{1.1.13}$$

is an initial value problem for a second order differential equation where y and y' are required to have specified values at the same point, in this case at $x = 0$. In general, an

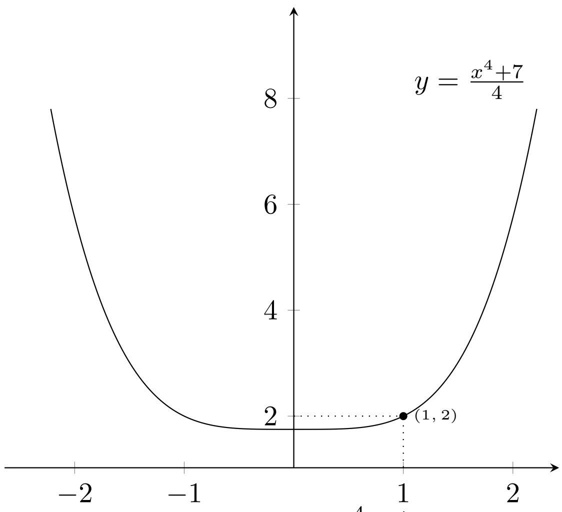


Figure 1.2 $y = \dfrac{x^4+7}{4}$

initial value problem for an n^{th} order differential equation requires y and its first $n-1$ derivatives to have specified values at some point x_0.

We'll denote an initial value problem for a differential equation by writing the initial conditions after the equation, as in (1.1.13). For example, we would write an initial value problem for an n^{th} order differential equation as

$$y^{(n)} = f(x, y, y', \dots, y^{(n-1)}),\ y(x_0) = k_0,\ y'(x_0) = k_1,\ \dots,\ y^{(n-1)}(x_0) = k_{n-1}. \tag{1.1.14}$$

Consistent with our earlier definition of a solution of a differential equation, we say that y is a solution of the initial value problem (1.1.14) on (a, b) if y is n times differentiable on the interval (a, b) that contains x_0,

$$y^{(n)}(x) = f(x, y(x), y'(x), \dots, y^{(n-1)}(x))$$

for all x in the interval (a, b), and y satisfies the initial conditions in (1.1.14). The domain of the solution is taken to be the largest open interval that contains x_0 on which y is defined and satisfies the differential equation.

Example 1.1.5 In Example 1.1.4 we saw that

$$y = \frac{x^4+7}{4}$$

is a solution of the initial value problem

$$y' = x^3, \quad y(1) = 2.$$

Since the function y is defined for all x, the domain of the solution is $(-\infty, \infty)$.

■

Example 1.1.6 In Example 1.1.1 we verified that

$$y = \frac{x^2}{3} + \frac{1}{x}$$

is a solution of

$$xy' + y = x^2$$

on $(0, \infty)$ and on $(-\infty, 0)$. Now consider the initial value problem

$$xy' + y = x^2, \quad y(-1) = -\frac{2}{3}. \tag{1.1.15}$$

The domain of the solution of (1.1.15) is $(-\infty, 0)$, since this is the largest interval that contains $x_0 = -1$ on which (1.1.15) is defined.

■

1.1 Exercises

1. State the order of the differential equation.

(a) $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx}\dfrac{d^3y}{dx^3} + x = 0$ **(b)** $y'' - 3y' + 2y = x^7$

(c) $y' - y^7 = 0$ **(d)** $y''y - (y')^2 = 2$

2. Verify that the function is a solution of the differential equation. Be sure to provide an appropriate domain for the solution.

(a) $y = ce^{2x}; \quad y' = 2y$

(b) $y = \dfrac{x^2}{3} + \dfrac{c}{x}; \quad xy' + y = x^2$

(c) $y = \dfrac{1}{2} + ce^{-x^2}; \quad y' + 2xy = x$

(d) $y = (1 + ce^{-x^2/2}); (1 - ce^{-x^2/2})^{-1} \quad 2y' + x(y^2 - 1) = 0$

(e) $y = \tan\left(\dfrac{x^3}{3} + c\right); \quad y' = x^2(1 + y^2)$

(f) $y = (c_1 + c_2 x)e^x + \sin x + x^2; \quad y'' - 2y' + y = -2\cos x + x^2 - 4x + 2$

(g) $y = c_1 e^x + c_2 x + \dfrac{2}{x}; \quad (1 - x)y'' + xy' - y = 4(1 - x - x^2)x^{-3}$

(h) $y = x^{-1/2}(c_1 \sin x + c_2 \cos x) + 4x + 8;$
$x^2 y'' + xy' + \left(x^2 - \dfrac{1}{4}\right)y = 4x^3 + 8x^2 + 3x - 2$

3. Find all solutions of the differential equation.

(a) $y' = -x$ **(b)** $y' = -x \sin x$

(c) $y' = x \ln x$ **(d)** $y'' = x \cos x$

(e) $y'' = 2xe^x$ **(f)** $y'' = 2x + \sin x + e^x$

(g) $y''' = -\cos x$ **(h)** $y''' = -x^2 + e^x$

(i) $y''' = 7e^{4x}$

4. Solve the initial value problem.

(a) $y' = -xe^x, \quad y(0) = 1$

(b) $y' = x \sin x^2, \quad y\left(\sqrt{\dfrac{\pi}{2}}\right) = 1$

(c) $y' = \tan x, \quad y(\pi/4) = 3$

(d) $y'' = x^4, \quad y(2) = -1, \quad y'(2) = -1$

(e) $y'' = xe^{2x}, \quad y(0) = 7, \quad y'(0) = 1$

(f) $y'' = -x \sin x, \quad y(0) = 1, \quad y'(0) = -3$

(g) $y''' = x^2 e^x, \quad y(0) = 1, \quad y'(0) = -2, \quad y''(0) = 3$

(h) $y''' = 2 + \sin 2x, \quad y(0) = 1, \quad y'(0) = -6, \quad y''(0) = 3$

(i) $y''' = 2x + 1, \quad y(2) = 1, \quad y'(2) = -4, \quad y''(2) = 7$

5. Verify that the function is a solution of the first order initial value problem.

(a) $y = x\cos x; \quad y' = \cos x - y\tan x, \quad y(\pi/4) = \dfrac{\pi}{4\sqrt{2}}$

(b) $y = \dfrac{1+2\ln x}{x^2} + \dfrac{1}{2}; \quad y' = \dfrac{x^2 - 2x^2y + 2}{x^3}, \quad y(1) = \dfrac{3}{2}$

(c) $y = \tan\left(\dfrac{x^2}{2}\right); \quad y' = x(1+y^2), \quad y(0) = 0$

(d) $y = \dfrac{2}{x-2}; \quad y' = \dfrac{-y(y+1)}{x}, \quad y(1) = -2$

6. Verify that the function is a solution of the second order initial value problem.

(a) $y = x^2(1+\ln x); \quad y'' = \dfrac{3xy' - 4y}{x^2}, \quad y(e) = 2e^2, \quad y'(e) = 5e$

(b) $y = \dfrac{x^2}{3} + x - 1; \quad y'' = \dfrac{x^2 - xy' + y + 1}{x^2}, \quad y(1) = \dfrac{1}{3}, \quad y'(1) = \dfrac{5}{3}$

(c) $y = (1+x^2)^{-1/2}; \quad y'' = \dfrac{(x^2-1)y - x(x^2+1)y'}{(x^2+1)^2}, \quad y(0) = 1,$
$y'(0) = 0$

(d) $y = \dfrac{x^2}{1-x}; \quad y'' = \dfrac{2(x+y)(xy'-y)}{x^3}, \quad y(1/2) = 1/2, \quad y'(1/2) = 3$

7. Let a be a nonzero real number.

(a) Verify that if c is an arbitrary constant then

$$y = (x-c)^a \tag{A}$$

is a solution of

$$y' = ay^{(a-1)/a} \tag{B}$$

on (c, ∞).

(b) Suppose $a < 0$ or $a > 1$. Can you think of a solution of (B) that isn't of the form (A)?

8. **(a)** Verify that if c is any real number then

$$y = c^2 + cx + 2c + 1 \tag{A}$$

satisfies

$$y' = \frac{-(x+2) + \sqrt{x^2 + 4x + 4y}}{2} \tag{B}$$

on some open interval. Identify the open interval.

(b) Verify that

$$y_1 = \frac{-x(x+4)}{4}$$

also satisfies (B) on some open interval, and identify the open interval. (Note that y_1 can't be obtained by selecting a value of c in (A); such an extra solution is called a *singular solution*.)

1.2 APPLICATIONS INVOLVING DIFFERENTIAL EQUATIONS

Differential equations can be used to model the behavior of a variety of real-life systems found in fields such as economics, physics, medicine, and sociology. (The Malthusian model discussed in the previous section is from a field referred to as *political economy*.) In this section, we use differential equations to model *dynamical systems* that change or evolve with the flow of time. In such applications, t is often used to represent time as the independent variable. So if y is a function of t, y' denotes the derivative of y with respect to t; that is,

$$y' = \frac{dy}{dt}.$$

For a dynamical system, a solution of its model gives the *state of the system*: different values of the independent variable t give values for the dependent variable (or variables) that describe the system in past, present, and future states. We will assume all variables are defined over a continuous range of time.

Mathematical models of bodies in motion provide a good example of how the independent variable t represents elapsed time. Construction of a mathematical model for the motion of a falling object requires the use of a second order differential equation. In this case, the initial conditions of the initial value problem are the position and the velocity of the object at the start of the experiment.

Free Fall Under Constant Gravity

When an object falls under the influence of gravity near Earth's surface, it can be assumed that the magnitude of the acceleration due to gravity is a constant, g. To simplify the model, we will assume that gravity is the only force acting on the object. If the altitude and velocity of the object at time $t = 0$ are known, then the model takes the form of an initial value problem.

Let's denote the altitude of the object at time t as $y(t)$. Since the acceleration of the object has constant magnitude g and is in the downward (negative) direction, y satisfies the second order equation

$$y'' = -g,$$

where the prime notation indicates differentiation with respect to t. If y_0 and v_0 denote the altitude and velocity when $t = 0$, then y is a solution of the initial value problem

$$y'' = -g, \quad y(0) = y_0, \quad y'(0) = v_0. \tag{1.2.1}$$

Although the emphasis in this section is on creating mathematical models rather than solving them, the solution to this initial value problem should be familiar from physics and/or calculus. Integrating (1.2.1) twice yields

$$\begin{aligned} y' &= -gt + c_1, \\ y &= -\frac{gt^2}{2} + c_1 t + c_2. \end{aligned}$$

By substituting the initial conditions $y(0) = y_0$ and $y'(0) = v_0$ into these two equations, we find that $c_1 = v_0$ and $c_2 = y_0$. (Try it!) Therefore the solution of the initial value problem (1.2.1) is

$$y = -\frac{gt^2}{2} + v_0 t + y_0.$$

This equation describes the altitude of the object as a function of time.

Spread of an Epidemic

An *epidemic* is a widespread occurrence of an infectious disease in a given community at a particular time. Consider a contagious illness such as the flu that is spread by interactions among different types of people: let $x(t)$ denote the number of infected people and let $y(t)$ denote the number of people who are susceptible but not yet infected. A reasonable model for the spread of a disease assumes that the number of people infected changes at a rate proportional to the number of encounters between these two groups of people; that is, assume the number of encounters is jointly proportional to $x(t)$ and $y(t)$. In this case

$$\frac{dx}{dt} = kxy, \tag{1.2.2}$$

where k is the constant of proportionality. Now suppose the community has a fixed population of n people and that one infected person enters the community. This means that $x + y = n + 1$ provides a relationship between the two groups of people. Solving this relationship for y and then substituting this into equation (1.2.2) gives us the model

$$\frac{dx}{dt} = kx(n + 1 - x). \tag{1.2.3}$$

This becomes an initial-value problem by noting the condition that $x(0) = 1$.

Radioactive Decay Combined with Growth

Experimental evidence shows that radioactive material decays at a rate proportional to the mass of the material present. This means that the mass $Q(t)$ of a radioactive material present at time t can be represented mathematically by the same model as the one we used for population growth. In this model, however, a negative constant of proportionality must be used. (This value for a given radioactive material must be determined by experimental observation.) For simplicity, we will replace the negative constant with a positive number k that we will call the *decay constant* of the material. In summary, if the mass of the material present at $t = t_0$ is Q_0, the mass present at time t is the solution of the initial value problem

$$Q' = -kQ, \quad Q(t_0) = Q_0.$$

Now suppose that the radioactive material is a drug administered intravenously at a constant rate of a units of mass per unit time. Assuming that $Q(0) = Q_0$, we can construct an initial value problem to model the mass $Q(t)$ of the substance present at time t. The basic concept is that

$$Q' = \text{ rate of increase of } Q - \text{ rate of decrease of } Q.$$

The rate of increase is the constant a. Since Q is radioactive with decay constant k, the rate of decrease is kQ. Therefore

$$Q' = a - kQ,$$

which is a first order differential equation. Rewriting it and imposing the initial condition shows that Q is the solution of the initial value problem

$$Q' + kQ = a, \quad Q(0) = Q_0. \tag{1.2.4}$$

Mixing Problems

In mixing problems, a saltwater solution with a given concentration (weight of salt per unit volume of solution) is added at a specified rate to a tank that initially contains saltwater with a different concentration. The problem is to determine the quantity of salt in the tank as a function of time. To construct a tractable mathematical model for these systems, we can assume that the mixture is stirred in such a way that the salt is always uniformly distributed throughout the mixture. We look at two scenarios where the newly mixed solution is then drained from the second tank at a constant rate: in the first scenario, the rate of water entering the tank is the same as the rate of water leaving the tank; in the second scenario, the two rates differ. Keep in mind that we are interested in the amount of salt present in the tank – not the rates at which solutions enter and drain (although these rates are an important part of the model). Similar to the previous discussion on radioactive decay combined with growth, we must account for simultaneous rates of increase and decrease in the amount of salt present.

For the first scenario, suppose a tank initially contains 40 pounds of salt dissolved in 600 gallons of water. Starting at $t_0 = 0$, water that contains $1/2$ pound of salt per gallon is poured into the tank at the rate of 4 gal/min while the mixture is drained from the tank at the same rate. (See Figure 1.1.)

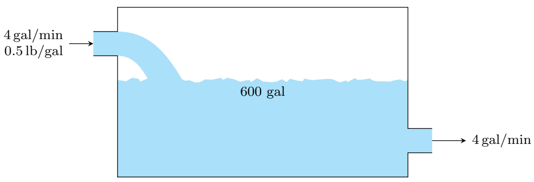


Figure 1.1 A mixing problem

To find a differential equation for the quantity $Q(t)$ of salt in the tank at time $t > 0$, we must use the given information to derive an expression for Q'. Here Q' is the rate of change of the quantity of salt in the tank that changes with respect to time; thus, if *rate in* denotes the rate at which salt enters the tank and *rate out* denotes the rate by which it leaves, then

$$Q' = \text{rate in} - \text{rate out}. \tag{1.2.5}$$

The rate in is

$$\left(\frac{1}{2}\ \text{lb/gal}\right) \times (4\ \text{gal/min}) = 2\ \text{lb/min}.$$

Determining the rate out requires a little more thought. Dimensional analysis is useful to see that

$$\begin{aligned} \text{rate out} &= (\text{concentration}) \times (\text{rate of flow out}) \\ &= (\text{lb/gal}) \times (\text{gal/min}) \\ &= \frac{Q(t)}{600} \times 4. \end{aligned}$$

In words, we're removing 4 gallons of the mixture per minute, and there are always 600 gallons in the tank (since the amount of water coming in and the amount of water going out are the same). Reduce the fraction to lowest terms to see that we're removing $1/150$ of the mixture per minute, and - since the salt is evenly distributed in the mixture - we are also removing $1/150$ of the salt per minute. Therefore, if there are $Q(t)$ pounds of salt in the tank at time t, the rate out at any time t is $Q(t)/150$. We can now rewrite (1.2.5) as a first order differential equation with the initial condition $Q(0) = 40$.

$$Q' = 2 - \frac{Q}{150}, \quad Q(0) = 40 \tag{1.2.6}$$

In the second scenario, we look at a mixing problem where the rate of water coming in and the rate of water going out are different. The basic model, however, is still the same.

Suppose a 500-$\mathtt{liter}$ tank initially contains 10 $\mathtt{grams}$ of salt dissolved in 200 $\mathtt{liters}$ of water. Starting at $t_0 = 0$, water that contains $1/4$ $\mathtt{grams}$ of salt per $\mathtt{liter}$ is poured into the tank at the rate of 4 $\mathtt{liters/min}$ and the mixture is drained from the tank at the rate of 2 $\mathtt{liters/min}$. (See Figure 1.2.) The task is to find an initial value problem whose solution describes the quantity $Q(t)$ of salt in the tank at any time t prior to the time when the tank overflows. (Notice that there is more water coming into the tank than there is going out, so that the model is valid for only a certain period of time.)

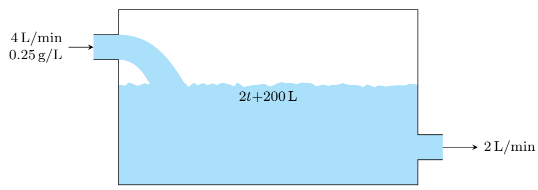


Figure 1.2 Another mixing problem

We first determine the amount $W(t)$ of solution in the tank at any time t prior to overflow. Since $W(0) = 200$ and we're adding 4 liters/min while removing only 2

liters/min, there's a net gain of 2 liters/min in the tank; therefore,

$$W(t) = 2t + 200.$$

Since $W(150) = 500$ (the capacity of the tank is 500 $\mathtt{liters}$), this formula is valid only when $0 \leqslant t \leqslant 150$.

Now let $Q(t)$ be the number of $\mathtt{grams}$ of salt in the tank at time t, where $0 \leqslant t \leqslant 150$. As in the previous example,

$$Q' = \text{rate in} - \text{rate out}.$$

The rate in is

$$\left(\frac{1}{4}\ \text{g/liter}\right) \times (4\ \text{liters/min}\,) = 1\ \text{g/min}. \tag{1.2.7}$$

To determine the rate out, we observe that since the mixture is being removed from the tank at the constant rate of 2 liters/min and there are $2t + 200$ $\mathtt{liters}$ in the tank at time t, the fraction of the mixture being removed per minute at time t is

$$\frac{2}{2t + 200} = \frac{1}{t + 100}.$$

Since the salt is evenly distributed in the mixture, we're removing this same fraction of the salt per minute. Therefore, since there are $Q(t)$ $\mathtt{grams}$ of salt in the tank at time t,

$$\text{rate out} = \frac{Q(t)}{t + 100}. \tag{1.2.8}$$

Substituting (1.2.7) and (1.2.8) into the basic model (1.2.5) and imposing the initial condition $Q(0) = 10$ gives us the desired initial value problem.

$$Q' = 1 - \frac{Q}{t + 100}, \quad Q(0) = 10 \tag{1.2.9}$$

The RLC Circuit

In an RLC *series circuit*, the letters R, L, and C represent resistance, inductance, and capacitance, respectively. The values of R, L, and C are generally constants; the changing quantities in the system are the current $I(t)$ and the charge $Q(t)$ on the capacitor. For reference, refer to the schematic shown in Figure 1.3.

A switch is used to control the flow of current in an RLC circuit: nothing happens while the switch is open, but current flows when the switch is closed to create a *closed circuit*. The current flows in a closed circuit due to a difference in electrical potential, or *voltage*. The battery or generator in Figure 1.3 creates a difference in electrical potential between its two terminals, one of which is negative and one of which is positive. This *impressed voltage* is represented by the function $E = E(t)$. Differences in potential also occur at the resistor, induction coil, and capacitor in Figure 1.3 and are referred to as *voltage drops*. (Note that the two sides of each of these components are also identified

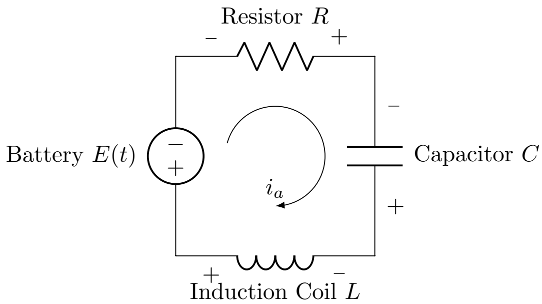


Figure 1.3 An RLC circuit

as positive and negative. The voltage drop across each component is defined to be the potential on the positive side of the component minus the potential on the negative side. This terminology is somewhat misleading, since "drop" suggests a decrease even though changes in potential are signed quantities and therefore may be increases. Nevertheless, we'll go along with tradition and call them voltage drops.)

The voltage drop across the resistor in Figure 1.3 is given by

$$V_R = IR, \tag{1.2.10}$$

where I is current and R is a positive constant that represents the *resistance* of the resistor.

The voltage drop across the induction coil is represented by V_I and is given by

$$L\frac{dI}{dt} = LI', \tag{1.2.11}$$

where L is a positive constant that represents the *inductance* of the coil.

A capacitor stores electrical charge $Q = Q(t)$, which is related to the current in the circuit by the equation

$$Q(t) = Q_0 + \int_0^t I(\tau)\, d\tau, \tag{1.2.12}$$

where Q_0 is the charge on the capacitor at the initial time $t = 0$. The voltage drop across a capacitor is given by

$$V_C = \frac{Q}{C}, \tag{1.2.13}$$

where C is a positive constant that represents the *capacitance* of the capacitor.

The table on Electrical Units lists the unit for each quantity needed to discuss an RLC circuit. The units are defined so that

$$\begin{aligned} 1\,\text{volt} &= 1\,\text{ampere} \cdot 1\,\text{ohm} \\ 1 volt &= 1 ampere \cdot 1 ohm \\ &= 1\,\text{henry} \cdot 1\,\text{ampere/second} \\ &= 1\,\text{coulomb/ farad} \end{aligned}$$

and

$$1\,\text{ampere} = 1\,\text{coulomb/second}.$$

Electrical Units

Symbol	Name	Unit
E	Impressed Voltage	volt
I	Current	ampere
Q	Charge	coulomb
R	Resistance	ohm
L	Inductance	henry
C	Capacitance	farad

According to *Kirchoff's law*, the sum of the voltage drops in a closed RLC circuit equals the impressed voltage. Therefore, from (1.2.10), (1.2.11), and (1.2.13),

$$LI' + RI + \frac{1}{C}Q = E(t). \tag{1.2.14}$$

This equation contains two unknowns, the current I in the circuit and the charge Q on the capacitor. However, (1.2.12) implies that $Q' = I$, so (1.2.14) can be converted into the second order differential equation

$$LQ'' + RQ' + \frac{1}{C}Q = E(t). \tag{1.2.15}$$

In summary, an initial value problem to represent an RLC circuit has the form

$$LQ'' + RQ' + \frac{1}{C}Q = E(t), \quad Q(0) = Q_0, \quad Q'(0) = I_0, \tag{1.2.16}$$

where Q_0 is the initial charge on the capacitor and I_0 is the initial current in the circuit. To find the current flowing in an RLC circuit, we solve (1.2.16) for Q and then differentiate the solution to obtain I.

For example, suppose an RLC circuit has resistance $R = 40$ ohms, inductance $L = .2$ henrys, and capacitance $C = 10^{-5}$ farads. If we know that a current of 2 amperes flows at time $t = 0$ and that the initial charge on the capacitor is 1 coulomb, we can create a mathematical model to find the current flowing in the circuit at any time $t > 0$.

The equation for the charge Q is

$$\frac{1}{5}Q'' + 40Q' + 10000Q = E(t).$$

Therefore, we must solve the initial value problem

$$\frac{1}{5}Q'' + 40Q' + 10000Q = E(t), \quad Q(0) = 1, \quad Q'(0) = 2.$$

The desired current is the derivative of the solution of this initial value problem.

1.2 Exercises

1. Use the Malthusian model from Section 1.1 to set up a differential equation for the population $P(t)$ of a colony of rabbits where the birth rate, represented by the constant $b > 0$, is ten times that of the death rate, represented by the constant $d > 0$.
2. Use the Malthusian model from Section 1.1 to set up a differential equation for the population $P(t)$ of a country where the birth rate $b > 0$ is proportional to the population present at time t but the death rate $d > 0$ is proportional to the square of the population present at time t.
3. Determine a differential equation for the population $P(t)$ of a small country where people immigrate into the country at a constant rate $a > 0$. (Use the Malthusian model from Section 1.1.)
4. Determine a differential equation for the population $P(t)$ of a large country where people emigrate out of the country at a constant rate $b > 0$. (Use the Malthusian model from Section 1.1.)
5. Suppose an object is launched from a point 320 feet above the earth with an initial velocity of 128 ft/sec upward, and the only force acting on it thereafter is gravity. (Use $g = 32$ ft/sec^2.) Construct an initial value problem that models the motion of the object.
6. Suppose a roofer accidentally drops a hammer from the roof of a two-story building that is 8 meters above the earth, and the only force acting on it thereafter is gravity. (Use $g = 9.8$ m/sec^2.) Construct an initial value problem that models the motion of the object.
7. After spring break, a student returns to campus infected with the flu. Suppose there are 4000 students on campus, none of which have been exposed to this flu. Construct an initial value problem for the number of people $x(t)$ who become infected if the rate at which the illness spreads is proportional to the number of encounters between those students who have the flu and those who have not yet been exposed to it.
8. A couple attends a large family reunion infected with Severe Acute Respiratory Syndrome (SARS). Suppose there are 84 family members at the reunion, none of which have been exposed to this airborne illness. Construct an initial value problem for the number of people $x(t)$ who become infected if the rate at which the illness spreads is proportional to the number of encounters between those family members who have SARS and those who have not yet been exposed to it.
9. An employee at a large company shares a private video by email with a co-worker, and the video goes "viral". Assume the company has a fixed population of n employees, none of whom have previously seen the video. Construct an initial value problem to represent the number of people $x(t)$ who have seen the video if we assume the rate at which the video is spread throughout the company is jointly proportional to the number of people who have seen it and the number of people $y(t)$ who have not seen it.

10. A pair of disgruntled employees begin a rumor about Company XYZ in the break room one day. Assume the company has a fixed population of n employees, none of whom have previously heard the rumor. Construct an initial value problem to represent the number of people $x(t)$ who have heard the rumor if we assume the rate at which the rumor is spread throughout the company is jointly proportional to the number of people who have heard it and the number of people $y(t)$ who have not heard it.

11. A candymaker makes 500 pounds of candy per week, while his large family eats the candy at a rate equal to $Q(t)/10$ pounds per week, where $Q(t)$ is the amount of candy present at time t. Find an initial value problem whose solution is $Q(t)$ for $t > 0$ if the candymaker has 250 pounds of candy at time $t = 0$.

12. A wizard creates gold continuously at the rate of 1 ounce per hour, but an assistant steals it continuously at the rate of 5% of however much is there per hour. Construct an initial value problem whose solution is $W(t)$, the number of ounces that the wizard has at time t. Assume the wizard begins with 1 ounce of gold.

13. A process creates a radioactive substance at the rate of 1 gram per hour, and the substance decays at an hourly rate equal to $1/10$ of the mass present (expressed in grams). Assuming that there are initially 20 grams, construct an initial value problem to find the mass $P(t)$ of the substance present at time t.

14. A tank initially contains 40 gallons of pure water. A water solution with 1 gram of salt per gallon is added to the tank at 3 gal/min, and the resulting solution drains out at the same rate. Find an initial value problem whose solution is the quantity $Q(t)$ of salt in the tank at time $t > 0$.

15. A tank initially contains a solution of 10 pounds of salt in 60 gallons of water. Water with $1/2$ pound of salt per gallon is added to the tank at 6 gal/min, and the resulting solution drains at the same rate. Find an initial value problem whose solution is the quantity $Q(t)$ of salt in the tank at time $t > 0$.

16. A 200 gallon tank initially contains 100 gallons of water with 20 pounds of salt. A salt solution with $1/4$ pound of salt per gallon is added to the tank at 4 gal/min, and the resulting mixture is drained out at 2 gal/min. Find an initial value problem whose solution is the quantity $Q(t)$ of salt in the tank any time before it overflows. Be sure to include the domain of the solution.

17. A 1200 gallon tank initially contains 40 pounds of salt dissolved in 600 gallons of water. Starting at $t_0 = 0$, water that contains $1/2$ pound of salt per gallon is added to the tank at the rate of 6 gal/min and the resulting mixture is drained from the tank at 4 gal/min. Find an initial value problem whose solution is the quantity $Q(t)$ of salt in the tank any time before it overflows. Be sure to include the domain of the solution.

18. Find an initial value problem that serves as a mathematical model for the RLC circuit with the values $R = 3$ ohms, $L = .1$ henrys, $C = .01$ farads, $Q_0 = 0$ coulombs, and $I_0 = 2$ amperes.

19. Find an initial value problem that serves as a mathematical model for the RLC circuit with the values $R = 2$ ohms, $L = .05$ henrys, $C = .01$ farads, $Q_0 = 2$ coulombs, and $I_0 = -2$ amperes.

20. Find an initial value problem that serves as a mathematical model for the RLC circuit with the values $R = 6$ ohms, $L = .1$ henrys, $C = .004$ farads, $Q_0 = 3$ coulombs, and $I_0 = -10$ amperes.

21. Find an initial value problem that serves as a mathematical model for the RLC circuit with the values $R = 4$ ohms, $L = .05$ henrys, $C = .008$ farads, $Q_0 = -1$ coulombs, and $I_0 = 2$ amperes.

1.3 ANALYZING SOLUTION CURVES WITHOUT SOLVING EQUATIONS

Some differential equations have no solutions; for others, it's impossible to find explicit formulas for solutions. Even if there are such formulas, they may be so complicated that they're useless. In such cases we may resort to graphical or numerical methods to get some idea of how the solutions to the given equation behave.

The next chapter will address the question of the existence of solutions of a first order equation

$$y' = f(x, y). \tag{1.3.1}$$

In this section we'll simply assume that (1.3.1) has solutions and discuss graphical methods for approximating them.

Direction Fields

Recall that a solution of (1.3.1) is a *function* $y = y(x)$ such that

$$y'(x) = f(x, y(x))$$

for all values of x in some interval, and that the graph of $y(x)$ is referred to as a solution curve. In the more general case, we may be interested in a graph of the solution(s) that need not be a function. Such a curve C is called an *integral curve* of a differential equation: that is, every function $y = y(x)$ whose graph is a segment of C is a solution of the differential equation. Thus, any solution curve of a differential equation is an integral curve, but an integral curve need not be a solution curve. This means an integral curve is either the graph of a solution or is made up of segments that are graphs of solutions.

Example 1.3.1 If a is any positive constant, the circle

$$x^2 + y^2 = a^2 \tag{1.3.2}$$

is an integral curve of

$$y' = -\frac{x}{y}. \tag{1.3.3}$$

To see this, consider the two functions whose graphs are segments of (1.3.2) are

$$y_1 = \sqrt{a^2 - x^2} \quad \text{and} \quad y_2 = -\sqrt{a^2 - x^2}.$$

We leave it to you to verify that these functions both satisfy (1.3.3) on the open interval $(-a, a)$. However, (1.3.2) is not a solution curve of (1.3.3), since it's not the graph of a function.

■

Not being able to solve an equation of the form (1.3.1) is equivalent to not knowing the equations of its integral curves. However, it is easy to calculate the slopes of these curves because they are first order differential equations. To be specific, the slope of an integral curve of (1.3.1) through a given point (x_0, y_0) is given by the number $f(x_0, y_0)$. This is the basic idea behind *direction fields*.

If f is defined on a region R, we can construct a direction field for (1.3.1) in R by drawing a short line segment through each point (x, y) in R with slope $f(x, y)$. As a practical matter, we can't draw line segments through every point in R; rather, we select a finite set of points to be representative of R. For example, suppose f is defined on the closed rectangular region

$$R : \{a \leqslant x \leqslant b, c \leqslant y \leqslant d\}.$$

Choose equally spaced points in $[a, b]$ so that

$$a = x_0 < x_1 < \cdots < x_m = b;$$

Similarly, choose equally spaced points in $[c, d]$ so that

$$c = y_0 < y_1 < \cdots < y_n = d.$$

This creates a finite set of ordered pairs

$$(x_i, y_j), \quad 0 \leqslant i \leqslant m, \quad 0 \leqslant j \leqslant n,$$

that form a *rectangular grid*. (See Figure 1.1.) Through each point in the grid we draw a short line segment with slope $f(x_i, y_j)$. The result is an approximation to a direction field for (1.3.1) in R. If the grid points are sufficiently numerous and close together, we can draw approximate integral curves of (1.3.1) by drawing curves through points in the grid. At each point, the solution curve should be tangent to the line segment associated with that point in the grid.

Unfortunately, approximating a direction field and graphing integral curves in this way is too tedious to be done effectively by hand. However, there is software for doing this.

The combination of direction fields and integral curves provides insight into the behavior of the solutions even if we can't solve the differential equation. Figures 1.2a and 1.2b show direction fields and solution curves for the differential equations

$$y' = \frac{x^2 - y^2}{1 + x^2 + y^2} \quad \text{and} \quad y' = 1 + xy^2,$$

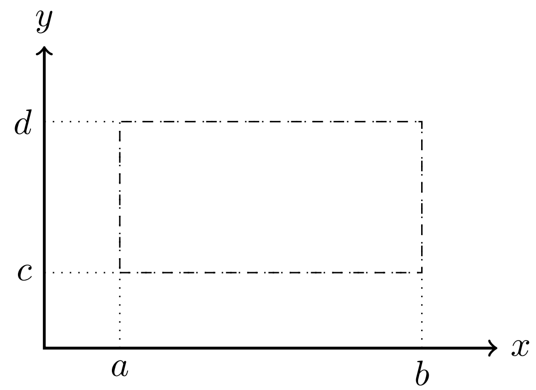
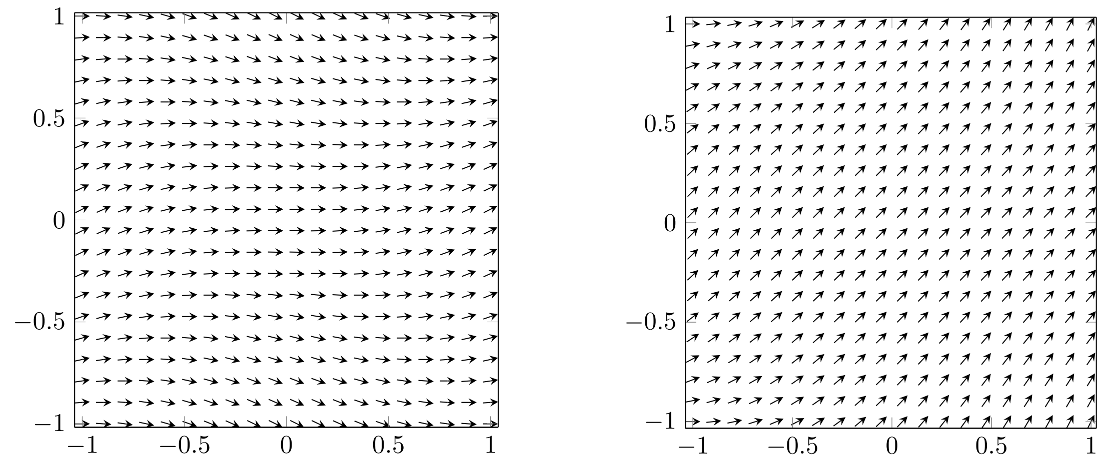


Figure 1.1 A rectangular grid



(a) A direction field and integral curves for $y = \frac{x^2-y^2}{1+x^2+y^2}$

(b) A direction field and integral curves for $y' = 1 + xy^2$

Figure 1.2 Direction Fields

which are both of the form (1.3.1). Notice that for both differential equations, $f(x, y)$ is continuous for all (x, y).

When a first order differential equation is such that $f(x, y)$ is *not* continuous for all (x, y), numerical methods can be limited. (A discussion of numerical methods is found in the Appendix.) For example, they do not work for the equation

$$y' = -x/y \tag{1.3.4}$$

if the region R contains any part of the x-axis, since $f(x, y) = -x/y$ is undefined when $y = 0$. Similarly, numerical methods will not work for the equation

$$y' = \frac{x^2}{1 - x^2 - y^2} \tag{1.3.5}$$

if R contains any part of the unit circle $x^2 + y^2 = 1$, because the right side of (1.3.5) is undefined if $x^2 + y^2 = 1$. However, we can still generate direction fields for these first order differential equations.

Figure 1.3 shows a direction field and some integral curves for (1.3.4). As we saw in Example 1.3.1, the integral curves of (1.3.4) are circles centered at the origin.

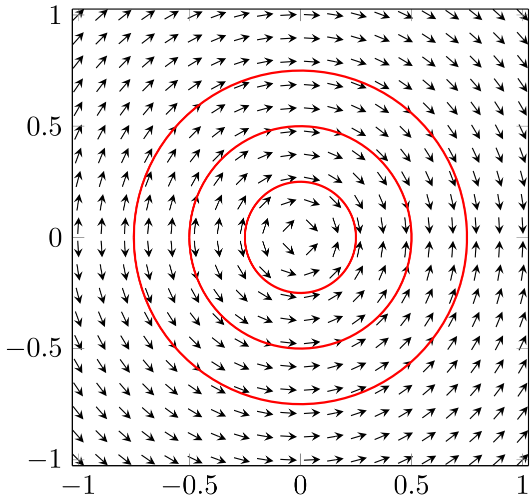


Figure 1.3 A direction field and integral curves for $y' = -\dfrac{x}{y}$

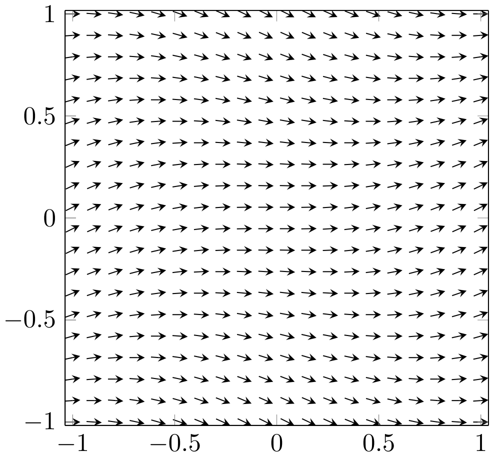
Figure 1.4a shows a direction field and some integral curves for (1.3.5). The integral curves near the top and bottom are solution curves. However, the integral curves near the middle are more complicated. For example, Figure 1.4b shows the integral curve through the origin. Two points from the circle $x^2 + y^2 = 1$ ($a \approx .846$, $b \approx .533$) are marked on this integral curve at (a, b) and $(-a, -b)$; at these points, the integral curve of (1.3.5) has infinite slope. The integral curve in Figure 1.4b is comprised of three solution curves of (1.3.5): the segment above the level $y = b$ is the graph of a solution on $(-\infty, a)$, the segment below the level $y = -b$ is the graph of a solution on $(-a, \infty)$, and the segment between these two levels is the graph of a solution on $(-a, a)$.

Phase Portraits

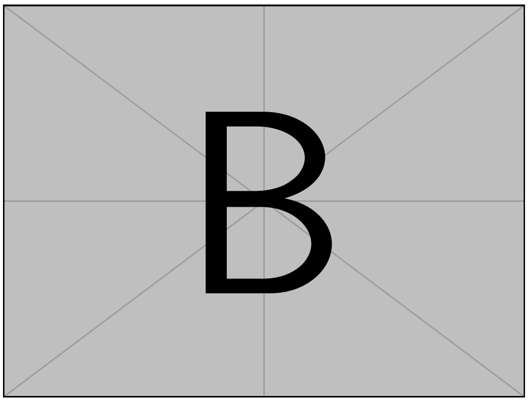
Now we consider a special type of differential equation where the independent variable does not appear in the equation. Such equations are said to be *autonomous*. For an autonomous first order differential equation, (1.3.1) takes the form $y' = f(y)$. For such equations, we can create a *phase portrait* that provides a geometric representation of the solution curves. First, we show how to create a phase portrait for an autonomous first order differential equation of the form

$$\frac{dy}{dx} = f(y), \tag{1.3.6}$$

and then we discuss what it tells us about the solution curves. (Although it may be



(a) A direction field for $y' = \frac{x^2}{1-x^2-y^2}$



(b) Integral curves for $y' = \frac{x^2}{1-x^2-y^2}$

Figure 1.4 A direction field and integral curves for $y' = \frac{x^2}{1-x^2-y^2}$

inconvenient to write, the *Leibniz notation* is helpful when representing these types of equations since it clarifies which variable is the independent variable.)

To begin, find the real values c such that $f(c) = 0$ in (1.3.6). Now consider the constant function $y(x) = c$: if we substitute this function into (1.3.6), we see that both sides of the equation will be zero. Therefore, $y(x) = c$ is a constant solution of the autonomous first order differential equation. In fact, the zeros of $f(y)$ are the only constant solutions of (1.3.6). A real value c that is a zero of $f(y)$ is referred to as an *equilibrium point*, and the corresponding function $y(x) = c$ is referred to as an *equilibrium solution*,

Now we graph a vertical line to represent the y-axis and mark the equilibrium points on it with a horizontal line. This divides the y-axis into intervals. To complete the phase portrait, we use a value from each interval on the y-axis to determine the algebraic sign of the derivative function $f(y)$ on that interval and mark an appropriate arrow on the corresponding interval of the y-axis.

Example 1.3.2 Find the equilibrium points of

$$\frac{dy}{dx} = y(2-y)(4-y) \tag{1.3.7}$$

and use these to create a phase portrait of the autonomous first order differential equation.

Solution Using the Zero Product Property with $f(y) = 0$ gives

$$y(2-y)(4-y) = 0,$$

so there are equilibrium points at $y = 0$, $y = 2$, and $y = 4$. This creates four intervals on the y-axis. For the interval where $y > 4$, we can test $y = 5$ in $f(y)$ to get

$$\frac{dy}{dx} = 5(2-5)(4-5).$$

Since the derivative has the value $15 > 0$, any solution curve passing through $y = 5$ must have the same positive slope, regardless of the value of x. In fact, any solution curve in the region where $y > 4$ must have positive slope, although not necessarily with a value of 15. (Test a few values of y to convince yourself, if needed.) We indicate this by drawing an arrow that points up on the y-axis above $y = 4$. In a similar fashion, we can test values in the other three intervals. (For example, $f(3) < 0$, $f(1) > 0$, and $f(-1) < 0$.) Adding arrows to the remaining intervals completes the phase portrait. (See PHASE PORTRAIT.) ■

INSERT FIGURE: PHASE PORTRAIT

The equilibrium points in a phase portrait divide the y-axis into intervals, and these intervals on the y-axis divide the xy-plane into corresponding subregions. Within a subregion, any nonconstant solution $y = y(x)$ of (1.3.6) must be continuous and therefore cannot change signs algebraically. (Recall that the equilibrium points mark the location of the zeros). This means that a nonconstant solution must be *strictly monotonic* – that is, either continually increasing or continually decreasing – within the subregion. Functions that are strictly monotonic cannot have relative extrema (maximum or minimum values), nor can they be oscillatory. Knowing these facts about the nonconstant solutions of an autonomous first order differential equation tells us a great deal about the solution curves – without actually solving the equation!

Knowing that the graph of a nonconstant solution cannot cross the graph of an equilibrium solution and that a nonconstant solution must be strictly monotonic suggests asymptotic behavior near the equilibrium points. For example, in PHASE PORTRAIT consider a nonconstant solution $y(x)$ that is bounded above by the equilibrium point $c = 2$ and bounded below by the equilibrium point $c = 0$. In this region, the graph of $y(x)$ must approach the graph of the equilibrium solution $y(x) = 2$ as $x \to \infty$ and must approach the graph of the equilibrium solution $y(x) = 0$ as $x \to -\infty$ since we know that $y(x)$ is continually increasing.

SAMPLE SOLUTION CURVES shows the phase portrait for (1.3.7) with the subregions it creates in the xy-plane. (The subregions have been labeled for ease of reference.) Sample solution curves are shown for each region.

INSERT FIGURE: SAMPLE SOLUTION CURVES

In the phase portrait for (1.3.7), the equilibrium point at $y = 4$ has arrows on either side pointing away from $y = 4$. This means all nonconstant solutions of $y(x)$ that start from any initial point in R_4 or R_3 will move away from $y = 4$ as x values increase. Equilibrium points such as this are said to be *unstable*. (For obvious reasons, this type of point is also called a *repeller*.) On the other hand, the equilibrium point at $y = 2$ has arrows on either side pointing toward $y = 2$. This means all nonconstant solutions of $y(x)$ that start from any initial point in R_3 or R_2 will move toward $y = 2$ as x values increase. Equilibrium points such as this are said to be *asymptotically stable*. (This type of point is also called an *attractor*.) Other equilibrium points that attract from one side and repel from the other are referred to as *semi stable*.

Many differential equations that model physical laws are autonomous because the laws themselves do not change with the passing of time. The Malthusian model of population growth discussed in the first section is an autonomous first order differential equation where

$$\frac{dP}{dt} = aP.$$

Another model of population growth that accounts for limitations of space and resources is the *Verhulst model*. This model uses the autonomous first order differential equation

$$\frac{dP}{dt} = aP(1 - \alpha P), \tag{1.3.8}$$

where both a and α are positive constants.

Recall that a flaw in the Malthusian model was that there was no limiting value to the size of the population with the passing of time. We can use a phase portrait of the Verhulst model to determine the limiting value of a population that grows according to the model, without solving the differential equation itself. (You will learn how to solve it later.)

Example 1.3.3 Find the equilibrium points of the Verhulst model

$$\frac{dP}{dt} = aP(1 - \alpha P),$$

and use these to create a phase portrait of the autonomous first order differential equation. Then identify any asymptotically stable equilibrium points.

Solution First set

$$aP(1 - \alpha P) = 0$$

to find equilibrium points at $P = 0$ and $P = 1/\alpha$. For this application based on population, we need only concern ourselves with positive values of the dependent variable. This means we need only two test values: P_1 between 0 and $1/\alpha$, and $P_2 > 1/\alpha$. For P_1, we choose half of $1/\alpha$ since we do not know the numerical value of α. Substituting $1/2\alpha$ gives

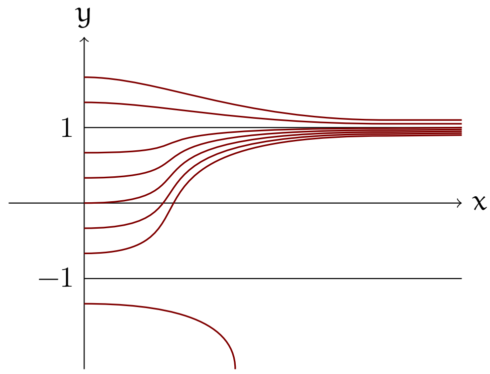
$$aP_1(1 - \alpha P_1) = 1/2aP_1$$

which we know is positive since a and P_1 are both positive. For P_2, we choose to double $1/\alpha$, which is $2/\alpha$. Testing this value gives

$$aP_2(1 - \alpha P_2) = -aP_2$$

which we know is negative since a and P_1 are both positive. The phase portrait shows that $1/\alpha$ is an asymptotically stable equilibrium point. This means that all solution curves will approach the horizontal asymptote $P = 1/\alpha$ as t increases, regardless of the value of the initial population, P_0. Sample solution curves are shown with the phase portrait for the Verhulst model in VERHULST.

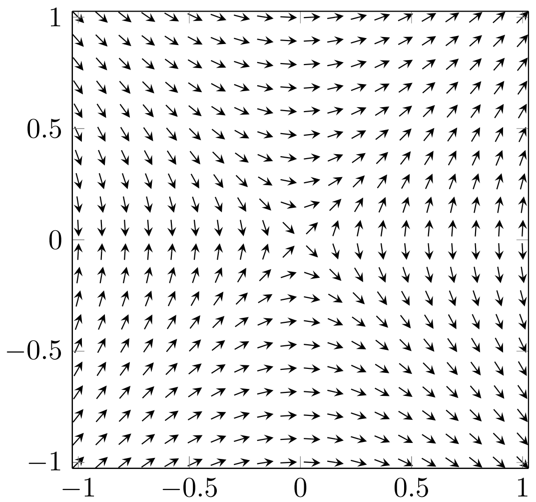
INSERT FIGURE: VERHULST



1.3 Exercises

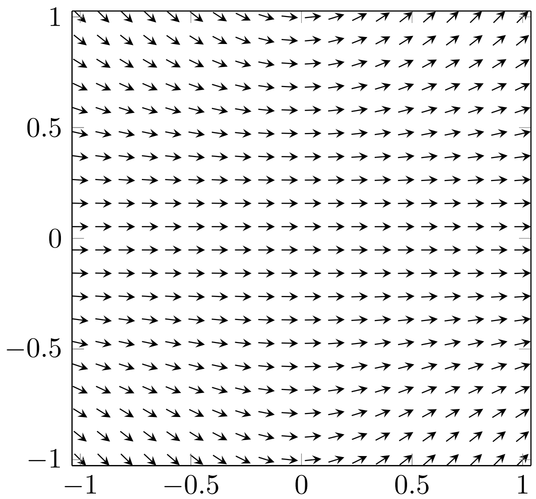
In Exercises 1–11 a direction field is drawn for the given equation. Sketch some integral curves.

1. $y' = \dfrac{x}{y}$



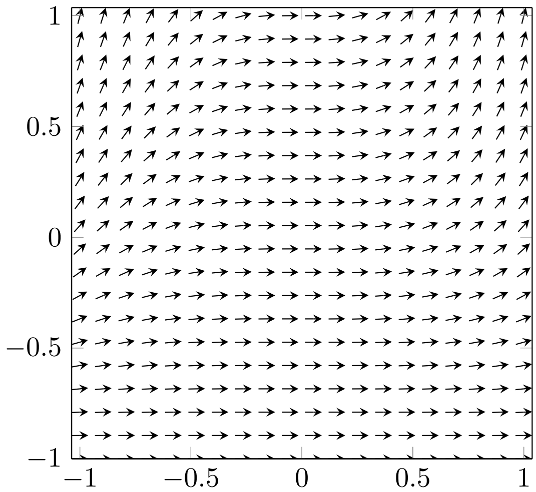
A direction field for $y' = \dfrac{x}{y}$

2. $y' = \dfrac{2xy^2}{1+x^2}$



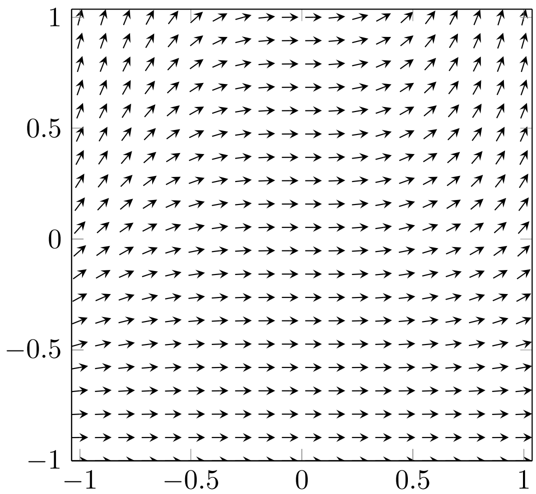
A direction field for $y' = \dfrac{2xy^2}{1+x^2}$

3. $y' = x^2(1 + y^2)$



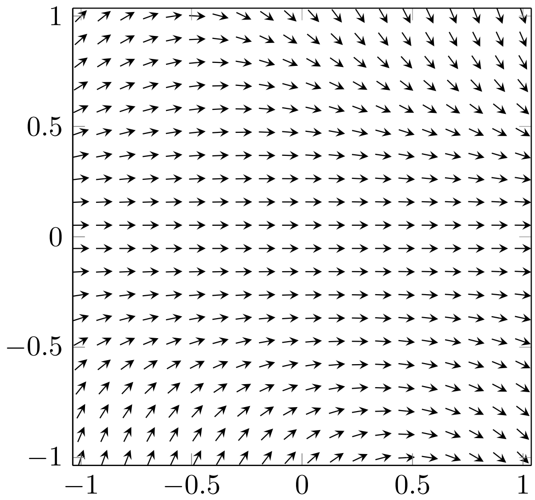
A direction field for $y' = x^2(1 + y^2)$

4. $y' = \dfrac{1}{1 + x^2 + y^2}$



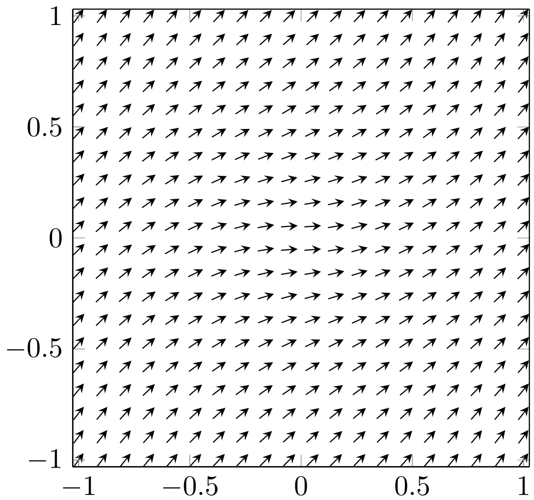
A direction field for $y' = \dfrac{1}{1 + x^2 + y^2}$

5. $y' = -(2xy^2 + y^3)$



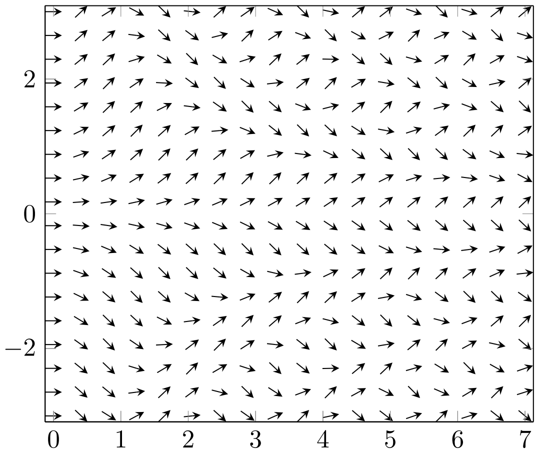
A direction field for $y' = -(2xy^2 + y^3)$

6. $y' = (x^2 + y^2)^{1/2}$



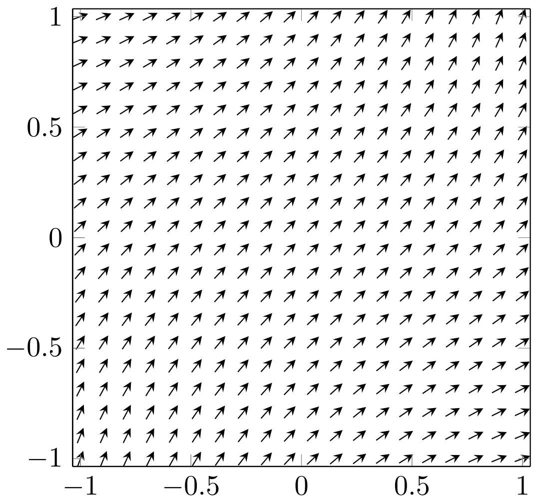
A direction field for $y' = (x^2 + y^2)^{1/2}$

7. $y' = \sin xy$



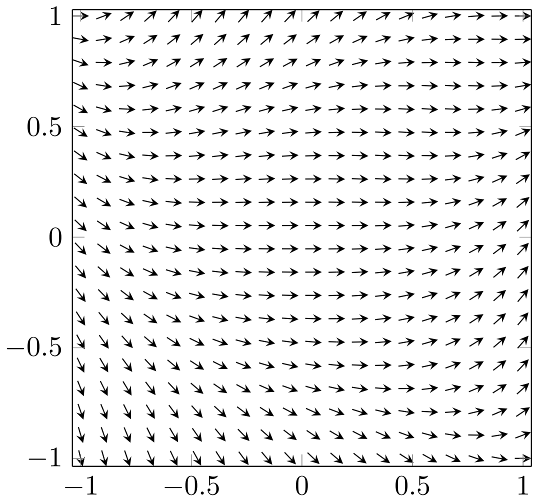
A direction field for $y' = \sin xy$

8. $y' = e^{xy}$



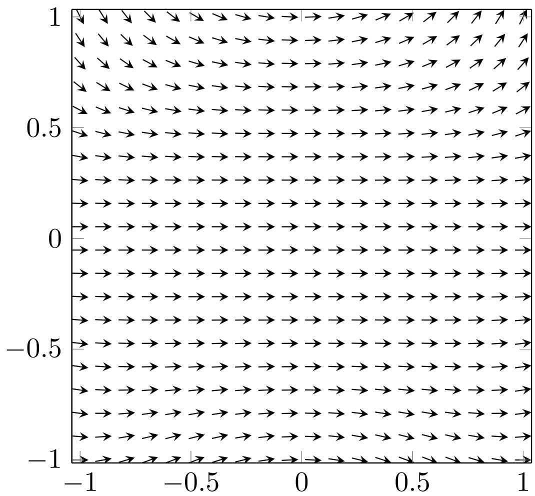
A direction field for $y' = e^{xy}$

9. $y' = (x - y^2)(x^2 - y)$



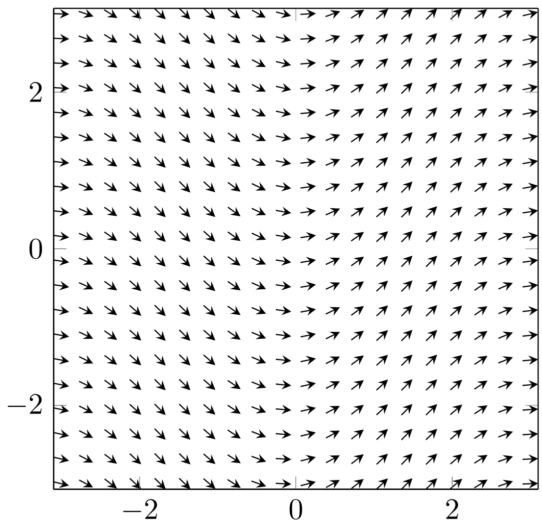
A direction field for $y' = (x - y^2)(x^2 - y)$

10. $y' = x^3y^2 + xy^3$



A direction field for $y' = x^3y^2 + xy^3$

11. $y' = \sin(x - 2y)$



A direction field for $y' = \sin(x - 2y)$

In Exercises 12-13 *construct a direction field in the indicated rectangular region.*

12. $y' = y(y - 1); \quad \{-1 \leqslant x \leqslant 2,\ -2 \leqslant y \leqslant 2\}$

13. $y' = 2 - 3xy; \quad \{-1 \leqslant x \leqslant 4,\ -4 \leqslant y \leqslant 4\}$

In Exercises 14-21 *find the equilibrium points and phase portrait of the given autonomous first order differential equation. Classify each equilibrium point as asymptotically stable, unstable, or semi-stable. By hand, sketch typical solution curves in each region of the plane created by the graph of the equilibrium solution.*

14. $\frac{dy}{dx} = 4y - y^2$

15. $\frac{dy}{dx} = y^3 - 2y^2$

16. $\frac{dy}{dx} = y^2 - 5x + 6$

17. $\frac{dy}{dx} = 10 + 3y - y^2$

18. $\frac{dP}{dt} = P(a - bP)$

19. $\frac{dR}{dt} = k(a - R)(b - R)$

20. $\frac{dF}{dt} = kF(n + 1 - F)$

21. $\frac{dM}{dt} = 4 - \frac{M}{100}$

CHAPTER 2

FIRST ORDER EQUATIONS

"Begin at the beginning,¨ the King said gravely, "and go on till you come to the end: then stop."

— Lewis Carroll, *Alice in Wonderland*

IN THIS CHAPTER we study first order equations for which there are general methods of solution.

SECTION 2.1 deals with linear equations, the simplest kind of first order equations. In this section we introduce the method of variation of parameters. The idea underlying this method will be a unifying theme for our approach to solving many different kinds of differential equations throughout the book.

SECTION 2.2 deals with separable equations, the simplest nonlinear equations. In this section we introduce the idea of implicit and constant solutions of differential equations, and we point out some differences between the properties of linear and nonlinear equations.

SECTION 2.3 discusses existence and uniqueness of solutions of nonlinear equations. Although it may seem logical to place this section before Section 2.2, Section 2.2 is presented first so that we could have illustrative examples in Section 2.3.

SECTION 2.4 deals with nonlinear equations that are not separable, although they can be transformed into separable equations by a procedure similar to variation of parameters.

SECTION 2.5 covers exact differential equations, which are given this name because the method for solving them uses the idea of an exact differential from calculus.

SECTION 2.6 deals with equations that are not exact, although they can be made exact by multiplying them by a function known as an integrating factor.

2.1 LINEAR FIRST ORDER EQUATIONS

A first order differential equation is said to be *linear* if it can be written in *standard form* as

$$y' + p(x)y = f(x). \tag{2.1.1}$$

A first order differential equation that cannot be written like this is *nonlinear*. We say that (2.1.1) is *homogeneous* if $f \equiv 0$; otherwise it is *nonhomogeneous*. Since $y \equiv 0$ is obviously a solution of the homogeneous equation

$$y' + p(x)y = 0,$$

we call it the *trivial solution*. Any other solution is *nontrivial*.

Example 2.1.1 These first order equations are not in standard form (2.1.1), but they are linear.

$$\begin{aligned} x^2y' + 3y &= x^2 \\ xy' - 8x^2y &= \sin x \\ xy' + (\ln x)y &= 0 \\ y' &= x^2y - 2 \end{aligned}$$

Rewritten in standard form, they have these forms.

$$\begin{aligned} y' + \frac{3}{x^2}y &= 1, \\ y' - 8xy &= \frac{\sin x}{x}, \\ y' + \frac{\ln x}{x}y &= 0, \\ y' - x^2y &= -2. \end{aligned}$$

■

Example 2.1.2 Here are some nonlinear first order equations.

$$\begin{aligned} xy' + 3y^2 &= 2x && \text{(because } y^2 \text{ is not of first degree),} \\ yy' &= 3 && \text{(because } y \text{ in the } y' \text{ term is not a function of } x\text{),} \\ y' + xe^y &= 12 && \text{(because } e^y \text{ is not linear).} \end{aligned}$$

■

General Solution of a Linear First Order Equation

To motivate an important definition, consider the simple linear first order equation

$$y' = \frac{1}{x^2}. \tag{2.1.2}$$

From calculus we know that y satisfies this equation if and only if

$$y = -\frac{1}{x} + c, \tag{2.1.3}$$

where c is an arbitrary constant. We call c a *parameter* and say that (2.1.3) defines a *one–parameter family* of functions. For each real number c, the function defined by (2.1.3) is a solution of (2.1.2) on $(-\infty, 0)$ and $(0, \infty)$; moreover, every solution of (2.1.2) on either of these intervals is of the form (2.1.3) for some choice of c.

A similar situation occurs in connection with any first order linear equation

$$y' + p(x)y = f(x); \tag{2.1.4}$$

that is, if p and f are continuous on some open interval (a, b) then there's a unique formula $y = y(x, c)$ analogous to (2.1.3) that involves a function of x and a parameter c which has these properties:

- For each fixed value of c, the resulting function of x is a solution of (2.1.4) on (a, b).
- If y is a solution of (2.1.4) on (a, b), then y can be obtained from the formula by choosing c appropriately.

We will call $y = y(x, c)$ the *general solution* of (2.1.4).

When this has been established, it will follow that an equation of the form

$$P_0(x)y' + P_1(x)y = F(x) \tag{2.1.5}$$

has a general solution on any open interval (a, b) on which P_0, P_1, and F are all continuous and P_0 has no zeros, since in this case we can rewrite (2.1.5) in the form (2.1.4) with $p = P_1/P_0$ and $f = F/P_0$, which are both continuous on (a, b).

To avoid awkward wording in examples and exercises, we will not specify the interval (a, b) when we ask for the general solution of a specific linear first order equation. Let us agree that this always means that we want the general solution on every open interval on which p and f are continuous if the equation is of the form (2.1.4), or on which P_0, P_1, and F are continuous and P_0 has no zeros, if the equation is of the form (2.1.5). We leave it to you to identify these intervals in specific examples and exercises.

For completeness, we point out that if P_0, P_1, and F are all continuous on an open interval (a, b), but P_0 *does* have a zero in (a, b), then (2.1.5) may fail to have a general solution on (a, b) in the sense just defined. Since this is not a major point that needs to be developed in depth, we will not discuss it further; however, see Exercise 44 for an example.

Homogeneous Linear First Order Equations

We begin with the problem of finding the general solution of a homogeneous linear first order equation. The next example recalls a familiar result from calculus.

Example 2.1.3 Let a be a constant, and let y' represent $\frac{dy}{dx}$. Find the general solution of

$$y' - ay = 0. \tag{2.1.6}$$

Solution You may remember from calculus that if c is any constant, then $y = ce^{ax}$ satisfies (2.1.6). Even without this knowledge, we can use this problem to illustrate a general method for solving a homogeneous linear first order equation.

We know that (2.1.6) has the trivial solution $y \equiv 0$. Now suppose y is a nontrivial solution of (2.1.6). Then, since a differentiable function must be continuous, there must be some open interval I on which y has no zeros. On this interval, we can rewrite (2.1.6) as

$$\frac{y'}{y} = a;$$

using the Leibniz notation for clarity (and some algebra) gives us

$$\frac{1}{y}\frac{dy}{dx} = a.$$

Finally, multiply both sides by the differential dx to obtain

$$\frac{1}{y}dy = adx.$$

Integrating both sides of this equation gives us

$$\ln|y| = ax + k.$$

(There is no need to use two constants in this type of integration. If we did use constants c_1 and c_2 on the left and right sides, respectively, then we could simply rewrite the equation using $k = c_2 - c_1$.)

Now we exponentiate both sides to get

$$|y| = e^k e^{ax}.$$

(Use rules of exponents to rewrite e^{ax+k} as $eaxe^k$.) Since e^{ax} can never equal zero, y has no zeros, which means that y is either always positive or always negative. Therefore we can rewrite y as

$$y = ce^{ax} \tag{2.1.7}$$

where

$$c = \begin{cases} e^k & \text{if } y > 0, \\ -e^k & \text{if } y < 0. \end{cases}$$

This shows that every nontrivial solution of (2.1.6) is of the form $y = ce^{ax}$ for some nonzero constant c. Since setting $c = 0$ yields the trivial solution, *all* solutions of (2.1.6) are of the form (2.1.7). Conversely, (2.1.7) is a solution of (2.1.6) for every choice of c, since differentiating (2.1.7) yields $y' = ace^{ax} = ay$. ∎

Rewriting a first order differential equation so that one side depends only on y and y' and the other depends only on x is called *separation of variables*. We will apply this method to nonlinear equations in Section 2.2.

Example 2.1.4 **(a)** Find the general solution of

$$xy' + y = 0. \tag{2.1.8}$$

(b) Solve the initial value problem

$$xy' + y = 0, \quad y(1) = 3. \tag{2.1.9}$$

(a) We rewrite (2.1.8) as

$$y' + \frac{1}{x}y = 0, \tag{2.1.10}$$

where x is restricted to either $(-\infty, 0)$ or $(0, \infty)$. If y is a nontrivial solution of (2.1.10), there must be some open interval I on which y has no zeros. We can rewrite (2.1.10) as

$$\frac{y'}{y} = -\frac{1}{x}$$

for all x in the specified interval I. Integration using separation of variables shows that

$$\ln|y| = -\ln|x| + k, \quad \text{so} \quad |y| = \frac{e^k}{|x|}.$$

Since a function that satisfies the last equation cannot change sign on either $(-\infty, 0)$ or $(0, \infty)$, we can rewrite this result more simply as

$$y = \frac{c}{x} \tag{2.1.11}$$

where

$$c = \begin{cases} e^k & \text{if } y > 0, \\ -e^k & \text{if } y < 0. \end{cases}$$

We have now shown that every solution of (2.1.10) is given by (2.1.11) for some choice of c. (Even though we assumed that y was nontrivial to derive (2.1.11), we can get the trivial solution by setting $c = 0$ in (2.1.11).) Conversely, any function of the form (2.1.11) is a solution of (2.1.10), since differentiating (2.1.11) yields

$$y' = -\frac{c}{x^2},$$

and substituting this and (2.1.11) into (2.1.10) yields

$$\begin{aligned} y' + \frac{1}{x}y &= -\frac{c}{x^2} + \frac{1}{x}\frac{c}{x} \\ &= -\frac{c}{x^2} + \frac{c}{x^2} = 0. \end{aligned}$$

(b) Imposing the initial condition $y(1) = 3$ in (2.1.11) yields $c = 3$. Therefore the solution of (2.1.9) is

$$y = \frac{3}{x}.$$

The domain of this solution is $(0, \infty)$.

The results in Examples 2.1.3 and 2.1.4**(b)** are special cases of the next theorem.

Theorem 2.1.1 *If p is continuous on (a,b), then the general solution of the homogeneous equation*

$$y' + p(x)y = 0 \tag{2.1.12}$$

on (a,b) is

$$y = ce^{-P(x)},$$

where

$$P(x) = \int p(x)\,dx \tag{2.1.13}$$

is any antiderivative of p on (a,b); that is,

$$P'(x) = p(x), \quad a < x < b. \tag{2.1.14}$$

Proof If $y = ce^{-P(x)}$, differentiating y and using (2.1.14) shows that

$$y' = -P'(x)ce^{-P(x)} = -p(x)ce^{-P(x)} = -p(x)y,$$

so $y' + p(x)y = 0$; that is, y is a solution of (2.1.12), for any choice of c.

Now we'll show that any solution of (2.1.12) can be written as $y = ce^{-P(x)}$ for some constant c. The trivial solution can be written this way, with $c = 0$. Now suppose y is a nontrivial solution. Then there's an open subinterval I of (a,b) on which y has no zeros. We can rewrite (2.1.12) as

$$\frac{y'}{y} = -p(x) \tag{2.1.15}$$

for x in I. Integrating (2.1.15) and recalling (2.1.13) yields

$$\ln|y| = -P(x) + k,$$

where k is a constant. This implies that

$$|y| = e^k e^{-P(x)}.$$

Since P is defined for all x in (a,b) and an exponential can never equal zero, we can take $I = (a,b)$, and can rewrite the last equation as $y = ce^{-P(x)}$, where

$$c = \begin{cases} e^k & \text{if } y > 0 \text{ on } (a,b), \\ -e^k & \text{if } y < 0 \text{ on } (a,b). \end{cases}$$

■

Linear Nonhomogeneous First Order Equations

We now solve the nonhomogeneous equation

$$y' + p(x)y = f(x). \tag{2.1.16}$$

When considering this equation we call

$$y' + p(x)y = 0$$

the *complementary equation*.

We will find solutions of (2.1.16) in the form $y = uy_1$, where y_1 is a nontrivial solution of the complementary equation and u is to be determined. This method of using a solution of the complementary equation to obtain solutions of a nonhomogeneous equation is a special case of a method called *variation of parameters*, which you will encounter several times in this book.

If

$$y = uy_1, \quad \text{then} \quad y' = u'y_1 + uy_1'.$$

Substituting these expressions for y and y' into (2.1.16) yields

$$u'y_1 + u(y_1' + p(x)y_1) = f(x),$$

which reduces to

$$u'y_1 = f(x), \tag{2.1.17}$$

since y_1 is a solution of the complementary equation; that is,

$$y_1' + p(x)y_1 = 0.$$

(Obviously, u cannot be constant, since if it were, the left side of (2.1.17) would be zero. Recognizing this, the early users of this method viewed u as a "parameter" that varies; hence, the name "variation of parameters.")

In the proof of Theorem 2.2.1 we saw that y_1 has no zeros on an interval where p is continuous. Therefore we can divide through by y_1 in (2.1.17) to obtain

$$u' = f(x)/y_1(x).$$

We can integrate this (introducing a constant of integration), and multiply the result by y_1 to get the general solution of (2.1.16). Before turning to the formal proof of this claim, let us look at some examples.

Example 2.1.5 Find the general solution of

$$y' + 2y = x^3e^{-2x}. \tag{2.1.18}$$

By applying Example 2.1.3 with $a = -2$, we see that $y_1 = e^{-2x}$ is a solution of the complementary equation $y' + 2y = 0$. Therefore we seek solutions of (2.1.18) in the form $y = ue^{-2x}$. Taking the derivative and then substituting gives

$$y' + 2y = u'e^{-2x} - 2ue^{-2x} + 2ue^{-2x}. \tag{2.1.19}$$

Since (2.1.19) reduces to $u'e^{-2x}$, y is a solution of (2.1.18) if and only if

$$u'e^{-2x} = x^3e^{-2x}.$$

Therefore, $u' = x^3$ and integrating gives

$$u = \frac{x^4}{4} + c.$$

So the general solution of (2.1.18) is

$$y = ue^{-2x} = e^{-2x}\left(\frac{x^4}{4} + c\right).$$

Example 2.1.6

(a) Find the general solution

$$y' + (\cot x)y = x\csc x. \tag{2.1.20}$$

(b) Solve the initial value problem

$$y' + (\cot x)y = x\csc x, \quad y(\pi/2) = 1. \tag{2.1.21}$$

a Here $p(x) = \cot x$ and $f(x) = x\csc x$ are both continuous except at the points $x = r\pi$, where r is an integer. Therefore we seek solutions of (2.1.20) on the intervals $(r\pi, (r+1)\pi)$. We need a nontrival solution y_1 of the complementary equation; thus, y_1 must satisfy $y_1' + (\cot x)y_1 = 0$, which we rewrite as

$$\frac{y_1'}{y_1} = -\cot x = -\frac{\cos x}{\sin x}. \tag{2.1.22}$$

Integrating this yields

$$\ln|y_1| = -\ln|\sin x| + c.$$

Keep in mind that we need only one function that satisfies (2.1.22). This means that we can take the constant of integration to be zero. After exponentiating both sides, we see that

$$|y_1| = |\sin x|^{-1}$$

and therefore $y_1 = 1/\sin x$ is a suitable choice. So we seek solutions of (2.1.20) in the form

$$y = \frac{u}{\sin x},$$

which has derivative

$$y' = \frac{u'}{\sin x} - \frac{u\cos x}{\sin^2 x} \tag{2.1.23}$$

so that

$$\begin{aligned} y' + (\cot x)y &= \frac{u'}{\sin x} - \frac{u\cos x}{\sin^2 x} + \frac{u\cot x}{\sin x} \\ &= \frac{u'}{\sin x} - \frac{u\cos x}{\sin^2 x} + \frac{u\cos x}{\sin^2 x} \\ &= \frac{u'}{\sin x}. \end{aligned} \tag{2.1.24}$$

Therefore y is a solution of (2.1.20) if and only if

$$u'/\sin x = x\csc x = x/\sin x.$$

Therefore, $u' = x$ and integrating gives

$$u = \frac{x^2}{2} + c, \quad \text{so that} \quad y = \frac{u}{\sin x} = \frac{x^2}{2\sin x} + \frac{c}{\sin x}. \tag{2.1.25}$$

is the general solution of (2.1.20) on every interval $(r\pi, (r+1)\pi)$ (r =integer).

b Imposing the initial condition $y(\pi/2) = 1$ in (2.1.25) yields

$$1 = \frac{\pi^2}{8} + c \quad \text{or} \quad c = 1 - \frac{\pi^2}{8}.$$

Thus,

$$y = \frac{x^2}{2\sin x} + \frac{(1 - \pi^2/8)}{\sin x}$$

is a solution of (2.1.21). The domain of this solution is $(0, \pi)$.

Figure 2.1 shows its graph.

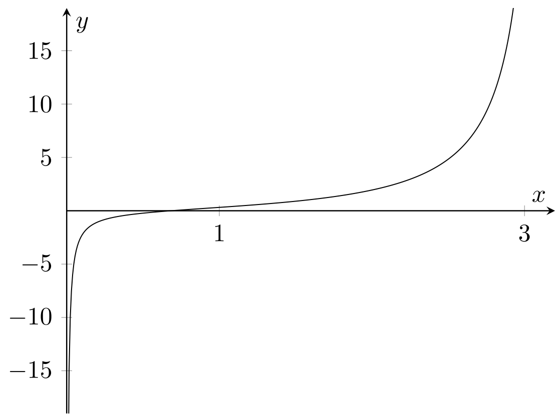


Figure 2.1 Solution of $y' + (\cot x)y = x\csc x, y(\pi/2) = 1$

It was not necessary to do the computations (2.1.23) and (2.1.24) in Example 2.1.6, since we showed in the discussion preceding Example 2.1.5 that if $y = uy_1$ where $y_1' + p(x)y_1 = 0$, then $y' + p(x)y = u'y_1$. We did these computations to show how the method works. We recommend that you include these "unnecesary" computations in doing exercises until you are confident that you understand the method. After that, omit them.

We summarize the method of variation of parameters for solving

$$y' + p(x)y = f(x) \tag{2.1.26}$$

as follows:

(a) Find a function y_1 such that

$$\frac{y_1'}{y_1} = -p(x).$$

For convenience, take the constant of integration to be zero.

(b) Write $u'y_1 = f$ and solve for u'. (So $u' = f/y_1$.)

(c) Integrate u' to obtain u with an arbitrary constant of integration.

(d) Substitute u into $y = uy_1$ to determine y.

To solve an equation written as

$$P_0(x)y' + P_1(x)y = F(x),$$

we recommend that you divide through by $P_0(x)$ to obtain an equation of the form (2.1.26) and then follow this procedure.

Solutions in Integral Form

Sometimes the integrals that arise in solving a linear first order equation cannot be evaluated in terms of elementary functions. In this case the solution must be left in terms of an integral.

Example 2.1.7

(a) Find the general solution of

$$y' - 2xy = 1.$$

(b) Solve the initial value problem

$$y' - 2xy = 1, \quad y(0) = y_0. \tag{2.1.27}$$

a To apply variation of parameters, we need a nontrivial solution y_1 of the complementary equation; thus, $y_1' - 2xy_1 = 0$, which we rewrite as

$$\frac{y_1'}{y_1} = 2x.$$

Integrating this and taking the constant of integration to be zero yields

$$\ln|y_1| = x^2, \quad \text{so} \quad |y_1| = e^{x^2}.$$

We choose $y_1 = e^{x^2}$ (with constant of integration equal to 0) and seek solutions of (2.1.27) in the form $y = ue^{x^2}$, where

$$u'e^{x^2} = 1, \quad \text{so} \quad u' = e^{-x^2}.$$

Therefore

$$u = c + \int e^{-x^2}\,dx.$$

However, we cannot simplify the integral on the right because there is no elementary function with derivative equal to e^{-x^2}. Therefore the best available form for the general solution of (2.1.27) is

$$y = ue^{x^2} = e^{x^2}\left(c + \int e^{-x^2}\,dx\right). \tag{2.1.28}$$

b Since the initial condition in (2.1.27) is imposed at $x_0 = 0$, it is convenient to rewrite (2.1.28) as

$$y = e^{x^2}\left(c + \int_0^x e^{-t^2}\,dt\right), \quad \text{since} \quad \int_0^0 e^{-t^2}\,dt = 0.$$

Setting $x = 0$ and $y = y_0$ here shows that $c = y_0$. Therefore the solution of the initial value problem is

$$y = e^{x^2}\left(y_0 + \int_0^x e^{-t^2}\,dt\right). \tag{2.1.29}$$

For a given value of y_0 and each fixed x, the integral on the right can be evaluated by numerical methods. An alternate procedure is to apply the numerical integration procedures discussed in Chapter 3 directly to the initial value problem (2.1.27).

An Existence and Uniqueness Theorem

The method of variation of parameters leads to this theorem.

Theorem 2.1.2 *Suppose p and f are continuous on an open interval (a,b), and let y_1 be any nontrivial solution of the complementary equation*

$$y' + p(x)y = 0$$

on (a,b). Then:

(a) *The general solution of the nonhomogeneous equation*

$$y' + p(x)y = f(x) \tag{2.1.30}$$

on (a,b) is

$$y = y_1(x)\left(c + \int f(x)/y_1(x)\,dx\right). \tag{2.1.31}$$

(b) *If x_0 is an arbitrary point in (a,b) and y_0 is an arbitrary real number, then the initial value problem*

$$y' + p(x)y = f(x), \quad y(x_0) = y_0$$

has the unique solution

$$y = y_1(x)\left(\frac{y_0}{y_1(x_0)} + \int_{x_0}^x \frac{f(t)}{y_1(t)}\,dt\right)$$

on (a,b).

Proof **(a)** To show that (2.1.31) is the general solution of (2.1.30) on (a, b), we must prove that:

(i) If c is any constant, the function y in (2.1.31) is a solution of (2.1.30) on (a, b).

(ii) If y is a solution of (2.1.30) on (a, b) then y is of the form (2.1.31) for some constant c.

To prove **(i)**, we first observe that any function of the form (2.1.31) is defined on (a, b) since p and f are continuous on (a, b). Differentiating (2.1.31) yields

$$y' = y_1'(x)\left(c + \int f(x)/y_1(x)\,dx\right) + f(x).$$

Since $y_1' = -p(x)y_1$, this and (2.1.31) imply that

$$\begin{aligned} y' &= -p(x)y_1(x)\left(c + \int f(x)/y_1(x)\,dx\right) + f(x) \\ &= -p(x)y(x) + f(x), \end{aligned}$$

which implies that y is a solution of (2.1.30).

To prove **(ii)**, suppose y is a solution of (2.1.30) on (a, b). From the proof of Theorem 2.1.1, we know that y_1 has no zeros on (a, b), so the function $u = y/y_1$ is defined on (a, b). Moreover, since $y' = -py + f$ and $y_1' = -py_1$,

$$\begin{aligned} u' &= \frac{y_1 y' - y_1' y}{y_1^2} \\ &= \frac{y_1(-py + f) - (-py_1)y}{y_1^2} = \frac{f}{y_1}. \end{aligned}$$

Integrating $u' = f/y_1$ yields

$$u = \left(c + \int f(x)/y_1(x)\,dx\right),$$

which implies (2.1.31), since $y = uy_1$.

(b) We've proved **(a)**, where $\int f(x)/y_1(x)\,dx$ in (2.1.31) is an arbitrary antiderivative of f/y_1. Now it's convenient to choose the antiderivative that equals zero when $x = x_0$, and write the general solution of (2.1.30) as

$$y = y_1(x)\left(c + \int_{x_0}^{x} \frac{f(t)}{y_1(t)}\,dt\right).$$

Since

$$y(x_0) = y_1(x_0)\left(c + \int_{x_0}^{x_0} \frac{f(t)}{y_1(t)}\,dt\right) = cy_1(x_0),$$

we see that $y(x_0) = y_0$ if and only if $c = y_0/y_1(x_0)$.

2.1 Exercises

In Exercises 1–5 find the general solution.

1. $y' + ay = 0$ (a=constant)

2. $y' + 3x^2y = 0$

3. $xy' + (\ln x)y = 0$

4. $xy' + 3y = 0$

5. $x^2y' + y = 0$

In Exercises 6–11 *solve the initial value problem.*

6. $y' + \left(\frac{1+x}{x}\right)y = 0, \quad y(1) = 1$

7. $xy' + \left(1 + \frac{1}{\ln x}\right)y = 0, \quad y(e) = 1$

8. $xy' + (1 + x\cot x)y = 0, \quad y\left(\frac{\pi}{2}\right) = 2$

9. $y' - \left(\frac{2x}{1+x^2}\right)y = 0, \quad y(0) = 2$

10. $y' + \frac{k}{x}y = 0, \quad y(1) = 3$ (k= constant)

11. $y' + (\tan kx)y = 0, \quad y(0) = 2$ ($k =$ constant)

In Exercises 12 –15 *find the general solution.*

12. $y' + 3y = 1$

13. $y' + \left(\frac{1}{x} - 1\right)y = -\frac{2}{x}$

14. $y' + 2xy = xe^{-x^2}$

15. $y' + \frac{2x}{1+x^2}y = \frac{e^{-x}}{1+x^2}$

In Exercises 16 –24 *find the general solution.*

16. $y' + \frac{1}{x}y = \frac{7}{x^2} + 3$

17. $y' + \frac{4}{x-1}y = \frac{1}{(x-1)^5} + \frac{\sin x}{(x-1)^4}$

18. $xy' + (1 + 2x^2)y = x^3e^{-x^2}$

19. $xy' + 2y = \frac{2}{x^2} + 1$

20. $y' + (\tan x)y = \cos x$

21. $(1 + x)y' + 2y = \frac{\sin x}{1+x}$

22. $(x-2)(x-1)y' - (4x-3)y = (x-2)^3$

23. $y' + (2\sin x\cos x)y = e^{-\sin^2 x}$ **24.** $x^2y' + 3xy = e^x$

In Exercises 25–29 solve the initial value problem and sketch the graph of the solution.

25. C/G

$y' + 7y = e^{3x}, \quad y(0) = 0$

26. C/G $(1+x^2)y' + 4xy = \dfrac{2}{1+x^2}, \quad y(0) = 1$

27. C/G $xy' + 3y = \dfrac{2}{x(1+x^2)}, \quad y(-1) = 0$

28. C/G $y' + (\cot x)y = \cos x, \quad y\left(\dfrac{\pi}{2}\right) = 1$

29. C/G $y' + \dfrac{1}{x}y = \dfrac{2}{x^2} + 1, \quad y(-1) = 0$

In Exercises 30–37 solve the initial value problem.

30. $(x-1)y' + 3y = \dfrac{1}{(x-1)^3} + \dfrac{\sin x}{(x-1)^2}, \quad y(0) = 1$

31. $xy' + 2y = 8x^2, \quad y(1) = 3$

32. $xy' - 2y = -x^2, \quad y(1) = 1$

33. $y' + 2xy = x, \quad y(0) = 3$

34. $(x-1)y' + 3y = \dfrac{1 + (x-1)\sec^2 x}{(x-1)^3}, \quad y(0) = -1$

35. $(x+2)y' + 4y = \dfrac{1+2x^2}{x(x+2)^3}, \quad y(-1) = 2$

36. $(x^2-1)y' - 2xy = x(x^2-1), \quad y(0) = 4$

37. $(x^2-5)y' - 2xy = -2x(x^2-5), \quad y(2) = 7$

In Exercises 38–42 solve the initial value problem and leave the answer in a form involving a definite integral.

38. $y' + 2xy = x^2, \quad y(0) = 3$

39. $y' + \dfrac{1}{x}y = \dfrac{\sin x}{x^2}, \quad y(1) = 2$

40. $y' + y = \dfrac{e^{-x}\tan x}{x}, \quad y(1) = 0$

41. $y' + \dfrac{2x}{1+x^2}y = \dfrac{e^x}{(1+x^2)^2}, \quad y(0) = 1$

42. $xy' + (x+1)y = e^{x^2}, \quad y(1) = 2$

43. Experiments indicate that glucose is absorbed by the body at a rate proportional to the amount of glucose present in the bloodstream. Let λ denote the (positive) constant of proportionality. Now suppose glucose is injected into a patient's bloodstream at a constant rate of r units per unit of time. Let $G = G(t)$ be the number of units in the patient's bloodstream at time $t > 0$. Then

$$G' = -\lambda G + r,$$

where the first term on the right is due to the absorption of the glucose by the patient's body and the second term is due to the injection. Determine G for $t > 0$, given that $G(0) = G_0$. Also, find $\lim_{t\to\infty} G(t)$.

44. Some nonlinear equations can be transformed into linear equations by changing the dependent variable. Show that if

$$g'(y)y' + p(x)g(y) = f(x)$$

where y is a function of x and g is a function of y, then the new dependent variable $z = g(y)$ satisfies the linear equation

$$z' + p(x)z = f(x).$$

45. Solve by the method discussed in Exercise 44.

(a) $(\sec^2 y)y' - 3\tan y = -1$ **(b)** $e^{y^2}\left(2yy' + \frac{2}{x}\right) = \frac{1}{x^2}$

(c) $\frac{xy'}{y} + 2\ln y = 4x^2$ **(d)** $\frac{y'}{(1+y)^2} - \frac{1}{x(1+y)} = -\frac{3}{x^2}$

2.2 SEPARABLE EQUATIONS

A first order differential equation is *separable* if it can be written as

$$h(y)y' = g(x), \tag{2.2.1}$$

where the left side is a product of y' and a function of y and the right side is a function of x. Rewriting a separable differential equation in this form is called *separation of variables.* In Section 2.1 we used separation of variables to solve homogeneous linear equations. In this section we'll apply this method to nonlinear equations.

To see how to solve (2.2.1), let's first assume that y is a solution. Let $G(x)$ and $H(y)$ be antiderivatives of $g(x)$ and $h(y)$; that is,

$$H'(y) = h(y) \quad \text{and} \quad G'(x) = g(x). \tag{2.2.2}$$

Then, from the chain rule,

$$\frac{d}{dx}H(y(x)) = H'(y(x))y'(x) = h(y)y'(x).$$

Therefore (2.2.1) is equivalent to

$$\frac{d}{dx}H(y(x)) = \frac{d}{dx}G(x).$$

Integrating both sides of this equation and combining the constants of integration yields

$$H(y(x)) = G(x) + c. \tag{2.2.3}$$

Although we derived this equation on the assumption that y is a solution of (2.2.1), we can now view it differently: Any differentiable function y that satisfies (2.2.3) for some constant c is a solution of (2.2.1). To see this, we differentiate both sides of (2.2.3), using the chain rule on the left, to obtain

$$H'(y(x))y'(x) = G'(x),$$

which is equivalent to

$$h(y(x))y'(x) = g(x)$$

because of (2.2.2).

In conclusion, to solve (2.2.1) it suffices to find functions $G = G(x)$ and $H = H(y)$ that satisfy (2.2.2). Then any differentiable function $y = y(x)$ that satisfies (2.2.3) is a solution of (2.2.1).

Example 2.2.1 Solve the equation

$$y' = x(1 + y^2).$$

Solution Separating variables yields

$$\frac{y'}{1 + y^2} = x.$$

Integrating yields

$$\tan^{-1} y = \frac{x^2}{2} + c$$

Therefore

$$y = \tan\left(\frac{x^2}{2} + c\right).$$

■

Example 2.2.2

(a) Solve the equation

$$y' = -\frac{x}{y}. \tag{2.2.4}$$

(b) Solve the initial value problem

$$y' = -\frac{x}{y}, \quad y(1) = 1. \tag{2.2.5}$$

(c) Solve the initial value problem

$$y' = -\frac{x}{y}, \quad y(1) = -2. \tag{2.2.6}$$

a Separating variables in (2.2.4) yields $yy' = -x$. Integrating yields

$$\frac{y^2}{2} = -\frac{x^2}{2} + c, \quad \text{or, equivalently,} \quad x^2 + y^2 = 2c.$$

The last equation shows that c must be positive if y is to be a solution of (2.2.4) on an open interval. Therefore we let $2c = a^2$ (with $a > 0$) and rewrite the last equation as

$$x^2 + y^2 = a^2. \tag{2.2.7}$$

This equation has two differentiable solutions for y in terms of x:

$$y = \sqrt{a^2 - x^2}, \quad -a < x < a, \tag{2.2.8}$$

and

$$y = -\sqrt{a^2 - x^2}, \quad -a < x < a. \tag{2.2.9}$$

The solution curves defined by (2.2.8) are semicircles above the x-axis and those defined by (2.2.9) are semicircles below the x-axis.

b The solution of (2.2.5) is positive when $x = 1$; hence, it is of the form (2.2.8). Substituting $x = 1$ and $y = 1$ into (2.2.7) to satisfy the initial condition yields $a^2 = 2$; hence, the solution of (2.2.5) is

$$y = \sqrt{2 - x^2}, \quad -\sqrt{2} < x < \sqrt{2}.$$

c The solution of (2.2.6) is negative when $x = 1$ and is therefore of the form (2.2.9). Substituting $x = 1$ and $y = -2$ into (2.2.7) to satisfy the initial condition yields $a^2 = 5$. Hence, the solution of (2.2.6) is

$$y = -\sqrt{5 - x^2}, \quad -\sqrt{5} < x < \sqrt{5}.$$

Figure 2.1 shows the solution curves for the initial value problems. ■

Implicit Solutions of Separable Equations

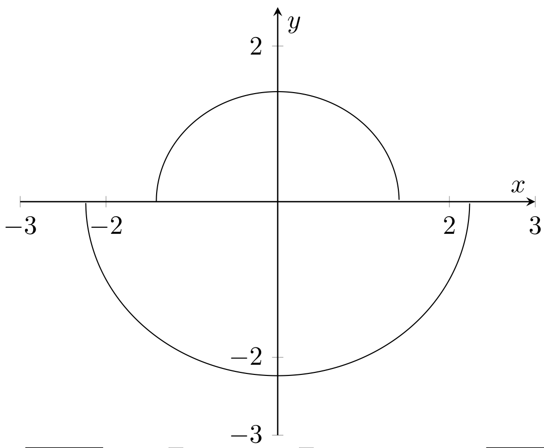


Figure 2.1 **(a)** $y = \sqrt{2 - x^2}$, $-\sqrt{2} < x < \sqrt{2}$; **(b)** $y = -\sqrt{5 - x^2}$, $-\sqrt{5} < x < \sqrt{5}$

In Examples 2.2.1 and 2.2.2 we were able to solve the equation $H(y) = G(x) + c$ to obtain explicit formulas for solutions of the given separable differential equations. The next example shows that this is not always possible. In this situation we must broaden our definition of a solution of a separable equation. The next theorem provides the basis for this modification. We omit the proof, which requires a result from advanced calculus called as the *implicit function theorem*.

Theorem 2.2.1 *Suppose $g = g(x)$ is continous on (a, b) and $h = h(y)$ is continuous on (c, d). Let G be an antiderivative of g on (a, b) and let H be an antiderivative of h on (c, d). Let x_0 be an arbitrary point in (a, b), let y_0 be a point in (c, d) such that $h(y_0) \neq 0$, and define*

$$c = H(y_0) - G(x_0). \tag{2.2.10}$$

Then there is a function $y = y(x)$ defined on some open interval (a_1, b_1), where $a \leqslant a_1 < x_0 < b_1 \leqslant b$, such that $y(x_0) = y_0$ and

$$H(y) = G(x) + c \tag{2.2.11}$$

for $a_1 < x < b_1$. Therefore y is a solution of the initial value problem

$$h(y)y' = g(x), \quad y(x_0) = x_0. \tag{2.2.12}$$

We sometimes say that a solution with the form (2.2.11) with a specific but arbitrary value of c is an *implicit solution* of $h(y)y' = g(x)$.

In the case where c satisfies (2.2.10), we say that (2.2.11) is an *implicit solution of the initial value problem* (2.2.12). However, keep these points in mind:

- For some choices of c there may not be any differentiable functions y that satisfy (2.2.11).

- The function y in (2.2.11) – not (2.2.11) itself – is a solution of $h(y)y' = g(x)$.

Example 2.2.3

(a) Find implicit solutions of

$$y' = \frac{2x+1}{5y^4+1}. \tag{2.2.13}$$

(b) Find an implicit solution of

$$y' = \frac{2x+1}{5y^4+1}, \quad y(2) = 1. \tag{2.2.14}$$

a Separating variables yields

$$(5y^4 + 1)y' = 2x + 1.$$

Integrating yields the implicit solutions

$$y^5 + y = x^2 + x + c. \tag{2.2.15}$$

of (2.2.13). (There are multiple solutions corresponding to multiple choices of the constant c.)

b Imposing the initial condition $y(2) = 1$ in (2.2.15) yields $1 + 1 = 4 + 2 + c$, so $c = -4$. Therefore

$$y^5 + y = x^2 + x - 4$$

is an implicit solution of the initial value problem (2.2.14). Although more than one differentiable function $y = y(x)$ satisfies (2.2.13) near $x = 1$, it can be shown that there is only one such function that satisfies the initial condition $y(1) = 2$. ■

Curves defined by (2.2.11) are integral curves of $h(y)y' = g(x)$. However, since the function y is an implicit solution, the appearance of the graph for the solution is not apparent. The problem of seeing what an implicit solution looks like can be overcome by using technology to generate a direction field. Figure 2.3 shows a direction field and some integral curves for (2.2.13).

Constant Solutions of Separable Equations

An equation of the form

$$y' = g(x)p(y)$$

is separable, since it can be rewritten as

$$\frac{1}{p(y)}y' = g(x).$$

However, the division by $p(y)$ is not legitimate if $p(y) = 0$ for some values of y. The next two examples show how to deal with this problem.

Example 2.2.4 Find all solutions of

$$y' = 2xy^2. \tag{2.2.16}$$

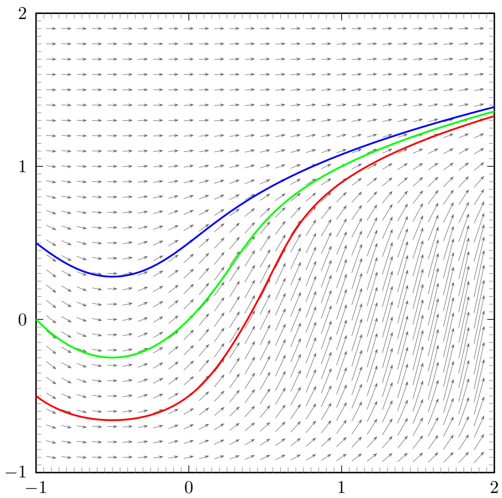


Figure 2.2 A direction field and integral curves for $y' = \dfrac{2x+1}{5y^4+1}$

Solution Here we must divide by $p(y) = y^2$ to separate variables. This is not legitimate if y is a solution of (2.2.16) that equals zero for some value of x. One such solution can be found by inspection: $y \equiv 0$. Now suppose y is a solution of (2.2.16) that isn't identically zero. Since y is continuous there must be an interval on which y is never zero. Since division by y^2 is legitimate for x in this interval, we can separate variables in (2.2.16) to obtain

$$\frac{y'}{y^2} = 2x.$$

Integrating this yields

$$-\frac{1}{y} = x^2 + c,$$

which is equivalent to

$$y = -\frac{1}{x^2 + c}. \tag{2.2.17}$$

We have now shown that if y is a solution of (2.2.16) that is not identically zero, then y must be of the form (2.2.17). By substituting (2.2.17) into (2.2.16), you can verify that (2.2.17) is a solution of (2.2.16). Thus solutions of (2.2.16) are $y \equiv 0$ and the functions of the form (2.2.17). Note that the solution $y \equiv 0$ is not of the form (2.2.17) for any value of c.

Figure 2.3 shows a direction field and some integral curves for (2.2.16).

■

Example 2.2.5 Find all solutions of

$$y' = \frac{1}{2}x(1 - y^2). \tag{2.2.18}$$

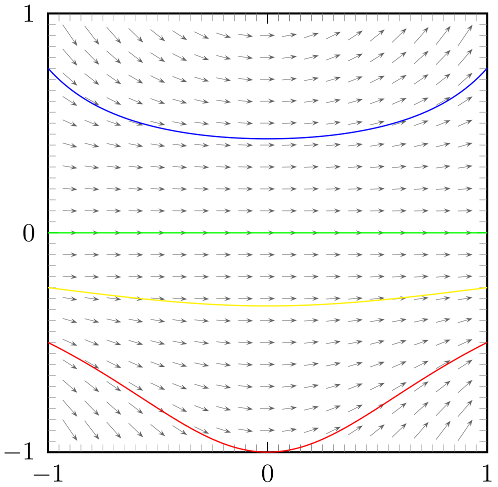


Figure 2.3 A direction field and integral curves for $y' = 2xy^2$

Solution Here we must divide by $p(y) = 1 - y^2$ to separate variables. This is not legitimate if y is a solution of (2.2.18) that equals ± 1 for some value of x. Two such solutions can be found by inspection: $y \equiv 1$ and $y \equiv -1$. Now suppose y is a solution of (2.2.18) such that $1 - y^2$ isn't identically zero. Since $1 - y^2$ is continuous there must be an interval on which $1 - y^2$ is never zero. Since division by $1 - y^2$ is legitimate for x in this interval, we can separate variables in (2.2.18) to obtain

$$\frac{2y'}{y^2 - 1} = -x.$$

A partial fraction expansion on the left yields

$$\left[\frac{1}{y-1} - \frac{1}{y+1}\right] y' = -x,$$

and integrating yields

$$\ln\left|\frac{y-1}{y+1}\right| = -\frac{x^2}{2} + k;$$

hence,

$$\left|\frac{y-1}{y+1}\right| = e^k e^{-x^2/2}.$$

Since $y(x) \neq \pm 1$ for x on the interval under discussion, the quantity $(y-1)/(y+1)$ cannot change sign in this interval. Therefore we can rewrite the last equation as

$$\frac{y-1}{y+1} = ce^{-x^2/2},$$

where $c = \pm e^k$, depending upon the sign of $(y-1)/(y+1)$ on the interval. Solving for y yields

$$y = \frac{1 + ce^{-x^2/2}}{1 - ce^{-x^2/2}}. \tag{2.2.19}$$

We have now shown that if y is a solution of (2.2.18) that is not identically equal to ± 1, then y must be as in (2.2.19). By substituting (2.2.19) into (2.2.18) you can verify that (2.2.19) is a solution of (2.2.18). Thus, the solutions of (2.2.18) are $y \equiv 1$, $y \equiv -1$ and the functions of the form (2.2.19). Note that the constant solution $y \equiv 1$ can be obtained from this formula by taking $c = 0$; however, the other constant solution, $y \equiv -1$, cannot be obtained in this way.

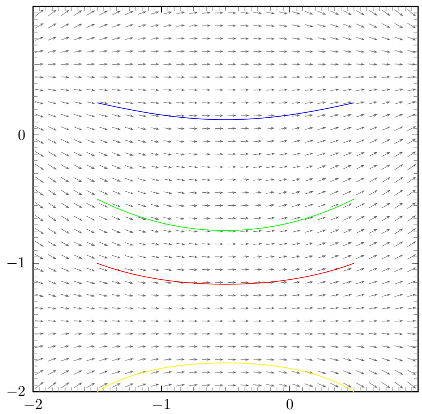


Figure 2.4 A direction field and integral curves for $y' = \frac{1}{2}x(1-y^2)$

■

Differences Between Linear and Nonlinear Equations

Theorem 2.1.2 states that if p and f are continuous on (a, b) then every solution of the linear equation

$$y' + p(x)y = f(x)$$

on (a, b) can be obtained by choosing a value for the constant c in the general solution, and if x_0 is any point in (a, b) and y_0 is arbitrary, then the initial value problem

$$y' + p(x)y = f(x), \quad y(x_0) = y_0$$

has a solution on (a, b).

This theorem does not hold true for nonlinear equations. First, we saw in Examples 2.2.4 and 2.2.5 that a nonlinear equation may have solutions that cannot be obtained by choosing a specific value of a constant appearing in a one-parameter family of solutions. (Such a solution is called a *singular solution*.) Second, it is generally impossible to determine the domain of a solution for an initial value problem for a nonlinear equation by simply examining the equation, since the domain may depend on the initial condition. For instance, in Example 2.2.2 we saw that the solution of

$$\frac{dy}{dx} = -\frac{x}{y}, \quad y(x_0) = y_0$$

has domain $(-a, a)$, where $a = \sqrt{x_0^2 + y_0^2}$. In other words, the domain of the solution depends on the point (x_0, y_0). Let us revisit Example 2.2.4 to see another example where the domain of the solution depends on the initial condition.

Example 2.2.6 Solve the initial value problem

$$y' = 2xy^2, \quad y(0) = y_0$$

and determine the domain of the solution.

Solution From Example 2.2.4, we know that y must be of the form

$$y = -\frac{1}{x^2 + c}. \tag{2.2.20}$$

Imposing the initial condition shows that $c = -1/y_0$. Substituting this into (2.2.20) and rearranging terms yields the solution

$$y = \frac{y_0}{1 - y_0 x^2}.$$

This is the solution if $y_0 = 0$. If $y_0 < 0$, the denominator cannot be zero for any value of x, so the solution has domain $(-\infty, \infty)$. If $y_0 > 0$, however, the domain of the solution must be restricted to $(-1/\sqrt{y_0}, 1/\sqrt{y_0})$. ■

2.2 Exercises

In Exercises 1–6 find all solutions.

1. $y' = \frac{3x^2 + 2x + 1}{y - 2}$

2. $(\sin x)(\sin y) + (\cos y)y' = 0$

3. $xy' + y^2 + y = 0$

4. $y' \ln|y| + x^2 y = 0$

5. $(3y^3 + 3y\cos y + 1)y' + \frac{(2x+1)y}{1+x^2} = 0$

6. $x^2yy' = (y^2-1)^{3/2}$

In Exercises 7–10 *find all solutions.*

7. $y' = x^2(1+y^2)$; $\{-1 \leqslant x \leqslant 1,\ -1 \leqslant y \leqslant 1\}$

8. $y'(1+x^2) + xy = 0$; $\{-2 \leqslant x \leqslant 2,\ -1 \leqslant y \leqslant 1\}$

9. $y' = (x-1)(y-1)(y-2)$; $\{-2 \leqslant x \leqslant 2,\ -3 \leqslant y \leqslant 3\}$

10. $(y-1)^2y' = 2x+3$; $\{-2 \leqslant x \leqslant 2,\ -2 \leqslant y \leqslant 5\}$

In Exercises 11 *and* 12 *solve the initial value problem.*

11. $y' = \dfrac{x^2+3x+2}{y-2}, \quad y(1) = 4$

12. $y' + x(y^2+y) = 0, \quad y(2) = 1$

In Exercises 13-16 *solve the initial value problem and graph the solution.*

13. $(3y^2+4y)y' + 2x + \cos x = 0, \quad y(0) = 1$

14. $y' + \dfrac{(y+1)(y-1)(y-2)}{x+1} = 0, \quad y(1) = 0$

15. $y' + 2x(y+1) = 0, \quad y(0) = 2$

16. $y' = 2xy(1+y^2), \quad y(0) = 1$

In Exercises 17–23 *solve the initial value problem and find the domain of the solution.*

17. $y'(x^2+2) + 4x(y^2+2y+1) = 0, \quad y(1) = -1$

18. $y' = -2x(y^2-3y+2), \quad y(0) = 3$

19. $y' = \dfrac{2x}{1+2y}, \quad y(2) = 0$

20. $y' = 2y - y^2, \quad y(0) = 1$

21. $x + yy' = 0, \quad y(3) = -4$

22. $y' + x^2(y+1)(y-2)^2 = 0, \quad y(4) = 2$

23. $(x+1)(x-2)y' + y = 0, \quad y(1) = -3$

24. Solve $y' = \dfrac{(1+y^2)}{(1+x^2)}$ explicitly. HINT: *Use the identity* $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$.

25. Solve $y'\sqrt{1-x^2} + \sqrt{1-y^2} = 0$ explicitly. HINT: *Use the identity* $\sin(A-B) = \sin A \cos B - \cos A \sin B$.

26. Solve $y' = \dfrac{\cos x}{\sin y}, \quad y(\pi) = \dfrac{\pi}{2}$ explicitly. HINT: *Use the identity* $\cos(x+\pi/2) = -\sin x$ *and the periodicity of the cosine function.*

27. The population $P = P(t)$ of a species satisfies the logistic equation

$$P' = aP(1 - \alpha P)$$

and $P(0) = P_0 > 0$. Find P for $t > 0$, and find $\lim_{t\to\infty} P(t)$.

28. An epidemic spreads through a population at a rate proportional to the product of the number of people already infected and the number of people susceptible, but not yet infected. Therefore, if S denotes the total population of susceptible people and $I = I(t)$ denotes the number of infected people at time t, then

$$I' = rI(S - I),$$

where r is a positive constant. Assuming that $I(0) = I_0$, find $I(t)$ for $t > 0$, and show that $\lim_{t\to\infty} I(t) = S$.

29. The result of Exercise 28 is discouraging: if any susceptible member of the group is initially infected, then in the long run all susceptible members are infected! On a more hopeful note, suppose the disease spreads according to the model of Exercise 28, but there is a medication that cures the infected population at a rate proportional to the number of infected individuals. Now the equation for the number of infected individuals becomes

$$I' = rI(S - I) - qI \tag{A}$$

where q is a positive constant.

(a) Assume r and S are positive. By drawing a phase portrait, verify that if I is any solution of (A) such that $I(0) > 0$, then $\lim_{t\to\infty} I(t) = S - q/r$ if $q < rS$ and $\lim_{t\to\infty} I(t) = 0$ if $q \geqslant rS$.

(b) To verify the experimental results of **(a)**, use separation of variables to solve (A) with initial condition $I(0) = I_0 > 0$, and find $\lim_{t\to\infty} I(t)$. HINT: *There are three cases to consider:* **(i)** $q < rS$; **(ii)** $q > rS$; **(iii)** $q = rS$.

Solve the equations in Exercises 30–33 *using variation of parameters followed by separation of variables.*

30. $y' + y = \dfrac{2xe^{-x}}{1 + ye^x}$

31. $xy' - 2y = \dfrac{x^6}{y + x^2}$

32. $y' - y = \dfrac{(x+1)e^{4x}}{(y + e^x)^2}$

33. $y' - 2y = \dfrac{xe^{2x}}{1 - ye^{-2x}}$

34. Use variation of parameters to show that the solutions of the following equations are of the form $y = uy_1$, where u satisfies a separable equation $u' = g(x)p(u)$. Find y_1 and g for each equation.

(a) $xy' + y = h(x)p(xy)$ **(b)** $xy' - y = h(x)p\left(\frac{y}{x}\right)$

(c) $y' + y = h(x)p(e^x y)$ **(d)** $xy' + ry = h(x)p(x^r y)$

(e) $y' + \frac{v'(x)}{v(x)}y = h(x)p\,(v(x)y)$

2.3 EXISTENCE AND UNIQUENESS OF SOLUTIONS OF NONLINEAR EQUATIONS

Although there are methods for solving some nonlinear equations, it is impossible to find useful formulas for the solutions of most. Whether we are looking for exact solutions or numerical approximations, it is useful to know conditions that imply the existence and uniqueness of solutions of initial value problems for nonlinear equations. In this section we state such a condition and illustrate it with examples.

Some terminology: an *open rectangle* R is a set of points (x, y) such that

$$a < x < b \quad \text{and} \quad c < y < d$$

(Figure 2.1). We will denote this set by $R = \{a < x < b, c < y < d\}$. "Open" means that the boundary rectangle (indicated by the dashed lines in Figure 2.1) is not included in R.

The next theorem gives sufficient conditions for existence and uniqueness of solutions of initial value problems for first order nonlinear differential equations. We omit the proof, which is beyond the scope of this text.

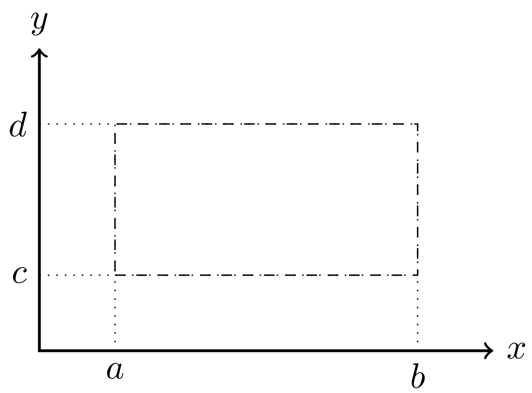


Figure 2.1 An open rectangular grid

Theorem 2.3.1

(a) *If* f *is continuous on an open rectangle*

$$R = \{a < x < b, c < y < d\}$$

that contains (x_0, y_0) *then the initial value problem*

$$y' = f(x, y), \quad y(x_0) = y_0 \tag{2.3.1}$$

has at least one solution on some open subinterval of (a,b) *that contains* x_0.

(b) *If both* f *and* f_y *are continuous on* R *then* (2.3.1) *has a unique solution on some open subinterval of* (a,b) *that contains* x_0. *(Recall that* f_y *denotes the partial derivative of* $f(x,y)$ *with respect to* y.*)*

It is important to understand exactly what Theorem 2.3.1 says.

(a) is an *existence theorem*. It guarantees that a solution exists on some open interval that contains x_0, but provides no information on how to find the solution nor on how to determine the open interval on which it exists. Moreover, **(a)** provides no information on the number of solutions that (2.3.1) may have. It leaves open the possibility that (2.3.1) may have two or more solutions that differ for values of x arbitrarily close to x_0. We will see in Example 2.3.6 that this can happen.

(b) is a *uniqueness theorem*. It guarantees that (2.3.1) has a unique solution on some open interval (a,b) that contains x_0. However, if $(a,b) \neq (-\infty,\infty)$, (2.3.1) may have more than one solution on a larger interval that contains (a,b). For example, it may happen that $b < \infty$ and all solutions have the same values on (a,b), but two solutions y_1 and y_2 are defined on some interval (a,b_1) with $b_1 > b$, and have different values for $b < x < b_1$; thus, the graphs of y_1 and y_2 "branch off" in different directions at $x = b$. (See Example **??** and Figure **??**). In this case, continuity implies that $y_1(b) = y_2(b)$ (call their common value $\overline{y}$), and y_1 and y_2 are both solutions of the initial value problem

$$y' = f(x,y), \quad y(b) = \overline{y} \tag{2.3.2}$$

that differ on every open interval that contains b. Therefore f or f_y must have a discontinuity at some point in each open rectangle that contains the point $(b,\overline{y})$, since if this were not so, (2.3.2) would have a unique solution on some open interval that contains b. We leave it to you to give a similar analysis of the case where $a > -\infty$.

Example 2.3.1 Consider the initial value problem

$$y' = \frac{x^2-y^2}{1+x^2+y^2}, \quad y(x_0) = y_0. \tag{2.3.3}$$

Since

$$f(x,y) = \frac{x^2-y^2}{1+x^2+y^2} \quad \text{and} \quad f_y(x,y) = -\frac{2y(1+2x^2)}{(1+x^2+y^2)^2}$$

are continuous for all (x,y), Theorem 2.3.1 implies that if (x_0,y_0) is arbitrary, then (2.3.3) has a unique solution on some open interval that contains x_0.

■

Example 2.3.2 Consider the initial value problem

$$y' = \frac{x^2-y^2}{x^2+y^2}, \quad y(x_0) = y_0. \tag{2.3.4}$$

Here

$$f(x, y) = \frac{x^2 - y^2}{x^2 + y^2} \quad \text{and} \quad f_y(x, y) = -\frac{4x^2y}{(x^2 + y^2)^2}$$

are continuous everywhere except at $(0, 0)$. If $(x_0, y_0) \neq (0, 0)$, there's an open rectangle R that contains (x_0, y_0) that does not contain $(0, 0)$. Since f and f_y are continuous on R, Theorem 2.3.1 implies that if $(x_0, y_0) \neq (0, 0)$ then (2.3.4) has a unique solution on some open interval that contains x_0.

■

Example 2.3.3 Consider the initial value problem

$$y' = \frac{x + y}{x - y}, \quad y(x_0) = y_0. \tag{2.3.5}$$

Here

$$f(x, y) = \frac{x + y}{x - y} \quad \text{and} \quad f_y(x, y) = \frac{2x}{(x - y)^2}$$

are continuous everywhere except on the line $y = x$. If $y_0 \neq x_0$, there's an open rectangle R that contains (x_0, y_0) that does not intersect the line $y = x$. Since f and f_y are continuous on R, Theorem 2.3.1 implies that if $y_0 \neq x_0$, (2.3.5) has a unique solution on some open interval that contains x_0.

■

Example 2.3.4 In Example 2.2.4 we saw that the solutions of

$$y' = 2xy^2 \tag{2.3.6}$$

are

$$y \equiv 0 \quad \text{and} \quad y = -\frac{1}{x^2 + c},$$

where c is an arbitrary constant. In particular, this implies that no solution of (2.3.6) other than $y \equiv 0$ can equal zero for any value of x. Show that Theorem 2.3.1**(b)** implies this.

Solution We will obtain a contradiction by assuming that (2.3.6) has a solution y_1 that equals zero for some value of x, but isn't identically zero. If y_1 has this property, there's a point x_0 such that $y_1(x_0) = 0$, but $y_1(x) \neq 0$ for some value of x in every open interval that contains x_0. This means that the initial value problem

$$y' = 2xy^2, \quad y(x_0) = 0 \tag{2.3.7}$$

has two solutions $y \equiv 0$ and $y = y_1$ that differ for some value of x on every open interval that contains x_0. This contradicts Theorem 2.3.1(b), since in (2.3.6) the functions

$$f(x, y) = 2xy^2 \quad \text{and} \quad f_y(x, y) = 4xy.$$

are both continuous for all (x, y), which implies that (2.3.7) has a unique solution on some open interval that contains x_0. ■

Example 2.3.5 Consider the initial value problem

$$y' = \frac{10}{3}xy^{2/5}, \quad y(x_0) = y_0. \tag{2.3.8}$$

(a) For what points (x_0, y_0) does Theorem 2.3.1**(a)** imply that (2.3.8) has a solution?

(b) For what points (x_0, y_0) does Theorem 2.3.1**(b)** imply that (2.3.8) has a unique solution on some open interval that contains x_0?

(a) Since

$$f(x, y) = \frac{10}{3}xy^{2/5}$$

is continuous for all (x, y), Theorem 2.3.1 implies that (2.3.8) has a solution for every (x_0, y_0).

(b) Here

$$f_y(x, y) = \frac{4}{3}xy^{-3/5}$$

is continuous for all (x, y) with $y \neq 0$. Therefore, if $y_0 \neq 0$ there's an open rectangle on which both f and f_y are continuous, and Theorem 2.3.1 implies that (2.3.8) has a unique solution on some open interval that contains x_0.

If $y = 0$ then $f_y(x, y)$ is undefined, and therefore discontinuous; hence, Theorem 2.3.1 does not apply to (2.3.8) if $y_0 = 0$. ■

Example 2.3.6 Example 2.3.5 leaves open the possibility that the initial value problem

$$y' = \frac{10}{3}xy^{2/5}, \quad y(0) = 0 \tag{2.3.9}$$

has more than one solution on every open interval that contains $x_0 = 0$. Show that this is true.

Solution By inspection, $y \equiv 0$ is a solution of the differential equation

$$y' = \frac{10}{3}xy^{2/5}. \tag{2.3.10}$$

Since $y \equiv 0$ satisfies the initial condition $y(0) = 0$, it is a solution of (2.3.9).

Now suppose y is a solution of (2.3.10) that isn't identically zero. Separating variables in (2.3.10) yields

$$y^{-2/5}y' = \frac{10}{3}x$$

on any open interval where y has no zeros. Integrating this and rewriting the arbitrary constant as $5c/3$ yields

$$\frac{5}{3}y^{3/5} = \frac{5}{3}(x^2 + c).$$

Therefore

$$y = (x^2 + c)^{5/3}. \tag{2.3.11}$$

Since we divided by y to separate variables in (2.3.10), our derivation of (2.3.11) is legitimate only on open intervals where y has no zeros. However, (2.3.11) actually defines y for all x, and differentiating (2.3.11) shows that

$$y' = \frac{10}{3}x(x^2 + c)^{2/3} = \frac{10}{3}xy^{2/5}, \quad -\infty < x < \infty.$$

Therefore (2.3.11) satisfies (2.3.10) on $(-\infty, \infty)$ even if $c \leqslant 0$, so that $y(\sqrt{|c|}) = y(-\sqrt{|c|}) = 0$. In particular, taking $c = 0$ in (2.3.11) yields

$$y = x^{10/3}$$

as a second solution of (2.3.9). Both solutions are defined on $(-\infty, \infty)$, but they differ on every open interval that contains $x_0 = 0$ (see Figure 2.2.) In fact, there are *four* distinct solutions of (2.3.9) defined on $(-\infty, \infty)$ that differ from each other on every open interval that contains $x_0 = 0$. Can you identify the other two? ■

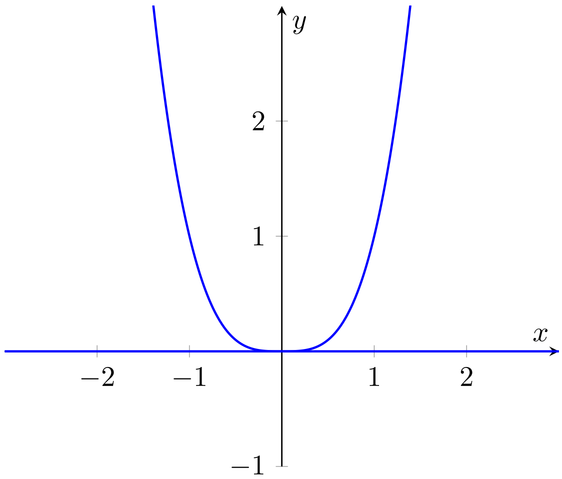


Figure 2.2 Two solutions of (2.3.9) that differ on every interval containing $x_0 = 0$

2.3 Exercises

In Exercises 1-13 find all (x_0, y_0) for which Theorem 2.3.1 implies that the initial value problem $y' = f(x, y),\ y(x_0) = y_0$ has **(a)** a solution **(b)** a unique solution on some open interval that contains x_0.

1. $y' = \dfrac{x^2 + y^2}{\sin x}$

2. $y' = \dfrac{e^x + y}{x^2 + y^2}$

3. $y' = \tan xy$

4. $y' = \dfrac{x^2 + y^2}{\ln xy}$

5. $y' = (x^2 + y^2)y^{1/3}$

6. $y' = 2xy$

7. $y' = \ln(1 + x^2 + y^2)$

8. $y' = \dfrac{2x + 3y}{x - 4y}$

9. $y' = (x^2 + y^2)^{1/2}$

10. $y' = x(y^2 - 1)^{2/3}$

11. $y' = (x^2 + y^2)^2$

12. $y' = (x + y)^{1/2}$

13. $y' = \dfrac{\tan y}{x - 1}$

2.4 TRANSFORMATION OF NONLINEAR EQUATIONS INTO SEPARABLE EQUATIONS

In Section 2.1 we found that the solutions of a linear nonhomogeneous equation

$$y' + p(x)y = f(x)$$

are of the form $y = uy_1$, where y_1 is a nontrivial solution of the complementary equation

$$y' + p(x)y = 0 \tag{2.4.1}$$

and u is a solution of

$$u'y_1(x) = f(x).$$

Note that this last equation is separable, since it can be rewritten as

$$u' = \frac{f(x)}{y_1(x)}.$$

In this section we will consider nonlinear differential equations that are not separable to begin with, but that can be solved in a similar fashion. This is done by writing their solutions in the form $y = uy_1$, where y_1 is a suitably chosen known function and u satisfies a separable equation. In this case, we will say that we *transformed* the given equation into a separable equation.

Bernoulli Equations

A *Bernoulli equation* is an equation of the form

$$y' + p(x)y = f(x)y^r, \tag{2.4.2}$$

where r can be any real number other than 0 or 1. (Note that (2.4.2) is linear if and only if $r = 0$ or $r = 1$.) We can transform (2.4.2) into a separable equation by variation of parameters: if y_1 is a nontrivial solution of (2.4.1), substituting $y = uy_1$ into (2.4.2) and applying the product rule for derivatives yields

$$u'y_1 + u(y_1' + p(x)y_1) = f(x)(uy_1)^r,$$

which is equivalent to the separable equation

$$u'y_1(x) = f(x)\left(y_1(x)\right)^r u^r, \tag{2.4.3}$$

since $y_1' + p(x)y_1 = 0$.

Example 2.4.1 Solve the Bernoulli equation

$$y' - y = xy^2.$$

Solution By inspection, $y_1 = e^x$ is a solution of $y' - y = 0$. We can use this fact to look for solutions in the form $y = ue^x$, where we can substitute into (2.4.3) to obtain

$$u'e^x = xu^2e^{2x} \quad \text{or, equivalently,} \quad u' = xu^2e^x.$$

Separating variables yields

$$\frac{u'}{u^2} = xe^x.$$

Now we integrate on both sides (use integration by parts on the right side) to obtain

$$-\frac{1}{u} = (x-1)e^x + c.$$

Hence,

$$u = -\frac{1}{(x-1)e^x + c}$$

and

$$y = -\frac{1}{x - 1 + ce^{-x}}.$$

■

Other Nonlinear Equations That Can be Transformed Into Separable Equations

We have seen that the nonlinear Bernoulli equation can be transformed into a separable equation by the substitution $y = uy_1$ if y_1 is suitably chosen. Now we discuss a sufficient condition for a nonlinear first order differential equation

$$y' = f(x, y) \tag{2.4.4}$$

to be transformable into a separable equation in the same way. Substituting $y = uy_1$ into (2.4.4) yields

$$u'y_1(x) + uy_1'(x) = f(x, uy_1(x)),$$

which is equivalent to

$$u'y_1(x) = f(x, uy_1(x)) - uy_1'(x). \tag{2.4.5}$$

If

$$f(x, uy_1(x)) = q(u)y_1'(x)$$

for some function q, then (2.4.5) becomes

$$u'y_1(x) = (q(u) - u)y_1'(x), \tag{2.4.6}$$

which is separable. After checking for constant solutions $u \equiv u_0$ such that $q(u_0) = u_0$, we can separate variables to obtain

$$\frac{u'}{q(u) - u} = \frac{y_1'(x)}{y_1(x)}.$$

In the next two examples, we consider only the most widely studied class of equations for which this method of transformation works. In these examples, x and y occur in f in such a way that $f(x, y)$ depends only on the ratio y/x; that is, (2.4.4) can be written as

$$y' = q(y/x), \tag{2.4.7}$$

where $q = q(u)$ is a function of a single variable. For the first example,

$$y' = \frac{y + xe^{-y/x}}{x} = \frac{y}{x} + e^{-y/x}$$

has

$$q(u) = u + e^{-u};$$

and for the second example,

$$y' = \frac{y^2 + xy - x^2}{x^2} = \left(\frac{y}{x}\right)^2 + \frac{y}{x} - 1$$

has

$$q(u) = u + e^{-u} \quad \text{and} \quad q(u) = u^2 + u - 1.$$

(Historically, these types of equations were referred to as *homogeneous equations*, but this is not the same as the definition given in Section 2.1, where we said that a linear equation of the form

$$y' + p(x)y = 0$$

is homogeneous. Unfortunately, homogeneous has been used in these two inconsistent ways. The one having to do with linear equations is the most important, and this is the only section where the meaning defined here will apply.)

The general method of transformation can be applied to (2.4.7) with $y_1 = x$ (and therefore $y_1' = 1$). Thus, substituting $y = ux$ in (2.4.7) yields

$$u'x + u = q(u),$$

and separation of variables (after checking for constant solutions such that $q(u) = u$) yields

$$\frac{u'}{q(u) - u} = \frac{1}{x}.$$

Since y/x is in general undefined if $x = 0$, we will consider solutions of equations only on open intervals that do not contain the point $x = 0$.

Example 2.4.2 Solve

$$y' = \frac{y + xe^{-y/x}}{x}. \tag{2.4.8}$$

Solution Substituting $y = ux$ into (2.4.8) yields

$$u'x + u = \frac{ux + xe^{-ux/x}}{x}.$$

We can simplify the fraction on the right to get

$$u'x + u = u + e^{-u},$$

then separate variables to arrive at

$$e^u u' = \frac{1}{x}.$$

Integrating yields $e^u = \ln|x| + c$. Therefore $u = \ln(\ln|x| + c)$ and the solution $y = ux$ is given by $y = x\ln(\ln|x| + c)$. ■

Example 2.4.3

(a) Solve

$$x^2y' = y^2 + xy - x^2. \tag{2.4.9}$$

(b) Solve the initial value problem

$$x^2y' = y^2 + xy - x^2, \quad y(1) = 2. \tag{2.4.10}$$

(a) We find solutions of (2.4.9) on open intervals that do not contain $x = 0$. We can rewrite (2.4.9) as

$$y' = \frac{y^2 + xy - x^2}{x^2}$$

for x in any such interval. Substituting $y = ux$ yields

$$u'x + u = \frac{(ux)^2 + x(ux) - x^2}{x^2},$$

which reduces to

$$u'x + u = u^2 + u - 1$$

after reducing the fraction on the right side. This equation simplifies to

$$u'x = u^2 - 1, \tag{2.4.11}$$

which has the constant solutions $u \equiv 1$ and $u \equiv -1$. (The constant solutions can be found by applying the Zero Product Property to the right side of the equation.) Therefore $y = x$ and $y = -x$ are solutions of (2.4.9). If u is a solution of (2.4.11) that does not assume the values ± 1 on some interval, separating variables yields

$$\frac{u'}{u^2 - 1} = \frac{1}{x},$$

or, after a partial fraction expansion,

$$\frac{1}{2}\left[\frac{1}{u-1} - \frac{1}{u+1}\right] u' = \frac{1}{x}.$$

Multiplying by 2 and integrating yields

$$\ln\left|\frac{u-1}{u+1}\right| = 2\ln|x| + k,$$

or

$$\left|\frac{u-1}{u+1}\right| = e^k x^2,$$

which holds if

$$\frac{u-1}{u+1} = cx^2 \tag{2.4.12}$$

where c is an arbitrary constant. Solving for u yields

$$u = \frac{1 + cx^2}{1 - cx^2}.$$

Therefore, we can substitute into $y = ux$ to find

$$y = \frac{x(1 + cx^2)}{1 - cx^2} \tag{2.4.13}$$

is a solution of (2.4.10) for any choice of the constant c. Setting $c = 0$ in (2.4.13) yields the solution $y = x$. However, the solution $y = -x$ can't be obtained from (2.4.13). Thus,

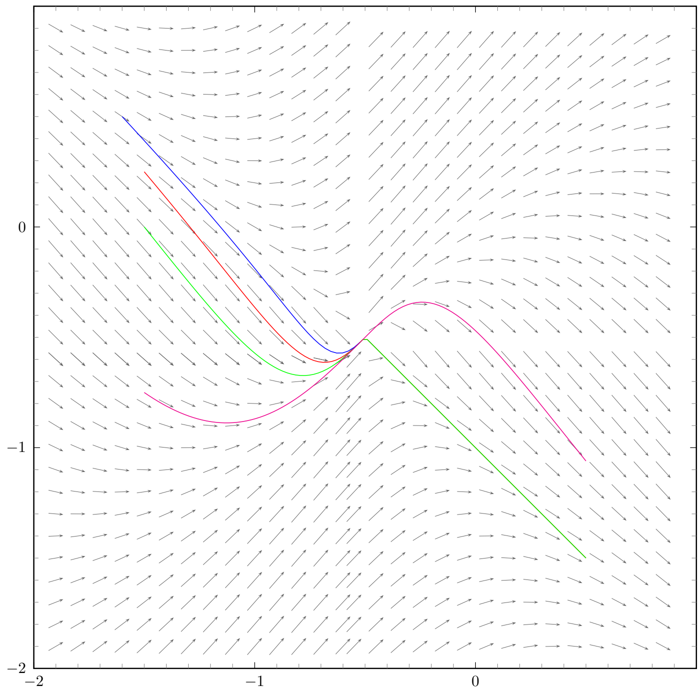


Figure 2.1 A direction field and integral curves for $x^2y' = y^2 + xy - x^2$

the solutions of (2.4.9) on intervals that do not contain $x = 0$ are $y = -x$ and functions of the form (2.4.13).

Figure 2.1 shows a direction field and some integral curves for (2.4.9).

(b) We could obtain c by imposing the initial condition $y(1) = 2$ in (2.4.13), and then solving for c. However, it is easier to use (2.4.12). Since $u = y/x$, the initial condition $y(1) = 2$ implies that $u(1) = 2$. Substituting this into (2.4.12) yields $c = 1/3$. Hence, the solution of (2.4.10) is

$$y = \frac{x(1 + x^2/3)}{1 - x^2/3}.$$

The domain of this solution is $(-\sqrt{3}, \sqrt{3})$. However, the largest interval on which (2.4.10) has a unique solution is $(0, \sqrt{3})$.

Figure 2.1 shows several solutions of the initial value problem (2.4.10). Note that these solutions coincide on $(0, \sqrt{3})$.

In the last two examples we were able to solve the given equations explicitly. However, this is not always possible, as you will see in the exercises.

2.4 Exercises

In Exercises 1–4 solve the given Bernoulli equation.

1. $y' + y = y^2$

2. $7xy' - 2y = -\dfrac{x^2}{y^6}$

3. $x^2y' + 2y = 2e^{1/x}y^{1/2}$

4. $(1+x^2)y' + 2xy = \dfrac{1}{(1+x^2)y}$

In Exercises 5 *and* 6 *find all solutions.*

5. $y' - xy = x^3y^3$

6. $y' - \dfrac{1+x}{3x}y = y^4$

In Exercises 7–11 *solve the initial value problem.*

7. $y' - 2y = xy^3, \quad y(0) = 2\sqrt{2}$

8. $y' - xy = xy^{3/2}, \quad y(1) = 4$

9. $xy' + y = x^4y^4, \quad y(1) = 1/2$

10. $y' - 2y = 2y^{1/2}, \quad y(0) = 1$

11. $y' - 4y = \dfrac{48x}{y^2}, \quad y(0) = 1$

In Exercises 12 *and* 13 *solve the initial value problem and graph the solution.*

12. $x^2y' + 2xy = y^3, \quad y(1) = 1/\sqrt{2}$

13. $y' - y = xy^{1/2}, \quad y(0) = 4$

14. You may have noticed that the logistic equation

$$P' = aP(1 - \alpha P)$$

from Verhulst's model for population growth can be written in Bernoulli form as

$$P' - aP = -a\alpha P^2.$$

The logistic equation is separable, and therefore solvable by the method studied in Section 2.2. Solve the logistic equation by the method of your choice.

In Exercises 15–18 *solve the equation explicitly.*

15. $y' = \dfrac{y + x}{x}$

16. $y' = \dfrac{y^2 + 2xy}{x^2}$

17. $xy^3y' = y^4 + x^4$

18. $y' = \dfrac{y}{x} + \sec\dfrac{y}{x}$

In Exercises 19-21 *solve the equation explicitly.*

19. $x^2y' = xy + x^2 + y^2$

20. $xyy' = x^2 + 2y^2$

21. $y' = \dfrac{2y^2 + x^2e^{-(y/x)^2}}{2xy}$

In Exercises 22–27 *solve the initial value problem.*

22. $y' = \dfrac{xy + y^2}{x^2}, \quad y(-1) = 2$

23. $y' = \dfrac{x^3 + y^3}{xy^2}, \quad y(1) = 3$

24. $xyy' + x^2 + y^2 = 0, \quad y(1) = 2$

25. $y' = \dfrac{y^2 - 3xy - 5x^2}{x^2}, \quad y(1) = -1$

26. $x^2y' = 2x^2 + y^2 + 4xy, \quad y(1) = 1$

27. $xyy' = 3x^2 + 4y^2, \quad y(1) = \sqrt{3}$

In Exercises 28–34 *solve the given "homogeneous" equation implicitly.*

28. $y' = \dfrac{x + y}{x - y}$

29. $(y'x - y)(\ln|y| - \ln|x|) = x$

30. $y' = \dfrac{y^3 + 2xy^2 + x^2y + x^3}{x(y + x)^2}$

31. $y' = \dfrac{x + 2y}{2x + y}$

32. $y' = \dfrac{y}{y - 2x}$

33. $y' = \dfrac{xy^2 + 2y^3}{x^3 + x^2y + xy^2}$

34. $y' = \dfrac{x^3 + x^2y + 3y^3}{x^3 + 3xy^2}$

2.5 EXACT EQUATIONS

In this section it will be convenient to write first order differential equations in the form

$$M(x, y)\,dx + N(x, y)\,dy = 0. \tag{2.5.1}$$

This type of equation can be interpreted as

$$M(x, y) + N(x, y)\,\frac{dy}{dx} = 0, \tag{2.5.2}$$

where x is the independent variable and y is the dependent variable, or as

$$M(x,y)\,\frac{dx}{dy}+N(x,y)=0, \tag{2.5.3}$$

where y is the independent variable and x is the dependent variable. Since the solutions of (2.5.2) and (2.5.3) will often need to be left in implicit form, we will say that $F(x,y)=c$ is an implicit solution of (2.5.1) if every differentiable function $y=y(x)$ that satisfies $F(x,y)=c$ is a solution of (2.5.2) and every differentiable function $x=x(y)$ that satisfies $F(x,y)=c$ is a solution of (2.5.3).

Some examples are shown in the table. Each differential equation is shown in three forms.

Form (2.5.1)	Form (2.5.2)	Form (2.5.3)
$3x^2y^2\,dx+2x^3y\,dy=0$	$3x^2y^2+2x^3y\,\frac{dy}{dx}=0$	$3x^2y^2\,\frac{dx}{dy}+2x^3y=0$
$(x^2+y^2)\,dx+2xy\,dy=0$	$(x^2+y^2)+2xy\,\frac{dy}{dx}=0$	$(x^2+y^2)\,\frac{dx}{dy}+2xy=0$
$3y\sin x\,dx-2xy\cos x\,dy=0$	$3y\sin x-2xy\cos x\,\frac{dy}{dx}=0$	$3y\sin x\,\frac{dx}{dy}-2xy\cos x=0$

Note that a separable equation can be written as (2.5.1) as

$$M(x)\,dx+N(y)\,dy=0.$$

We will develop a method for solving equations of this form under appropriate assumptions on M and N. This method is an extension of the method of separation of variables (Exercise **??**). Before discussing the method, we consider an example.

Example 2.5.1 Show that

$$x^4y^3+x^2y^5+2xy=c \tag{2.5.4}$$

is an implicit solution of

$$(4x^3y^3+2xy^5+2y)\,dx+(3x^4y^2+5x^2y^4+2x)\,dy=0. \tag{2.5.5}$$

Solution Regarding y as a function of x and differentiating (2.5.4) implicitly with respect to x yields

$$(4x^3y^3+2xy^5+2y)+(3x^4y^2+5x^2y^4+2x)\,\frac{dy}{dx}=0.$$

Similarly, regarding x as a function of y and differentiating (2.5.4) implicitly with respect to y yields

$$(4x^3y^3 + 2xy^5 + 2y)\frac{dx}{dy} + (3x^4y^2 + 5x^2y^4 + 2x) = 0.$$

Therefore (2.5.4) is an implicit solution of (2.5.5) in either of its two possible interpretations. ■

You may think this example is pointless, since concocting a differential equation that has a given implicit solution is difficult to do and not particularly interesting. However, it illustrates the next important theorem, which we will prove by using implicit differentiation, as in Example 2.5.1.

Theorem 2.5.1 *If* $F = F(x, y)$ *has continuous partial derivatives* F_x *and* F_y*, then*

$$F(x, y) = c \tag{2.5.6}$$

is an implicit solution of the differential equation

$$F_x(x, y)\,dx + F_y(x, y)\,dy = 0. \tag{2.5.7}$$

(Here, c *is an arbitrary constant.)*

Proof Regarding y as a function of x and differentiating (2.5.6) implicitly with respect to x yields

$$F_x(x, y) + F_y(x, y)\frac{dy}{dx} = 0.$$

On the other hand, regarding x as a function of y and differentiating (2.5.6) implicitly with respect to y yields

$$F_x(x, y)\frac{dx}{dy} + F_y(x, y) = 0.$$

Thus, (2.5.6) is an implicit solution of (2.5.7) in either of its two possible interpretations. ■

We will say that the equation

$$M(x, y)\,dx + N(x, y)\,dy = 0 \tag{2.5.8}$$

is *exact* on an an open rectangle R if there is a function $F = F(x, y)$ such F_x and F_y are continuous, and

$$F_x(x, y) = M(x, y) \quad \text{and} \quad F_y(x, y) = N(x, y) \tag{2.5.9}$$

for all (x, y) in R. This usage of "exact" is related to its usage in calculus, where the expression

$$F_x(x, y)\,dx + F_y(x, y)\,dy$$

is the *exact differential* of F. (This can be obtained by substituting (2.5.9) into the left side of (2.5.8).)

Example 2.5.1 shows that it is easy to solve (2.5.8) if it is exact *and* we know a function F that satisfies (2.5.9). The important questions are:

QUESTION 1. Given an equation (2.5.8), how can we determine whether it is exact?

QUESTION 2. If (2.5.8) is exact, how do we find a function F satisfying (2.5.9)?

To discover the answer to Question 1, assume that there is a function F that satisfies (2.5.9) on some open rectangle R, and in addition that F has continuous mixed partial derivatives F_{xy} and F_{yx}. Then a theorem from calculus implies that

$$F_{xy} = F_{yx}. \tag{2.5.10}$$

If $F_x = M$ and $F_y = N$, differentiating the first of these equations with respect to y and the second with respect to x yields

$$F_{xy} = M_y \quad \text{and} \quad F_{yx} = N_x. \tag{2.5.11}$$

From (2.5.10) and (2.5.11), we conclude that a necessary condition for exactness is that $M_y = N_x$. This motivates the next theorem, which we state without proof.

Theorem 2.5.2 *[The Exactness Condition] Suppose M and N are continuous and have continuous partial derivatives M_y and N_x on an open rectangle R. Then*

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

is exact on R if and only if

$$M_y(x,y) = N_x(x,y) \tag{2.5.12}$$

for all (x,y) in R..

To help you remember the exactness condition, observe that the coefficients of dx and dy are differentiated in (2.5.12) with respect to the "opposite" variables; that is, the coefficient of dx is differentiated with respect to y, while the coefficient of dy is differentiated with respect to x.

Example 2.5.2 Show that the equation

$$3x^2y\,dx + 4x^3\,dy = 0$$

is not exact on any open rectangle.

Solution Here

$$M(x,y) = 3x^2y \quad \text{and} \quad N(x,y) = 4x^3$$

so

$$M_y(x,y) = 3x^2 \quad \text{and} \quad N_x(x,y) = 12x^2.$$

Therefore $M_y = N_x$ on the line $x = 0$, but not on any open rectangle, so there is no function F such that $F_x(x,y) = M(x,y)$ and $F_y(x,y) = N(x,y)$ for all (x,y) on any open rectangle. ■

The next example illustrates two possible methods for finding a function F that satisfies the condition $F_x = M$ and $F_y = N$ if $M\,dx + N\,dy = 0$ is exact.

Example 2.5.3 Solve

$$(4x^3y^3 + 3x^2)\,dx + (3x^4y^2 + 6y^2)\,dy = 0. \tag{2.5.13}$$

Solution (Method 1) Here

$$M(x,y) = 4x^3y^3 + 3x^2, \quad N(x,y) = 3x^4y^2 + 6y^2,$$

and

$$M_y(x,y) = N_x(x,y) = 12x^3y^2$$

for all (x,y). Therefore Theorem 2.5.2 implies that there is a function F such that

$$F_x(x,y) = M(x,y) = 4x^3y^3 + 3x^2 \tag{2.5.14}$$

and

$$F_y(x,y) = N(x,y) = 3x^4y^2 + 6y^2 \tag{2.5.15}$$

for all (x,y). To find F, we integrate (2.5.14) with respect to x to obtain

$$F(x,y) = x^4y^3 + x^3 + \phi(y), \tag{2.5.16}$$

where $\phi(y)$ is the "constant" of integration. (Here ϕ is "constant" in that it is independent of x, the variable of integration.) If ϕ is any differentiable function of y then F satisfies (2.5.14). To determine ϕ so that F also satisfies (2.5.15), assume that ϕ is differentiable and differentiate F with respect to y. This yields

$$F_y(x,y) = 3x^4y^2 + \phi'(y).$$

Comparing this with (2.5.15) shows that

$$\phi'(y) = 6y^2.$$

We integrate this with respect to y and take the constant of integration to be zero because we are interested only in finding *some* F that satisfies (2.5.14) and (2.5.15). This yields

$$\phi(y) = 2y^3.$$

Substituting this into (2.5.16) yields

$$F(x,y) = x^4y^3 + x^3 + 2y^3. \tag{2.5.17}$$

Now Theorem 2.5.1 implies that

$$x^4y^3 + x^3 + 2y^3 = c$$

is an implicit solution of (2.5.13). Solving this for y yields the explicit solution

$$y = \left(\frac{c - x^3}{2 + x^4}\right)^{1/3}.$$

Solution (Method 2) Instead of first integrating (2.5.14) with respect to x, we could begin by integrating (2.5.15) with respect to y to obtain

$$F(x,y) = x^4y^3 + 2y^3 + \psi(x), \tag{2.5.18}$$

where ψ is an arbitrary function of x. To determine ψ, we assume that ψ is differentiable and differentiate F with respect to x, which yields

$$F_x(x,y) = 4x^3y^3 + \psi'(x).$$

Comparing this with (2.5.14) shows that

$$\psi'(x) = 3x^2.$$

Integrating this and again taking the constant of integration to be zero yields

$$\psi(x) = x^3.$$

Substituting this into (2.5.18) yields (2.5.17). ■

Here's a summary of the procedure used in Method 1 of this example. A summary of the procedure used in Method 2 is similar.

Procedure For Solving An Exact Equation

Step 1. Check that the equation

$$M(x,y)\,dx + N(x,y)\,dy = 0 \tag{2.5.19}$$

satisfies the exactness condition $M_y = N_x$. If not, don't go further with this procedure.

Step 2. Integrate

$$\frac{\partial F(x,y)}{\partial x} = M(x,y)$$

with respect to x to obtain

$$F(x,y) = G(x,y) + \phi(y), \tag{2.5.20}$$

where G is an antiderivative of M with respect to x, and ϕ is an unknown function of y.

Step 3. Differentiate (2.5.20) with respect to y to obtain

$$\frac{\partial F(x,y)}{\partial y} = \frac{\partial G(x,y)}{\partial y} + \phi'(y).$$

Step 4. Equate the right side of this equation to N and solve for ϕ'; thus,

$$\frac{\partial G(x,y)}{\partial y} + \phi'(y) = N(x,y), \quad \text{so} \quad \phi'(y) = N(x,y) - \frac{\partial G(x,y)}{\partial y}.$$

Step 5. Integrate ϕ' with respect to y (taking the constant of integration to be zero), and substitute the result into (2.5.20) to obtain $F(x, y)$.

Step 6. Set $F(x, y) = c$ to obtain an implicit solution of (2.5.19). If possible, solve for y explicitly as a function of x.

It is a common mistake to omit Step 6. However, it is important to include this step, since F is not itself a solution of (2.5.19).

Many equations can be conveniently solved by either of the two methods used in Example 2.5.3. However, sometimes the integration required in one approach is more difficult than in the other. In such cases we choose the approach that requires the easier integration.

Example 2.5.4 Solve the equation

$$(ye^{xy}\tan x + e^{xy}\sec^2 x)\,dx + xe^{xy}\tan x\,dy = 0. \tag{2.5.21}$$

Solution We leave it to you to check that $M_y = N_x$ on any open rectangle where $\tan x$ and $\sec x$ are defined. Here we must find a function F such that

$$F_x(x, y) = ye^{xy}\tan x + e^{xy}\sec^2 x \tag{2.5.22}$$

and

$$F_y(x, y) = xe^{xy}\tan x. \tag{2.5.23}$$

It is difficult to integrate (2.5.22) with respect to x, but easy to integrate (2.5.23) with respect to y. This yields

$$F(x, y) = e^{xy}\tan x + \psi(x). \tag{2.5.24}$$

Differentiating this with respect to x yields

$$F_x(x, y) = ye^{xy}\tan x + e^{xy}\sec^2 x + \psi'(x).$$

Comparing this with (2.5.22) shows that $\psi'(x) = 0$. Hence, ψ is a constant, which we can take to be zero in (2.5.24), and

$$e^{xy}\tan x = c$$

is an implicit solution of (2.5.21). ■

Attempting to apply our procedure to an equation that is not exact will lead to failure in Step 4, since the function

$$N - \frac{\partial G}{\partial y}$$

will not be independent of x if $M_y \neq N_x$ (Exercise **??**), and therefore cannot be the derivative of a function of y alone. Here is an example that illustrates this.

Example 2.5.5 Verify that the equation

$$3x^2y^2\,dx + 6x^3y\,dy = 0 \tag{2.5.25}$$

is not exact, and show that the procedure for solving exact equations fails when applied to (2.5.25).

Solution Here

$$M_y(x,y) = 6x^2y \quad \text{and} \quad N_x(x,y) = 18x^2y,$$

so (2.5.25) is not exact. Nevertheless, let us try to find a function F such that

$$F_x(x,y) = 3x^2y^2 \tag{2.5.26}$$

and

$$F_y(x,y) = 6x^3y. \tag{2.5.27}$$

Integrating (2.5.26) with respect to x yields

$$F(x,y) = x^3y^2 + \phi(y),$$

and differentiating this with respect to y yields

$$F_y(x,y) = 2x^3y + \phi'(y).$$

For this equation to be consistent with (2.5.27),

$$6x^3y = 2x^3y + \phi'(y),$$

or

$$\phi'(y) = 4x^3y.$$

This is a contradiction, since ϕ' must be independent of x. Therefore the procedure fails.

2.5 Exercises

In Exercises 1–17 determine which equations are exact and solve them.

1. $6x^2y^2\,dx + 4x^3y\,dy = 0$
2. $(3y\cos x + 4xe^x + 2x^2e^x)\,dx + (3\sin x + 3)\,dy = 0$
3. $14x^2y^3\,dx + 21x^2y^2\,dy = 0$
4. $(2x - 2y^2)\,dx + (12y^2 - 4xy)\,dy = 0$
5. $(x+y)^2\,dx + (x+y)^2\,dy = 0$ **6.** $(4x + 7y)\,dx + (3x + 4y)\,dy = 0$
7. $(-2y^2\sin x + 3y^3 - 2x)\,dx + (4y\cos x + 9xy^2)\,dy = 0$

8. $(2x + y)\,dx + (2y + 2x)\,dy = 0$

9. $(3x^2 + 2xy + 4y^2)\,dx + (x^2 + 8xy + 18y)\,dy = 0$

10. $(2x^2 + 8xy + y^2)\,dx + (2x^2 + xy^3/3)\,dy = 0$

11. $\left(\frac{1}{x} + 2x\right)\,dx + \left(\frac{1}{y} + 2y\right)\,dy = 0$

12. $(y \sin xy + xy^2 \cos xy)\,dx + (x \sin xy + xy^2 \cos xy)\,dy = 0$

13. $\frac{x\,dx}{(x^2 + y^2)^{3/2}} + \frac{y\,dy}{(x^2 + y^2)^{3/2}} = 0$

14. $\left(e^x(x^2y^2 + 2xy^2) + 6x\right)\,dx + (2x^2ye^x + 2)\,dy = 0$

15. $\left(x^2e^{x^2+y}(2x^2 + 3) + 4x\right)\,dx + (x^3e^{x^2+y} - 12y^2)\,dy = 0$

16. $\left(e^{xy}(x^4y + 4x^3) + 3y\right)\,dx + (x^5e^{xy} + 3x)\,dy = 0$

17. $(3x^2 \cos xy - x^3y \sin xy + 4x)\,dx + (8y - x^4 \sin xy)\,dy = 0$

In Exercises 18–22 *solve the initial value problem.*

18. $(4x^3y^2 - 6x^2y - 2x - 3)\,dx + (2x^4y - 2x^3)\,dy = 0, \quad y(1) = 3$

19. $(-4y \cos x + 4 \sin x \cos x + \sec^2 x)\,dx + (4y - 4 \sin x)\,dy = 0, \quad y(\pi/4) = 0$

20. $(y^3 - 1)e^x\,dx + 3y^2(e^x + 1)\,dy = 0, \quad y(0) = 0$

21. $(\sin x - y \sin x - 2 \cos x)\,dx + \cos x\,dy = 0, \quad y(0) = 1$

22. $(2x - 1)(y - 1)\,dx + (x + 2)(x - 3)\,dy = 0, \quad y(1) = -1$

23. Find all functions M such that the equation is exact.

(a) $M(x, y)\,dx + (x^2 - y^2)\,dy = 0$

(b) $M(x, y)\,dx + 2xy \sin x \cos y\,dy = 0$

(c) $M(x, y)\,dx + (e^x - e^y \sin x)\,dy = 0$

24. Find all functions N such that the equation is exact.

(a) $(x^3y^2 + 2xy + 3y^2)\,dx + N(x, y)\,dy = 0$

(b) $(\ln xy + 2y \sin x)\,dx + N(x, y)\,dy = 0$

(c) $(x \sin x + y \sin y)\,dx + N(x, y)\,dy = 0$

25. Rewrite the separable equation

$$h(y)y' = g(x) \tag{A}$$

as an exact equation

$$M(x, y)\,dx + N(x, y)\,dy = 0. \tag{B}$$

Show that applying the method of this section to (B) yields the same solutions that would be obtained by applying the method of separation of variables to (A)

2.6 INTEGRATING FACTORS

In Section 2.5 we saw that if M, N, M_y and N_x are continuous and $M_y = N_x$ on an open rectangle R then

$$M(x,y)\,dx + N(x,y)\,dy = 0 \tag{2.6.1}$$

is exact on R. Sometimes an equation that is not exact can be made exact by multiplying it by an appropriate function. For example,

$$(3x + 2y^2)\,dx + 2xy\,dy = 0 \tag{2.6.2}$$

is not exact, since $M_y(x,y) = 4y \neq N_x(x,y) = 2y$ in (2.6.2). However, multiplying (2.6.2) by x yields

$$(3x^2 + 2xy^2)\,dx + 2x^2y\,dy = 0, \tag{2.6.3}$$

which is exact, since $M_y(x,y) = N_x(x,y) = 4xy$ in (2.6.3). Solving (2.6.3) by the procedure given in Section 2.5 yields the implicit solution

$$x^3 + x^2y^2 = c.$$

In Section 2.4, we transformed equations into separable equations by use of a substitution. In this section, we transform equations into exact equations by using the multiplication property of equality. More specifically, a function $\mu = \mu(x,y)$ is called an *integrating factor* for (2.6.1) if

$$\mu(x,y)M(x,y)\,dx + \mu(x,y)N(x,y)\,dy = 0 \tag{2.6.4}$$

is exact. If we know an integrating factor μ for (2.6.1), we can solve the exact equation (2.6.4) by the method of Section 2.5. (It would be nice if we could say that (2.6.1) and (2.6.4) always have the same solutions, but this is not always true. However, if $\mu(x,y)$ is defined and nonzero for all (x,y), (2.6.1) and (2.6.4) are equivalent; that is, they have the same solutions.)

Finding Integrating Factors

By applying Theorem 2.5.2 (with M and N replaced by μM and μN), we see that (2.6.4) is exact on an open rectangle R if μM, μN, $(\mu M)_y$, and $(\mu N)_x$ are continuous and

$$\frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial x}(\mu N) \quad \text{or, equivalently,} \quad \mu_y M + \mu M_y = \mu_x N + \mu N_x$$

on R. It is better to rewrite the last equation as

$$\mu(M_y - N_x) = \mu_x N - \mu_y M, \tag{2.6.5}$$

which reduces to the known result for exact equations; that is, if $M_y = N_x$ then (2.6.5) holds with $\mu = 1$, so (2.6.1) is exact.

You may think (2.6.5) is of little value, since it involves *partial* derivatives of the unknown integrating factor μ, and we have not studied methods for solving such equations. However, we will now show that (2.6.5) is useful if we restrict our search to integrating factors that are products of a function of x and a function of y; that is, $\mu(x,y) = P(x)Q(y)$. We are not saying that *every* equation $M\,dx + N\,dy = 0$ has an integrating factor of this form; rather, we are saying that *some* equations have such integrating factors. We will now develop a way to determine whether a given equation has such an integrating factor, and a method for finding the integrating factor in this case.

If $\mu(x,y) = P(x)Q(y)$, then $\mu_x(x,y) = P'(x)Q(y)$ and $\mu_y(x,y) = P(x)Q'(y)$, so (2.6.5) becomes

$$P(x)Q(y)(M_y - N_x) = P'(x)Q(y)N - P(x)Q'(y)M, \tag{2.6.6}$$

or, after dividing through by $P(x)Q(y)$,

$$M_y - N_x = \frac{P'(x)}{P(x)}N - \frac{Q'(y)}{Q(y)}M. \tag{2.6.7}$$

Now let

$$p(x) = \frac{P'(x)}{P(x)} \quad \text{and} \quad q(y) = \frac{Q'(y)}{Q(y)},$$

so (2.6.7) becomes

$$M_y - N_x = p(x)N - q(y)M. \tag{2.6.8}$$

We obtained (2.6.8) by *assuming* that $M\,dx + N\,dy = 0$ has an integrating factor $\mu(x,y) = P(x)Q(y)$. However, we can now view (2.6.7) differently: If there are functions $p = p(x)$ and $q = q(y)$ that satisfy (2.6.8) and we define

$$P(x) = \pm e^{\int p(x)\,dx} \quad \text{and} \quad Q(y) = \pm e^{\int q(y)\,dy}, \tag{2.6.9}$$

then reversing the steps that led from (2.6.6) to (2.6.8) shows that $\mu(x,y) = P(x)Q(y)$ is an integrating factor for $M\,dx + N\,dy = 0$. In using this result, we take the constants of integration in (2.6.9) to be zero and choose the signs conveniently so the integrating factor has the simplest form.

There is no simple general method for ascertaining whether functions $p = p(x)$ and $q = q(y)$ satisfying (2.6.8) exist. However, the next theorem gives simple sufficient conditions for the given equation to have an integrating factor that depends on only one of the independent variables x and y, and for finding an integrating factor in this case.

Theorem 2.6.1 *Let M, N, M_y, and N_x be continuous on an open rectangle R. Then:*

(a) *If $(M_y - N_x)/N$ is independent of y on R and we define*

$$p(x) = \frac{M_y - N_x}{N}$$

then

$$\mu(x) = \pm e^{\int p(x)\,dx} \tag{2.6.10}$$

is an integrating factor for

$$M(x,y)\,dx + N(x,y)\,dy = 0 \tag{2.6.11}$$

on R.

(b) *If* $(N_x - M_y)/M$ *is independent of* x *on* R *and we define*

$$q(y) = \frac{N_x - M_y}{M},$$

then

$$\mu(y) = \pm e^{\int q(y)\,dy} \tag{2.6.12}$$

is an integrating factor for (2.6.11) *on* R.

Proof (a) If $(M_y - N_x)/N$ is independent of y, then (2.6.8) holds with $p = (M_y - N_x)/N$ and $q \equiv 0$. Therefore

$$P(x) = \pm e^{\int p(x)\,dx} \quad \text{and} \quad Q(y) = \pm e^{\int q(y)\,dy} = \pm e^0 = \pm 1,$$

so (2.6.10) is an integrating factor for (2.6.11) on R.

(b) If $(N_x - M_y)/M$ is independent of x then (2.6.8) holds with $p \equiv 0$ and $q = (N_x - M_y)/M$, and a similar argument shows that (2.6.12) is an integrating factor for (2.6.11) on R. ■

The next two examples show how to apply Theorem 2.6.1.

Example 2.6.1 Find an integrating factor for the equation

$$(2xy^3 - 2x^3y^3 - 4xy^2 + 2x)\,dx + (3x^2y^2 + 4y)\,dy = 0 \tag{2.6.13}$$

and solve the equation.

Solution In (2.6.13)

$$M = 2xy^3 - 2x^3y^3 - 4xy^2 + 2x, \;\; N = 3x^2y^2 + 4y,$$

and

$$(M_y) - N_x = (6xy^2 - 6x^3y^2 - 8xy) - 6xy^2 = -6x^3y^2 - 8xy,$$

so (2.6.13) is not exact. However,

$$\frac{M_y - N_x}{N} = -\frac{6x^3y^2 + 8xy}{3x^2y^2 + 4y} = -2x$$

is independent of y, so Theorem 2.6.1**(a)** applies with $p(x) = -2x$. Since

$$\int p(x)\,dx = -\int 2x\,dx = -x^2,$$

$\mu(x) = e^{-x^2}$ is an integrating factor. Multiplying (2.6.13) by μ yields the exact equation

$$e^{-x^2}(2xy^3 - 2x^3y^3 - 4xy^2 + 2x)\,dx + e^{-x^2}(3x^2y^2 + 4y)\,dy = 0. \tag{2.6.14}$$

To solve this equation, we must find a function F such that

$$F_x(x,y) = e^{-x^2}(2xy^3 - 2x^3y^3 - 4xy^2 + 2x) \tag{2.6.15}$$

and

$$F_y(x,y) = e^{-x^2}(3x^2y^2 + 4y). \tag{2.6.16}$$

Integrating (2.6.16) with respect to y yields

$$F(x,y) = e^{-x^2}(x^2y^3 + 2y^2) + \psi(x). \tag{2.6.17}$$

Differentiating this with respect to x yields

$$F_x(x,y) = e^{-x^2}(2xy^3 - 2x^3y^3 - 4xy^2) + \psi'(x).$$

Comparing this with (2.6.15) shows that $\psi'(x) = 2xe^{-x^2}$; therefore, we can let $\psi(x) = -e^{-x^2}$ in (2.6.17) and conclude that

$$e^{-x^2}\left(y^2(x^2y + 2) - 1\right) = c$$

is an implicit solution of (2.6.14). It is also an implicit solution of (2.6.13). ■

Example 2.6.2 Find an integrating factor for

$$2xy^3\,dx + (3x^2y^2 + x^2y^3 + 1)\,dy = 0 \tag{2.6.18}$$

and solve the equation.

Solution In (2.6.18),

$$M = 2xy^3, \quad N = 3x^2y^2 + x^2y^3 + 1,$$

and

$$M_y - N_x = 6xy^2 - (6xy^2 + 2xy^3) = -2xy^3,$$

so (2.6.18) is not exact. Moreover,

$$\frac{M_y - N_x}{N} = -\frac{2xy^3}{3x^2y^2 + x^2y^2 + 1}$$

is not independent of y, so Theorem 2.6.1(a) does not apply. However, Theorem 2.6.1(b) does apply, since

$$\frac{N_x - M_y}{M} = \frac{2xy^3}{2xy^3} = 1$$

is independent of x, so we can take $q(y) = 1$. Since

$$\int q(y)\,dy = \int dy = y,$$

$\mu(y) = e^y$ is an integrating factor. Multiplying (2.6.18) by μ yields the exact equation

$$2xy^3e^y\,dx + (3x^2y^2 + x^2y^3 + 1)e^y\,dy = 0. \tag{2.6.19}$$

To solve this equation, we must find a function F such that

$$F_x(x, y) = 2xy^3e^y \tag{2.6.20}$$

and

$$F_y(x, y) = (3x^2y^2 + x^2y^3 + 1)e^y. \tag{2.6.21}$$

Integrating (2.6.20) with respect to x yields

$$F(x, y) = x^2y^3e^y + \phi(y). \tag{2.6.22}$$

Differentiating this with respect to y yields

$$F_y = (3x^2y^2 + x^2y^3)e^y + \phi'(y),$$

and comparing this with (2.6.21) shows that $\phi'(y) = e^y$. Therefore we set $\phi(y) = e^y$ in (2.6.22) and conclude that

$$(x^2y^3 + 1)e^y = c$$

is an implicit solution of (2.6.19). It is also an implicit solution of (2.6.18). ■

When working with exact equations, be sure to use the form $M(x, y)dx + N(x, y)dy = 0$. For example, suppose an equation is given as $G(x, y)dx = H(x, y)dy$; in this case, we would first rewrite it as $G(x, y)dx - H(x, y)dy = 0$ and then identify $N(x, y) = -H(x, y)$ before applying the method of solving.

2.6 Exercises

In Exercises 1–14, find an integrating factor that is a function of only one variable, and then solve the given equation.

1. $y\,dx - x\,dy = 0$

2. $3x^2y\,dx + 2x^3\,dy = 0$

3. $2y^3\,dx + 3y^2\,dy = 0$

4. $(5xy + 2y + 5)\,dx + 2x\,dy = 0$

5. $(xy + x + 2y + 1)\,dx = -(x + 1)\,dy$

6. $(27xy^2 + 8y^3)\,dx + (18x^2y + 12xy^2)\,dy = 0$

7. $(6xy^2 + 2y)\,dx + (12x^2y + 6x + 3)\,dy = 0$

8. $-y^2\,dx = \left(xy^2 + 3xy + \frac{1}{y}\right)\,dy$

9. $(12x^3y + 24x^2y^2)\,dx + (9x^4 + 32x^3y + 4y)\,dy = 0$

10. $(x^2y + 4xy + 2y)\,dx + (x^2 + x)\,dy = 0$

11. $-y\,dx = -(x^4 - x)\,dy$

12. $\cos x \cos y\,dx + (\sin x \cos y - \sin x \sin y + y)\,dy = 0$

13. $(2xy + y^2)\,dx + (2xy + x^2 - 2x^2y^2 - 2xy^3)\,dy = 0$

14. $y \sin y\,dx + x(\sin y - y \cos y)\,dy = 0$

CHAPTER 3

LINEAR HIGHER ORDER EQUATIONS

IN THIS CHAPTER we study higher order equations, primarily second order equations that can be written in the form

$$P_2(x)y'' + P_1(x)y' + P_0(x)y = F(x).$$

Such equations are said to be *linear*. As in the case of first order linear equations, an equation is said to be *homogeneous* if $F \equiv 0$, or *nonhomogeneous* if $F \not\equiv 0$. Because of their many applications in science and engineering, second order differential equations have historically been the most thoroughly studied class of differential equations. We will look at a few of these applications at the end of the chapter. Throughout the chapter, we will also encounter a few differential equations of order three or higher.

SECTION 3.1 is devoted to the theory of homogeneous linear equations.

SECTION 3.2 deals primarily with homogeneous equations of the special form

$$ay'' + by' + cy = 0,$$

where a, b, and c are constant ($a \neq 0$).

SECTION 3.3 presents the theory of nonhomogeneous linear equations.

SECTIONS 3.4 AND 3.5 present the *method of undetermined coefficients*, which can be used to solve nonhomogeneous equations of the form

$$ay'' + by' + cy = F(x),$$

where a, b, and c are constants and F has a special form that is still sufficiently general to occur in many applications. In this section we make extensive use of the idea of variation of parameters introduced in Chapter 2.

SECTION 3.6 deals with *reduction of order*, a technique based on the idea of variation of parameters, which enables us to find the general solution of a nonhomogeneous linear second order equation provided that we know one nontrivial (not identically zero) solution of the associated homogeneous equation.

SECTION 3.7 deals with the method traditionally called *variation of parameters*, which enables us to find the general solution of a nonhomogeneous linear second order equation provided that we know two nontrivial solutions (with nonconstant ratio) of the associated homogeneous equation.

SECTION 3.8 looks at applications of linear higher order equations to spring—mass systems. In particular, we consider simple harmonic motion, undamped forced oscillation, and free vibrations with damping.

3.1 HOMOGENEOUS LINEAR EQUATIONS

A second order differential equation is said to be *linear* if it can be written as

$$y'' + p(x)y' + q(x)y = f(x). \tag{3.1.1}$$

We say that (3.1.1) is *homogeneous* if $f \equiv 0$ or *nonhomogeneous* if $f \not\equiv 0$. Since these definitions are like the corresponding definitions for the linear first order equation

$$y' + p(x)y = f(x), \tag{3.1.2}$$

it is natural to expect similarities between methods of solving (3.1.1) and (3.1.2). However, solving (3.1.1) is more difficult than solving (3.1.2). For example, while Theorem 2.1.1 gives a formula for the general solution of (3.1.2) in the case where $f \equiv 0$ and Theorem 2.1.2 gives a formula for the case where $f \not\equiv 0$, there are no formulas for the general solution of (3.1.1) in either case. Therefore we must be content to solve linear second order equations of special forms.

In Section 2.1, we first considered the homogeneous equation $y' + p(x)y = 0$ and then used a nontrivial solution of this equation to find the general solution of the nonhomogeneous equation $y' + p(x)y = f(x)$. Although the progression from the homogeneous to the nonhomogeneous case is not that simple for the linear second order equation, it is still necessary to solve the homogeneous equation

$$y'' + p(x)y' + q(x)y = 0 \tag{3.1.3}$$

in order to solve the nonhomogeneous equation (3.1.1). This section is devoted to solving homogeneous equations of this type.

The next theorem gives sufficient conditions for existence and uniqueness of solutions of initial value problems for (3.1.3). We omit the proof.

Theorem 3.1.1 *Suppose p and q are continuous on an open interval (a, b), let x_0 be any point in (a, b), and let k_0 and k_1 be arbitrary real numbers. Then the initial value problem*

$$y'' + p(x)y' + q(x)y = 0, \; y(x_0) = k_0, \; y'(x_0) = k_1$$

has a unique solution on (a, b).

Since $y \equiv 0$ is obviously a solution of (3.1.3) we call it the *trivial* solution. Any other solution is *nontrivial*. Notice that under the assumptions of Theorem 3.1.1, the only solution of the initial value problem

$$y'' + p(x)y' + q(x)y = 0, \; y(x_0) = 0, \; y'(x_0) = 0$$

on (a, b) is the trivial solution.

The next three examples illustrate concepts that we will develop later in this section. You should not be concerned with how to *find* the given solutions of the equations in these examples. This will be explained in later sections.

Example 3.1.1 The coefficients of y' and y in

$$y'' - y = 0 \tag{3.1.4}$$

are the constant functions $p \equiv 0$ and $q \equiv -1$, which are continuous on $(-\infty, \infty)$. Therefore Theorem 3.1.1 implies that every initial value problem for (3.1.4) has a unique solution on $(-\infty, \infty)$.

(a) Verify that $y_1 = e^x$ and $y_2 = e^{-x}$ are solutions of (3.1.4) on $(-\infty, \infty)$.

(b) Verify that if c_1 and c_2 are arbitrary constants, $y = c_1e^x + c_2e^{-x}$ is a solution of (3.1.4) on $(-\infty, \infty)$.

(c) Solve the initial value problem

$$y'' - y = 0, \quad y(0) = 1, \quad y'(0) = 3. \tag{3.1.5}$$

Solution (a) If $y_1 = e^x$ then both $y_1' = e^x$ and $y_1'' = e^x$, so that $y_1'' - y_1 = 0$. If $y_2 = e^{-x}$, then $y_2' = -e^{-x}$ and $y_2'' = e^{-x}$ so that $y_2'' = y_2$. This verifies that $y_2'' - y_2 = 0$.

(b) If

$$y = c_1e^x + c_2e^{-x} \tag{3.1.6}$$

then

$$y' = c_1e^x - c_2e^{-x} \tag{3.1.7}$$

and

$$y'' = c_1e^x + c_2e^{-x},$$

so

$$\begin{aligned} y'' - y &= (c_1e^x + c_2e^{-x}) - (c_1e^x + c_2e^{-x}) \\ &= c_1(e^x - e^x) + c_2(e^{-x} - e^{-x}) = 0 \end{aligned}$$

for all x. Therefore $y = c_1e^x + c_2e^{-x}$ is a solution of (3.1.4) on $(-\infty, \infty)$.

(c) We can solve (3.1.5) by choosing c_1 and c_2 in (3.1.6) so that $y(0) = 1$ and $y'(0) = 3$. Setting $x = 0$ in (3.1.6) and (3.1.7) shows that this is equivalent to

$$\begin{aligned} c_1 + c_2 &= 1 \\ c_1 - c_2 &= 3. \end{aligned}$$

Solving this system of equations yields $c_1 = 2$ and $c_2 = -1$. Therefore $y = 2e^x - e^{-x}$ is the unique solution of (3.1.5) on $(-\infty, \infty)$. ■

The next example will be a useful reference for the technique discussed in the next section.

Example 3.1.2 Let ω be a positive constant. The coefficients of y' and y in

$$y'' + \omega^2 y = 0 \tag{3.1.8}$$

are the constant functions $p \equiv 0$ and $q \equiv \omega^2$, which are continuous on $(-\infty, \infty)$. Therefore Theorem 3.1.1 implies that every initial value problem for (3.1.8) has a unique solution on $(-\infty, \infty)$.

(a) Verify that $y_1 = \cos\omega x$ and $y_2 = \sin\omega x$ are solutions of (3.1.8) on $(-\infty,\infty)$.

(b) Verify that if c_1 and c_2 are arbitrary constants then $y = c_1\cos\omega x + c_2\sin\omega x$ is a solution of (3.1.8) on $(-\infty,\infty)$.

(c) Solve the initial value problem

$$y'' + \omega^2 y = 0, \quad y(0) = 1, \quad y'(0) = 3. \tag{3.1.9}$$

Solution (a) If $y_1 = \cos\omega x$ then $y_1' = -\omega\sin\omega x$ and $y_1'' = -\omega^2\cos\omega x$. Substitution then verifies that $y_1'' + \omega^2 y_1 = 0$. If $y_2 = \sin\omega x$ then, $y_2' = \omega\cos\omega x$ and $y_2'' = -\omega^2\sin\omega x$. Again, substitution is used to verify that $y_2'' + \omega^2 y_2 = 0$.

(b) If

$$y = c_1\cos\omega x + c_2\sin\omega x \tag{3.1.10}$$

then

$$y' = \omega(-c_1\sin\omega x + c_2\cos\omega x) \tag{3.1.11}$$

and

$$y'' = -\omega^2(c_1\cos\omega x + c_2\sin\omega x),$$

so

$$\begin{aligned} y'' + \omega^2 y &= -\omega^2(c_1\cos\omega x + c_2\sin\omega x) + \omega^2(c_1\cos\omega x + c_2\sin\omega x) \\ &= c_1\omega^2(-\cos\omega x + \cos\omega x) + c_2\omega^2(-\sin\omega x + \sin\omega x) = 0 \end{aligned}$$

for all x. Therefore $y = c_1\cos\omega x + c_2\sin\omega x$ is a solution of (3.1.8) on $(-\infty,\infty)$.

(c) To solve (3.1.9), we must choose c_1 and c_2 in (3.1.10) so that $y(0) = 1$ and $y'(0) = 3$. Setting $x = 0$ in (3.1.10) and (3.1.11) shows that $c_1 = 1$ and $c_2 = 3/\omega$. Therefore

$$y = \cos\omega x + \frac{3}{\omega}\sin\omega x$$

is the unique solution of (3.1.9) on $(-\infty,\infty)$. ■

Theorem 3.1.1 implies that if k_0 and k_1 are arbitrary real numbers then the initial value problem

$$P_2(x)y'' + P_1(x)y' + P_0(x)y = 0, \quad y(x_0) = k_0, \quad y'(x_0) = k_1 \tag{3.1.12}$$

has a unique solution on an interval (a,b) that contains x_0, provided that P_2, P_1, and P_0 are continuous and P_2 has no zeros on (a,b). To see this, we rewrite the differential equation in (3.1.12) as

$$y'' + \frac{P_1(x)}{P_2(x)}y' + \frac{P_0(x)}{P_2(x)}y = 0$$

and apply Theorem 3.1.1 with $p = P_1/P_2$ and $q = P_0/P_2$.

Example 3.1.3 The equation

$$x^2y'' + xy' - 4y = 0 \tag{3.1.13}$$

has the form of the differential equation in (3.1.12), with $P_2(x) = x^2$, $P_1(x) = x$, and $P_0(x) = -4$, which are are all continuous on $(-\infty, \infty)$. However, since $P_2(0) = 0$ we must consider solutions of (3.1.13) on $(-\infty, 0)$ and $(0, \infty)$. Since P_2 has no zeros on these intervals, Theorem 3.1.1 implies that the initial value problem

$$x^2y'' + xy' - 4y = 0, \quad y(x_0) = k_0, \quad y'(x_0) = k_1$$

has a unique solution on $(0, \infty)$ if $x_0 > 0$, or on $(-\infty, 0)$ if $x_0 < 0$.

(a) Verify that $y_1 = x^2$ is a solution of (3.1.13) on $(-\infty, \infty)$ and $y_2 = 1/x^2$ is a solution of (3.1.13) on $(-\infty, 0)$ and $(0, \infty)$.

(b) Verify that if c_1 and c_2 are any constants then $y = c_1x^2 + c_2/x^2$ is a solution of (3.1.13) on $(-\infty, 0)$ and $(0, \infty)$.

(c) Solve the initial value problem

$$x^2y'' + xy' - 4y = 0, \quad y(1) = 2, \quad y'(1) = 0. \tag{3.1.14}$$

(d) Solve the initial value problem

$$x^2y'' + xy' - 4y = 0, \quad y(-1) = 2, \quad y'(-1) = 0. \tag{3.1.15}$$

Solution (a) If $y_1 = x^2$ then $y_1' = 2x$ and $y_1'' = 2$, so

$$x^2y_1'' + xy_1' - 4y_1 = x^2(2) + x(2x) - 4x^2,$$

which reduces to zero for x in $(-\infty, \infty)$. If $y_2 = 1/x^2$, then $y_2' = -2/x^3$ and $y_2'' = 6/x^4$, so

$$x^2y_2'' + xy_2' - 4y_2 = x^2\left(\frac{6}{x^4}\right) - x\left(\frac{2}{x^3}\right) - \frac{4}{x^2},$$

which reduces to zero for x in $(-\infty, 0)$ or $(0, \infty)$.

(b) If

$$y = c_1x^2 + \frac{c_2}{x^2} \tag{3.1.16}$$

then

$$y' = 2c_1x - \frac{2c_2}{x^3} \tag{3.1.17}$$

and

$$y'' = 2c_1 + \frac{6c_2}{x^4},$$

so

$$\begin{aligned} x^2y'' + xy' - 4y &= x^2\left(2c_1 + \frac{6c_2}{x^4}\right) + x\left(2c_1x - \frac{2c_2}{x^3}\right) - 4\left(c_1x^2 + \frac{c_2}{x^2}\right) \\ &= c_1(2x^2 + 2x^2 - 4x^2) + c_2\left(\frac{6}{x^2} - \frac{2}{x^2} - \frac{4}{x^2}\right) \\ &= c_1 \cdot 0 + c_2 \cdot 0 \\ &= 0 \end{aligned}$$

for x in $(-\infty, 0)$ or $(0, \infty)$.

(c) To solve (3.1.14), we choose c_1 and c_2 in (3.1.16) so that $y(1) = 2$ and $y'(1) = 0$. Setting $x = 1$ in (3.1.16) and (3.1.17) shows that this is equivalent to

$$\begin{aligned} c_1 + \ \ c_2 &= 2 \\ 2c_1 - 2c_2 &= 0. \end{aligned}$$

Solving this system of equations yields $c_1 = 1$ and $c_2 = 1$. Therefore, $y = x^2 + 1/x^2$ is the unique solution of (3.1.14) on $(0, \infty)$.

(d) We can solve (3.1.15) by choosing c_1 and c_2 in (3.1.16) so that $y(-1) = 2$ and $y'(-1) = 0$. Setting $x = -1$ in (3.1.16) and (3.1.17) shows that this is equivalent to

$$\begin{aligned} c_1 + \ \ c_2 &= 2 \\ -2c_1 + 2c_2 &= 0. \end{aligned}$$

Solving this system of equations yields $c_1 = 1$ and $c_2 = 1$. Therefore $y = x^2 + 1/x^2$ is the unique solution of (3.1.15) on $(-\infty, 0)$. ■

Although the formulas for the solutions of (3.1.14) and (3.1.15) are both $y = x^2 + 1/x^2$, you should not conclude that these two initial value problems have the same solution. Remember that a solution of an initial value problem is defined *on an interval that contains the initial point*; therefore, the solution of (3.1.14) is $y = x^2 + 1/x^2$ *on the interval* $(0, \infty)$, which contains the initial point $x_0 = 1$, while the solution of (3.1.15) is $y = x^2 + 1/x^2$ *on the interval* $(-\infty, 0)$, which contains the initial point $x_0 = -1$.

Initial value problems impose conditions on a single point x_0. However, many applications involve solving differential equations where conditions have been imposed on two different points x_0 and x_1. A *boundary value problem* is a differential equation together with conditions specified on the dependent variable or its derivatives at two different points. For example,

$$y'' - y = 0, \quad y(0) = 1, \quad y(2) = 3$$

is a boundary value problem. A solution of this problem is a function satisfying the differential equation on some interval that contains both $x = 0$ and $x = 2$; that is, the solution passes through the points $(0, 1)$ and $(2, 3)$. Unfortunately, even when the conditions of Theorem 3.1.1 are satisfied, it is not known whether a boundary value problem will have none, one, or multiple solutions.

Example 3.1.4
The homogeneous linear second order equation $y'' + 16y = 0$ has the two-parameter family of solutions

$$y = c_1 \cos 4x + c_2 \sin 4x.$$

(You should verify this.)

(a) Find the solution of $y'' + 16y = 0$ that satisfies the boundary conditions $y(0) = 0$ and $y(\pi/2) = 0$.

(b) Find the solution of $y'' + 16y = 0$ that satisfies the boundary conditions $y(0) = 0$ and $y(\pi/8) = 0$.

(c) Find the solution of $y'' + 16y = 0$ that satisfies the boundary conditions $y(0) = 0$ and $y(\pi/2) = 1$.

Solution (a) Setting $0 = c_1 \cos 0 + c_2 \sin 0$ implies that $c_1 = 0$ and therefore $y = c_2 \sin 4x$. Before applying the second condition, notice that substituting $x = \pi/2$ into $\sin 4x$ gives $\sin 2\pi = 0$. This means that $0 = c_2 \sin 4(\pi/2)$ is true for any choice of c_2. Therefore, the boundary value problem

$$y'' + 16y = 0, \quad y(0) = 0, \quad y(\pi/2) = 0$$

has infinitely many solutions. (b) As before, the first condition implies that $c_1 = 0$ and therefore $y = c_2 \sin 4x$. This time, notice that substituting $x = \pi/8$ into $\sin 4x$ gives $\sin \pi/2 = 1$. This means that $0 = c_2$ is required to fulfill the second condition. In fact, the boundary value problem

$$y'' + 16y = 0, \quad y(0) = 0, \quad y(\pi/8) = 0$$

has only the one solution $y = 0$. (c) Once more, the first condition implies that $c_1 = 0$ and therefore $y = c_2 \sin 4x$. Here however, the substitution of $x = \pi/2$ into $\sin 4x$ that gives $\sin 2\pi = 0$ leads to the contradiction $1 = 0$. This means that the boundary value problem

$$y'' + 16y = 0, \quad y(0) = 0, \quad y(\pi/8) = 0$$

has no solutions. ■

The General Solution of a Homogeneous Linear Second Order Equation

If y_1 and y_2 are defined on an interval (a, b) and c_1 and c_2 are constants, then

$$y = c_1 y_1 + c_2 y_2$$

is a *linear combination of y_1 and y_2*. For example, $y = 2\cos x + 7\sin x$ is a linear combination of $y_1 = \cos x$ and $y_2 = \sin x$, with $c_1 = 2$ and $c_2 = 7$.

The next theorem states a fact that we illustrated in Examples 3.1.1, 3.1.2, and 3.1.3.

Theorem 3.1.2 *If y_1 and y_2 are solutions of the homogeneous equation*

$$y'' + p(x)y' + q(x)y = 0 \tag{3.1.18}$$

on (a, b), then any linear combination

$$y = c_1y_1 + c_2y_2 \tag{3.1.19}$$

of y_1 and y_2 is also a solution of (3.1.18) on (a, b).

Proof If

$$y = c_1y_1 + c_2y_2$$

then

$$y' = c_1y_1' + c_2y_2' \quad \text{and} \quad y'' = c_1y_1'' + c_2y_2''.$$

Therefore

$$\begin{aligned} y'' + p(x)y' + q(x)y &= (c_1y_1'' + c_2y_2'') + p(x)(c_1y_1' + c_2y_2') + q(x)(c_1y_1 + c_2y_2) \\ &= c_1\left(y_1'' + p(x)y_1' + q(x)y_1\right) + c_2\left(y_2'' + p(x)y_2' + q(x)y_2\right) \\ &= c_1 \cdot 0 + c_2 \cdot 0 \\ &= 0, \end{aligned}$$

since y_1 and y_2 are solutions of (3.1.18). ■

We say that $\{y_1, y_2\}$ is a *fundamental set of solutions of (3.1.18) on (a, b)* if every solution of (3.1.18) on (a, b) can be written as a linear combination of y_1 and y_2 as in (3.1.19). In this case we say that (3.1.19) is the *general solution of (3.1.18) on (a, b)*.

Linear Independence

We need a way to determine whether a given set $\{y_1, y_2\}$ of solutions of (3.1.18) is a fundamental set. The next definition will enable us to state necessary and sufficient conditions for this.

We say that two functions y_1 and y_2 defined on an interval (a, b) are *linearly independent on (a, b)* if neither is a constant multiple of the other on (a, b). (In particular, this means that neither can be the trivial solution of (3.1.18), since, for example, if $y_1 \equiv 0$ we could write $y_1 = 0y_2$.) We will also say that the set $\{y_1, y_2\}$ *is linearly independent on (a, b)*.

Theorem 3.1.3 *Suppose p and q are continuous on (a, b). Then a set $\{y_1, y_2\}$ of solutions of*

$$y'' + p(x)y' + q(x)y = 0 \tag{3.1.20}$$

on (a, b) is a fundamental set if and only if $\{y_1, y_2\}$ is linearly independent on (a, b).

We will present the proof of Theorem 3.1.3 in steps worth regarding as theorems in their own right. However, let us first interpret Theorem 3.1.3 in terms of Examples 3.1.1, 3.1.2, and 3.1.3.

Example 3.1.5

(a) In Example 3.1.1, since $e^x/e^{-x} = e^{2x}$ is nonconstant, Theorem 3.1.3 implies that $y = c_1e^x + c_2e^{-x}$ is the general solution of $y'' - y = 0$ on $(-\infty, \infty)$.

(b) In Example 3.1.2, since $\cos\omega x/\sin\omega x = \cot\omega x$ is nonconstant, Theorem 3.1.3 implies that $y = c_1\cos\omega x + c_2\sin\omega x$ is the general solution of $y'' + \omega^2 y = 0$ on $(-\infty, \infty)$.

(c) In Example **??**, since $x^2/x^{-2} = x^4$ is nonconstant, Theorem 3.1.3 implies that $y = c_1x^2 + c_2/x^2$ is the general solution of $x^2y'' + xy' - 4y = 0$ on $(-\infty, 0)$ and $(0, \infty)$.

The Wronskian and Abel's Formula

To motivate a result that we need in order to prove Theorem 3.1.3, let us see what is required to prove that $\{y_1, y_2\}$ is a fundamental set of solutions of (3.1.20) on (a, b). Let x_0 be an arbitrary point in (a, b), and suppose y is an arbitrary solution of (3.1.20) on (a, b). Then y is the unique solution of the initial value problem

$$y'' + p(x)y' + q(x)y = 0, \quad y(x_0) = k_0, \quad y'(x_0) = k_1; \tag{3.1.21}$$

that is, k_0 and k_1 are the numbers obtained by evaluating y and y' at x_0. Moreover, k_0 and k_1 can be any real numbers, since Theorem 3.1.1 implies that (3.1.21) has a solution no matter how k_0 and k_1 are chosen. Therefore $\{y_1, y_2\}$ is a fundamental set of solutions of (3.1.20) on (a, b) if and only if it is possible to write the solution of an arbitrary initial value problem (3.1.21) as $y = c_1y_1 + c_2y_2$. This is equivalent to requiring that the system

$$\begin{aligned} c_1y_1(x_0) + c_2y_2(x_0) &= k_0 \\ c_1y_1'(x_0) + c_2y_2'(x_0) &= k_1 \end{aligned} \tag{3.1.22}$$

has a solution (c_1, c_2) for every choice of (k_0, k_1). Let us try to solve (3.1.22).

Multiplying the first equation in (3.1.22) by $y_2'(x_0)$ and the second by $y_2(x_0)$ yields

$$\begin{aligned} c_1y_1(x_0)y_2'(x_0) + c_2y_2(x_0)y_2'(x_0) &= y_2'(x_0)k_0 \\ c_1y_1'(x_0)y_2(x_0) + c_2y_2'(x_0)y_2(x_0) &= y_2(x_0)k_1, \end{aligned}$$

and subtracting the second equation here from the first yields

$$\left(y_1(x_0)y_2'(x_0) - y_1'(x_0)y_2(x_0)\right)c_1 = y_2'(x_0)k_0 - y_2(x_0)k_1. \tag{3.1.23}$$

Multiplying the first equation in (3.1.22) by $y_1'(x_0)$ and the second by $y_1(x_0)$ yields

$$\begin{aligned} c_1y_1(x_0)y_1'(x_0) + c_2y_2(x_0)y_1'(x_0) &= y_1'(x_0)k_0 \\ c_1y_1'(x_0)y_1(x_0) + c_2y_2'(x_0)y_1(x_0) &= y_1(x_0)k_1, \end{aligned}$$

and subtracting the first equation here from the second yields

$$\left(y_1(x_0)y_2'(x_0) - y_1'(x_0)y_2(x_0)\right)c_2 = y_1(x_0)k_1 - y_1'(x_0)k_0. \tag{3.1.24}$$

If

$$y_1(x_0)y_2'(x_0) - y_1'(x_0)y_2(x_0) = 0,$$

it is impossible to satisfy (3.1.23) and (3.1.24) (and therefore (3.1.22)) unless k_0 and k_1 happen to satisfy

$$\begin{aligned} y_1(x_0)k_1 - y_1'(x_0)k_0 &= 0 \\ y_2'(x_0)k_0 - y_2(x_0)k_1 &= 0. \end{aligned}$$

On the other hand, if

$$y_1(x_0)y_2'(x_0) - y_1'(x_0)y_2(x_0) \neq 0 \tag{3.1.25}$$

we can divide (3.1.23) and (3.1.24) through by the quantity on the left to obtain

$$\begin{aligned} c_1 &= \frac{y_2'(x_0)k_0 - y_2(x_0)k_1}{y_1(x_0)y_2'(x_0) - y_1'(x_0)y_2(x_0)} \\ c_2 &= \frac{y_1(x_0)k_1 - y_1'(x_0)k_0}{y_1(x_0)y_2'(x_0) - y_1'(x_0)y_2(x_0)}, \end{aligned} \tag{3.1.26}$$

no matter how k_0 and k_1 are chosen. This motivates us to consider conditions on y_1 and y_2 that imply (3.1.25).

Theorem 3.1.4 *Suppose p and q are continuous on (a,b), let y_1 and y_2 be solutions of*

$$y'' + p(x)y' + q(x)y = 0 \tag{3.1.27}$$

on (a,b), and define

$$W = y_1y_2' - y_1'y_2. \tag{3.1.28}$$

Let x_0 be any point in (a,b). Then

$$W(x) = W(x_0)e^{-\int_{x_0}^{x} p(t)\,dt}, \quad a < x < b. \tag{3.1.29}$$

Therefore either W has no zeros in (a,b) or $W \equiv 0$ on (a,b).

Proof Differentiating (3.1.28) yields

$$W' = y_1'y_2' + y_1y_2'' - y_1'y_2' - y_1''y_2 = y_1y_2'' - y_1''y_2. \tag{3.1.30}$$

Since y_1 and y_2 both satisfy (3.1.27),

$$y_1'' = -py_1' - qy_1 \quad \text{and} \quad y_2'' = -py_2' - qy_2.$$

Substituting these into (3.1.30) yields

$$\begin{aligned} W' &= -y_1\left(py_2' + qy_2\right) + y_2\left(py_1' + qy_1\right) \\ &= -p(y_1y_2' - y_2y_1') - q(y_1y_2 - y_2y_1) \\ &= -p(y_1y_2' - y_2y_1') \\ &= -pW. \end{aligned}$$

Therefore $W' + p(x)W = 0$; that is, W is the solution of the initial value problem

$$y' + p(x)y = 0, \quad y(x_0) = W(x_0).$$

We leave it to you to verify by separation of variables that this implies (3.1.29). If $W(x_0) \neq 0$, (3.1.29) implies that W has no zeros in (a, b), since an exponential is never zero. On the other hand, if $W(x_0) = 0$, (3.1.29) implies that $W(x) = 0$ for all x in (a, b). ■

The function W defined in (3.1.28) is the *Wronskian of* $\{y_1, y_2\}$. Formula (3.1.29) is *Abel's formula*.

The Wronskian of $\{y_1, y_2\}$ is usually written as the determinant

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}.$$

The expressions in (3.1.26) for c_1 and c_2 can be written in terms of determinants as

$$c_1 = \frac{1}{W(x_0)} \begin{vmatrix} k_0 & y_2(x_0) \\ k_1 & y_2'(x_0) \end{vmatrix} \quad \text{and} \quad c_2 = \frac{1}{W(x_0)} \begin{vmatrix} y_1(x_0) & k_0 \\ y_1'(x_0) & k_1 \end{vmatrix}.$$

If you have taken linear algebra you may recognize this as *Cramer's rule*.

Example 3.1.6 Verify Abel's formula for the differential equations and the corresponding solutions, from Examples 3.1.1, 3.1.2, and 3.1.3:

(a) $y'' - y = 0; \quad y_1 = e^x, \; y_2 = e^{-x}$

(b) $y'' + \omega^2 y = 0; \quad y_1 = \cos \omega x, \; y_2 = \sin \omega x$

(c) $x^2 y'' + xy' - 4y = 0; \quad y_1 = x^2, \; y_2 = 1/x^2$

Solution (a) Here there is no y' term, so $p \equiv 0$. Therefore we can verify Abel's formula by showing that W is constant. By computing the Wronskian as a determinant, we see that

$$\begin{aligned} W(x) &= \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} \\ &= e^x(-e^{-x}) - e^x e^{-x} \\ &= -2 \end{aligned}$$

for all x.

(b) Again, since $p \equiv 0$, we verify Abel's formula by showing that W is constant. In this case,

$$\begin{aligned} W(x) &= \begin{vmatrix} \cos \omega x & \sin \omega x \\ -\omega \sin \omega x & \omega \cos \omega x \end{vmatrix} \\ &= \cos \omega x(\omega \cos \omega x) - (-\omega \sin \omega x) \sin \omega x \\ &= \omega(\cos^2 \omega x + \sin^2 \omega x) \\ &= \omega \end{aligned}$$

for all x.

(c) Computing the Wronskian of $y_1 = x^2$ and $y_2 = 1/x^2$ directly yields

$$\begin{aligned} W(x) &= \begin{vmatrix} x^2 & 1/x^2 \\ 2x & -2/x^3 \end{vmatrix} \\ &= x^2\left(-\frac{2}{x^3}\right) - 2x\left(\frac{1}{x^2}\right) \\ &=) - \frac{4}{x}. \end{aligned}$$

To verify Abel's formula, we rewrite the differential equation as

$$y'' + \frac{1}{x}y' - \frac{4}{x^2}y = 0$$

to see that $p(x) = 1/x$. If x_0 and x are either both in $(-\infty, 0)$ or both in $(0, \infty)$ then

$$\int_{x_0}^{x} p(t)\,dt = \int_{x_0}^{x} \frac{dt}{t} = \ln\left(\frac{x}{x_0}\right),$$

so the right side of Abel's formula becomes

$$\begin{aligned} W(x) &= W(x_0)e^{-\ln(x/x_0)} \\ &= W(x_0)\frac{x_0}{x} \\ &= -\left(\frac{4}{x_0}\right)\left(\frac{x_0}{x}\right) \\ &= -\frac{4}{x}, \end{aligned}$$

which is consistent with the result we got from computing the Wronskian directly. ■

The next theorem will enable us to complete the proof of Theorem 3.1.3.

Theorem 3.1.5 *Suppose p and q are continuous on an open interval (a, b), let y_1 and y_2 be solutions of*

$$y'' + p(x)y' + q(x)y = 0 \tag{3.1.31}$$

on (a, b), and let $W = y_1y_2' - y_1'y_2$. Then y_1 and y_2 are linearly independent on (a, b) if and only if W has no zeros on (a, b).

Proof We first show that if $W(x_0) = 0$ for some x_0 in (a, b), then y_1 and y_2 are linearly dependent on (a, b). Let I be a subinterval of (a, b) on which y_1 has no zeros. (If there is no such subinterval, $y_1 \equiv 0$ on (a, b), so y_1 and y_2 are linearly independent, and we are finished with this part of the proof.) Then y_2/y_1 is defined on I, and

$$\left(\frac{y_2}{y_1}\right)' = \frac{y_1y_2' - y_1'y_2}{y_1^2} = \frac{W}{y_1^2}. \tag{3.1.32}$$

However, if $W(x_0) = 0$, Theorem 3.1.4 implies that $W \equiv 0$ on (a, b). Therefore (3.1.32) implies that $(y_2/y_1)' \equiv 0$, so $y_2/y_1 = c$ (constant) on I. This shows that $y_2(x) = cy_1(x)$ for all x in I. However, we want to show that $y_2 = cy_1(x)$ for all x in (a, b). Let $Y = y_2 - cy_1$. Then Y is a solution of (3.1.31) on (a, b) such that $Y \equiv 0$ on I, and therefore $Y' \equiv 0$ on I. Consequently, if x_0 is chosen arbitrarily in I then Y is a solution of the initial value problem

$$y'' + p(x)y' + q(x)y = 0, \quad y(x_0) = 0, \quad y'(x_0) = 0,$$

which implies that $Y \equiv 0$ on (a, b), by the paragraph following Theorem 3.1.1. Hence, $y_2 - cy_1 \equiv 0$ on (a, b), which implies that y_1 and y_2 are not linearly independent on (a, b).

Now suppose W has no zeros on (a, b). Then y_1 cannot be identically zero on (a, b) (why not?), and therefore there is a subinterval I of (a, b) on which y_1 has no zeros. Since (3.1.32) implies that y_2/y_1 is nonconstant on I, y_2 is not a constant multiple of y_1 on (a, b). This means that y_1 and y_2 are linearly independent on (a, b). A similar argument shows that y_1 is not a constant multiple of y_2 on (a, b), since

$$\left(\frac{y_1}{y_2}\right)' = \frac{y_1'y_2 - y_1y_2'}{y_2^2} = -\frac{W}{y_2^2}$$

on any subinterval of (a, b) where y_2 has no zeros. ■

We can now complete the proof of Theorem 3.1.3. From Theorem 3.1.5, two solutions y_1 and y_2 of (3.1.31) are linearly independent on (a, b) if and only if W has no zeros on (a, b). From Theorem 3.1.4 and the motivating comments preceding it, $\{y_1, y_2\}$ is a fundamental set of solutions of (3.1.31) if and only if W has no zeros on (a, b). Therefore $\{y_1, y_2\}$ is a fundamental set for (3.1.31) on (a, b) if and only if $\{y_1, y_2\}$ is linearly independent on (a, b). ■

The next theorem summarizes the relationships among the concepts discussed in this section.

Theorem 3.1.6 *Suppose p and q are continuous on an open interval (a, b) and let y_1 and y_2 be solutions of*

$$y'' + p(x)y' + q(x)y = 0 \tag{3.1.33}$$

on (a, b). Then the following statements are equivalent; that is, they are either all true or all false.

(a) *The general solution of (3.1.33) on (a, b) is $y = c_1y_1 + c_2y_2$.*

(b) *$\{y_1, y_2\}$ is a fundamental set of solutions of (3.1.33) on (a, b).*

(c) *$\{y_1, y_2\}$ is linearly independent on (a, b).*

(d) *The Wronskian of $\{y_1, y_2\}$ is nonzero at some point in (a, b).*

(e) *The Wronskian of $\{y_1, y_2\}$ is nonzero at all points in (a, b).*

We can apply this theorem to an equation written as

$$P_2(x)y'' + P_1(x)y' + P_0(x)y = 0$$

on an interval (a, b) where P_2, P_1, and P_0 are continuous and P_2 has no zeros.

3.1 Exercises

1. **(a)** Verify that $y_1 = e^{2x}$ and $y_2 = e^{5x}$ are solutions of

$$y'' - 7y' + 10y = 0 \tag{A}$$

on $(-\infty, \infty)$.

(b) Verify that if c_1 and c_2 are arbitrary constants then $y = c_1e^{2x} + c_2e^{5x}$ is a solution of (A) on $(-\infty, \infty)$.

(c) Solve the initial value problem

$$y'' - 7y' + 10y = 0, \quad y(0) = -1, \quad y'(0) = 1.$$

(d) Solve the initial value problem

$$y'' - 7y' + 10y = 0, \quad y(0) = k_0, \quad y'(0) = k_1.$$

2. **(a)** Verify that $y_1 = e^x \cos x$ and $y_2 = e^x \sin x$ are solutions of

$$y'' - 2y' + 2y = 0 \tag{A}$$

on $(-\infty, \infty)$.

(b) Verify that if c_1 and c_2 are arbitrary constants then $y = c_1e^x \cos x + c_2e^x \sin x$ is a solution of (A) on $(-\infty, \infty)$.

(c) Solve the initial value problem

$$y'' - 2y' + 2y = 0, \quad y(0) = 3, \quad y'(0) = -2.$$

(d) Solve the initial value problem

$$y'' - 2y' + 2y = 0, \quad y(0) = k_0, \quad y'(0) = k_1.$$

3. **(a)** Verify that $y_1 = e^x$ and $y_2 = xe^x$ are solutions of

$$y'' - 2y' + y = 0 \tag{A}$$

on $(-\infty, \infty)$.

(b) Verify that if c_1 and c_2 are arbitrary constants then $y = e^x(c_1 + c_2x)$ is a solution of (A) on $(-\infty, \infty)$.

(c) Solve the initial value problem

$$y'' - 2y' + y = 0, \quad y(0) = 7, \quad y'(0) = 4.$$

(d) Solve the initial value problem

$$y'' - 2y' + y = 0, \quad y(0) = k_0, \quad y'(0) = k_1.$$

4. $y = c_1 \cos 2x + c_2 \sin 2x$ is a two-parameter family of solutions for the second order differential equation $y''+4y = 0$. If possible, find a solution of the differential equation that satisfies the given boundary conditions. **(a)** $y(0) = 0, y(\pi/4) = 3$ **(b)** $y(0) = 0, y(\pi)$
(c) $y'(0) = 0, y'(\pi/6) = 0$ **(d)** $y'(\pi/2) = 1, y'(\pi) = 0$

5. $y = c_1 e^x \cos x + c_2 e^x \sin x$ is a two-parameter family of solutions for the second order differential equation $y''-2y'+2y = 0$ on the interval $(-\infty, \infty)$. If possible, find a solution of the differential equation that satisfies the given boundary conditions.
(a) $y(0) = 1, y'(\pi) = 0$ **(b)** $y(0) = 1, y(\pi) = -1$
(c) $y(0) = 1, y(\pi/2) = 1$ **(d)** $y(0) = 0, y(\pi) = 0$

6. $y = c_1 x^2 + c_2 x^4 + 3$ is a two-parameter family of solutions for the second order differential equation $x^2 y'' - 5xy' + 8y = 24$ on the interval $(-\infty, \infty)$. If possible, find a solution of the differential equation that satisfies the given boundary conditions.
(a) $y(-1) = 0, y(1) = 4$ **(b)** $y(0) = 1, y(1) = 2$
(c) $y(0) = 3, y(1) = 0$ **(d)** $y(1) = 3, y(2) = 15$

7. Compute the Wronskians of the given sets of functions.
(a) $\{1, e^x\}$ **(b)** $\{e^x, e^x \sin x\}$
(c) $\{x + 1, x^2 + 2\}$ **(d)** $\{x^{1/2}, x^{-1/3}\}$
(e) $\{\frac{\sin x}{x}, \frac{\cos x}{x}\}$ **(f)** $\{x \ln |x|, x^2 \ln |x|\}$
(g) $\{e^x \cos \sqrt{x}, e^x \sin \sqrt{x}\}$

8. **(a)** Verify that $y_1 = 1/(x - 1)$ and $y_2 = 1/(x + 1)$ are solutions of

$$(x^2 - 1)y'' + 4xy' + 2y = 0 \tag{A}$$

on $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$. What is the general solution of (A) on each of these intervals?

(b) Solve the initial value problem

$$(x^2 - 1)y'' + 4xy' + 2y = 0, \quad y(0) = -5, \quad y'(0) = 1.$$

What is the domain of the solution?

(c) Graph the solution of the initial value problem.

(d) Verify Abel's formula for y_1 and y_2, with $x_0 = 0$.

9. Use Abel's formula to find the Wronskian of a given set $\{y_1, y_2\}$ of solutions of

$$y'' + 3(x^2 + 1)y' - 2y = 0,$$

given that $W(\pi) = 0$.

10. Use Abel's formula to find the Wronskian of a given set $\{y_1, y_2\}$ of solutions of

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0,$$

given that $W(0) = 1$. (This is *Legendre's equation*.)

11. Use Abel's formula to find the Wronskian of a given set $\{y_1, y_2\}$ of solutions of

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0,$$

given that $W(1) = 1$. (This is *Bessel's equation*.)

12. (This exercise shows that if you know one nontrivial solution of $y'' + p(x)y' + q(x)y = 0$, you can use Abel's formula to find another.)

Suppose p and q are continuous and y_1 is a solution of

$$y'' + p(x)y' + q(x)y = 0 \tag{A}$$

that has no zeros on (a, b). Let $P(x) = \int p(x)\,dx$ be any antiderivative of p on (a, b).

(a) Show that if K is an arbitrary nonzero constant and y_2 satisfies

$$y_1y_2' - y_1'y_2 = Ke^{-P(x)} \tag{B}$$

on (a, b), then y_2 also satisfies (A) on (a, b), and $\{y_1, y_2\}$ is a fundamental set of solutions on (A) on (a, b).

(b) Conclude from **(a)** that if $y_2 = uy_1$ where $u' = K\dfrac{e^{-P(x)}}{y_1^2(x)}$, then $\{y_1, y_2\}$ is a fundamental set of solutions of (A) on (a, b).

In Exercises 13–26 *use the method suggested by Exercise* 12 *to find a second solution* y_2 *that is not a constant multiple of the solution* y_1. *Choose* K *conveniently to simplify* y_2.

13. $y'' - 2y' - 3y = 0; \quad y_1 = e^{3x}$

14. $y'' - 6y' + 9y = 0; \quad y_1 = e^{3x}$

15. $y'' - 2ay' + a^2y = 0$ (a = constant); $\quad y_1 = e^{ax}$

16. $x^2y'' + xy' - y = 0; \quad y_1 = x$

17. $x^2y'' - xy' + y = 0; \quad y_1 = x$

18. $x^2y'' - (2a - 1)xy' + a^2y = 0$ (a = nonzero constant); $x > 0; \quad y_1 = x^a$

19. $4x^2y'' - 4xy' + (3 - 16x^2)y = 0; \quad y_1 = x^{1/2}e^{2x}$

20. $(x-1)y''-xy'+y=0; \quad y_1=e^x$

21. $x^2y''-2xy'+(x^2+2)y=0; \quad y_1=x\cos x$

22. $4x^2(\sin x)y''-4x(x\cos x+\sin x)y'+(2x\cos x+3\sin x)y=0; \quad y_1=x^{1/2}$

23. $(3x-1)y''-(3x+2)y'-(6x-8)y=0; \quad y_1=e^{2x}$

24. $(x^2-4)y''+4xy'+2y=0; \quad y_1=\dfrac{1}{x-2}$

25. $(2x+1)xy''-2(2x^2-1)y'-4(x+1)y=0; \quad y_1=\dfrac{1}{x}$

26. $(x^2-2x)y''+(2-x^2)y'+(2x-2)y=0; \quad y_1=e^x$

3.2 CONSTANT COEFFICIENT HOMOGENEOUS EQUATIONS

If a, b, and c are real constants and $a \neq 0$, then

$$ay''+by'+cy=F(x)$$

is said to be a *constant coefficient equation*. We first consider the homogeneous constant coefficient equation

$$ay''+by'+cy=0. \tag{3.2.1}$$

As we will see, all solutions of (3.2.1) are defined on $(-\infty,\infty)$. This being the case, we will omit references to the interval on which solutions are defined, or on which a given set of solutions is a fundamental set, etc., since the interval will always be $(-\infty,\infty)$.

The key to solving (3.2.1) is that if $y=e^{rx}$ where r is a constant then the left side of (3.2.1) is a multiple of e^{rx}. So if $y=e^{rx}$, then $y'=re^{rx}$ and $y''=r^2e^{rx}$. We can substitute into (3.2.1) to get

$$ar^2e^{rx}+bre^{rx}+ce^{rx}=(ar^2+br+c)e^{rx}. \tag{3.2.2}$$

The quadratic polynomial

$$p(r)=ar^2+br+c$$

is the *characteristic polynomial* of (3.2.1), and $p(r)=0$ is the *characteristic equation*. From (3.2.2) we can see that $y=e^{rx}$ is a solution of (3.2.1) if and only if $p(r)=0$.

The roots of the characteristic equation are given by the quadratic formula

$$r=\frac{-b\pm\sqrt{b^2-4ac}}{2a}. \tag{3.2.3}$$

We consider three cases:

Case 1. $b^2-4ac>0$, so the characteristic equation has two distinct real roots.

Case 2. $b^2-4ac=0$, so the characteristic equation has a repeated real root.

CASE 3. $b^2 - 4ac < 0$, so the characteristic equation has complex roots.

In each case we will start with an example.

Case 1: Distinct Real Roots

Example 3.2.1

(a) Find the general solution of

$$y'' + 6y' + 5y = 0. \tag{3.2.4}$$

(b) Solve the initial value problem

$$y'' + 6y' + 5y = 0, \quad y(0) = 3, \; y'(0) = -1. \tag{3.2.5}$$

Solution (a) The characteristic polynomial $p(r)$ of (3.2.4) is

$$r^2 + 6r + 5 = (r + 1)(r + 5).$$

Since $p(-1) = p(-5) = 0$, $y_1 = e^{-x}$ and $y_2 = e^{-5x}$ are solutions of (3.2.4). Since $y_2/y_1 = e^{-4x}$ is nonconstant, Theorem 3.1.6 implies that the general solution of (3.2.4) is

$$y = c_1 e^{-x} + c_2 e^{-5x}. \tag{3.2.6}$$

(b) We must determine c_1 and c_2 in (3.2.6) so that y satisfies the initial conditions in (3.2.5). Differentiating (3.2.6) yields

$$y' = -c_1 e^{-x} - 5c_2 e^{-5x}. \tag{3.2.7}$$

Imposing the initial conditions $y(0) = 3$ and $y'(0) = -1$ in (3.2.6) and (3.2.7) yields

$$\begin{aligned} c_1 + c_2 &= 3 \\ -c_1 - 5c_2 &= -1. \end{aligned}$$

The solution of this system is $c_1 = 7/2, c_2 = -1/2$. Therefore the solution of (3.2.5) is

$$y = \frac{7}{2}e^{-x} - \frac{1}{2}e^{-5x}.$$

Figure 3.1 is a graph of this solution. ■

To summarize, if the characteristic equation has arbitrary distinct real roots r_1 and r_2, then $y_1 = e^{r_1x}$ and $y_2 = e^{r_2x}$ are solutions of $ay''+by'+cy = 0$. Since $y_2/y_1 = e^{(r_2-r_1)x}$ is nonconstant, Theorem 3.1.6 implies that $\{y_1, y_2\}$ is a fundamental set of solutions of $ay'' + by' + cy = 0$.

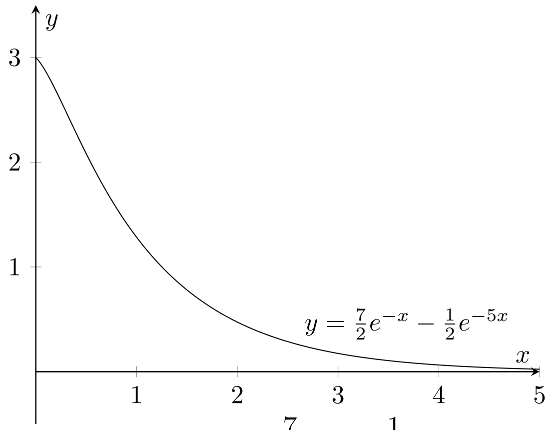


Figure 3.1 $y = \dfrac{7}{2}e^{-x} - \dfrac{1}{2}e^{-5x}$

Case 2: A Repeated Real Root

Example 3.2.2

(a) Find the general solution of

$$y'' + 6y' + 9y = 0. \tag{3.2.8}$$

(b) Solve the initial value problem

$$y'' + 6y' + 9y = 0, \quad y(0) = 3, \ y'(0) = -1. \tag{3.2.9}$$

Solution (a) The characteristic polynomial $p(r)$ of (3.2.8) is

$$r^2 + 6r + 9 = (r+3)^2,$$

so the characteristic equation has the repeated real root $r_1 = -3$. Therefore $y_1 = e^{-3x}$ is a solution of (3.2.8). Since the characteristic equation has no other roots, (3.2.8) has no other solutions of the form e^{rx}. We look for solutions of the form $y = uy_1 = ue^{-3x}$, where u is a function that we will now determine. (This should remind you of the method of variation of parameters that was used to solve the nonhomogeneous equation $y' + p(x)y = f(x)$, given a solution y_1 of the complementary equation $y' + p(x)y = 0$.)

If $y = ue^{-3x}$, then

$$y' = u'e^{-3x} - 3ue^{-3x} \quad \text{and} \quad y'' = u''e^{-3x} - 6u'e^{-3x} + 9ue^{-3x},$$

so

$$\begin{aligned} y'' + 6y' + 9y &= e^{-3x}\left[(u'' - 6u' + 9u) + 6(u' - 3u) + 9u\right] \\ &= e^{-3x}\left[u'' - (6-6)u' + (9 - 18 + 9)u\right] \\ &= u''e^{-3x}. \end{aligned}$$

Therefore $y = ue^{-3x}$ is a solution of (3.2.8) if and only if $u'' = 0$, which is equivalent to $u = c_1 + c_2x$, where c_1 and c_2 are constants. Therefore any function of the form

$$y = e^{-3x}(c_1 + c_2x) \tag{3.2.10}$$

is a solution of (3.2.8). Letting $c_1 = 1$ and $c_2 = 0$ yields the solution $y_1 = e^{-3x}$ that we already knew. Letting $c_1 = 0$ and $c_2 = 1$ yields the second solution $y_2 = xe^{-3x}$. Since $y_2/y_1 = x$ is nonconstant, Theorem 3.1.6 implies that $\{y_1, y_2\}$ is a fundamental set of solutions of (3.2.8), and (3.2.10) is the general solution.

(b) To solve the initial value problem, differentiate (3.2.10) to get

$$y' = -3e^{-3x}(c_1 + c_2x) + c_2e^{-3x}. \tag{3.2.11}$$

Now impose the initial conditions $y(0) = 3$ and $y'(0) = -1$ in (3.2.10) and (3.2.11) to obtain $c_1 = 3$ and $-3c_1 + c_2 = -1$. (So $c_2 = 8$.) Therefore the solution of (3.2.9) is

$$y = e^{-3x}(3 + 8x).$$

Figure 3.2 is a graph of this solution. ■

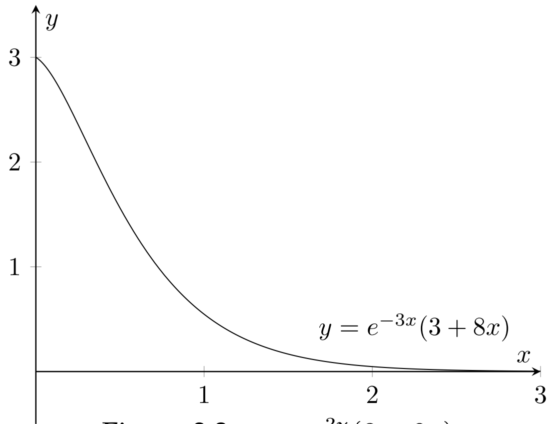


Figure 3.2 $y = e^{-3x}(3 + 8x)$

In general, if the characteristic equation of $ay''+by'+cy=0$ has an arbitrary repeated root r_1, the characteristic polynomial must be

$$p(r)=a(r-r_1)^2=a(r^2-2r_1r+r_1^2).$$

Therefore

$$ar^2+br+c=ar^2-(2ar_1)r+ar_1^2,$$

which implies that $b=-2ar_1$ and $c=ar_1^2$. Therefore $ay''+by'+cy=0$ can be written as $a(y''-2r_1y'+r_1^2y)=0$. Since $a\neq 0$ this equation has the same solutions as

$$y''-2r_1y'+r_1^2y=0. \tag{3.2.12}$$

Since $p(r_1)=0$, $y_1=e^{r_1x}$ is a solution of $ay''+by'+cy=0$, and therefore of (3.2.12). Proceeding as in Example 3.2.2, we look for other solutions of (3.2.12) of the form $y=ue^{r_1x}$; then

$$y'=u'e^{r_1x}+rue^{r_1x} \quad \text{and} \quad y''=u''e^{r_1x}+2r_1u'e^{r_1x}+r_1^2ue^{r_1x},$$

so

$$\begin{aligned} y''-2r_1y'+r_1^2y &= e^{rx}\left[(u''+2r_1u'+r_1^2u)-2r_1(u'+r_1u)+r_1^2u\right]\\ &= e^{r_1x}\left[u''+(2r_1-2r_1)u'+(r_1^2-2r_1^2+r_1^2)u\right]\\ &= u''e^{r_1x}. \end{aligned}$$

Therefore $y=ue^{r_1x}$ is a solution of (3.2.12) if and only if $u''=0$, which is equivalent to $u=c_1+c_2x$, where c_1 and c_2 are constants. Hence, any function of the form

$$y=e^{r_1x}(c_1+c_2x) \tag{3.2.13}$$

is a solution of (3.2.12). Letting $c_1=1$ and $c_2=0$ here yields the solution $y_1=e^{r_1x}$ that we already knew. Letting $c_1=0$ and $c_2=1$ yields the second solution $y_2=xe^{r_1x}$. Since $y_2/y_1=x$ is nonconstant, 3.1.6 implies that $\{y_1,y_2\}$ is a fundamental set of solutions of (3.2.12), and (3.2.13) is the general solution.

Case 3: Complex Conjugate Roots

Example 3.2.3

(a) Find the general solution of

$$y''+4y'+13y=0. \tag{3.2.14}$$

(b) Solve the initial value problem

$$y''+4y'+13y=0, \quad y(0)=2,\ y'(0)=-3. \tag{3.2.15}$$

Solution (a) The characteristic polynomial $p(r)$ of (3.2.14) is

$$r^2 + 4r + 13 = r^2 + 4r + 4 + 9 = (r+2)^2 + 9.$$

By the square root property, the roots of the characteristic equation are $r_1 = -2 + 3i$ and $r_2 = -2 - 3i$. (Alternatively, the quadratic formula may be employed to find the roots of the characteristic equation.) By analogy with Case 1, it is reasonable to expect that $e^{(-2+3i)x}$ and $e^{(-2-3i)x}$ are solutions of (3.2.14). This is true. However there are difficulties here, since you are probably not familiar with exponential functions involving the imaginary unit i. Such functions are inconvenient to work with, so we will take a simpler approach. Notice that

$$e^{(-2+3i)x} = e^{-2x}e^{3ix} \quad \text{and} \quad e^{(-2-3i)x} = e^{-2x}e^{-3ix};$$

even though we have not defined e^{3ix} and e^{-3ix}, it is reasonable to expect that every linear combination of $e^{(-2+3i)x}$ and $e^{(-2-3i)x}$ can be written as $y = ue^{-2x}$, where u depends upon x. To determine u, we note that if $y = ue^{-2x}$ then

$$y' = u'e^{-2x} - 2ue^{-2x} \quad \text{and} \quad y'' = u''e^{-2x} - 4u'e^{-2x} + 4ue^{-2x},$$

so

$$\begin{aligned} y'' + 4y' + 13y &= e^{-2x}\left[(u'' - 4u' + 4u) + 4(u' - 2u) + 13u\right] \\ &= e^{-2x}\left[u'' - (4-4)u' + (4 - 8 + 13)u\right] \\ &= e^{-2x}(u'' + 9u). \end{aligned}$$

Therefore $y = ue^{-2x}$ is a solution of (3.2.14) if and only if

$$u'' + 9u = 0.$$

From Example 3.1.2, the general solution of this equation is

$$u = c_1 \cos 3x + c_2 \sin 3x.$$

Therefore any function of the form

$$y = e^{-2x}(c_1 \cos 3x + c_2 \sin 3x) \tag{3.2.16}$$

is a solution of (3.2.14). Letting $c_1 = 1$ and $c_2 = 0$ yields the solution $y_1 = e^{-2x}\cos 3x$. Letting $c_1 = 0$ and $c_2 = 1$ yields the second solution $y_2 = e^{-2x}\sin 3x$. Since $y_2/y_1 = \tan 3x$ is nonconstant, 3.1.6 implies that $\{y_1, y_2\}$ is a fundamental set of solutions of (3.2.14), and (3.2.16) is the general solution.

(b) Imposing the condition $y(0) = 2$ in (3.2.16) shows that $c_1 = 2$. Differentiating (3.2.16) yields

$$y' = -2e^{-2x}(c_1 \cos 3x + c_2 \sin 3x) + 3e^{-2x}(-c_1 \sin 3x + c_2 \cos 3x),$$

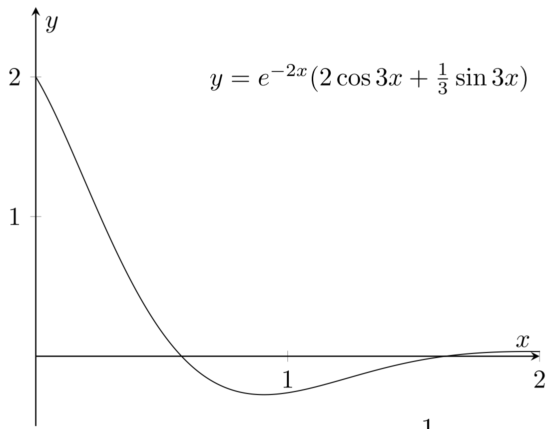


Figure 3.3 $y = e^{-2x}(2\cos 3x + \frac{1}{3}\sin 3x)$

and imposing the initial condition $y'(0) = -3$ here yields $-3 = -2c_1 + 3c_2 = -4 + 3c_2$, so $c_2 = 1/3$. Therefore the solution of (3.2.15) is

$$y = e^{-2x}(2\cos 3x + \frac{1}{3}\sin 3x).$$

Figure 3.3 is a graph of this function. ■

To generalize the preceding example, suppose the characteristic equation of $ay'' + by' + cy = 0$ is such that $b^2 - 4ac < 0$ with roots

$$r_1 = \lambda + i\omega, \quad r_2 = \lambda - i\omega, \tag{3.2.17}$$

where

$$\lambda = -\frac{b}{2a} \quad \text{and} \quad \omega = \frac{\sqrt{4ac - b^2}}{2a}.$$

Do not memorize these formulas. Just remember that r_1 and r_2 are of the form (3.2.17), where λ is an arbitrary real number and ω is positive. Recall that r_1 and r_2 are *complex conjugates*, which means that they have the same real part and their imaginary parts have the same absolute values, but opposite signs. Here, λ and ω are the *real* and *imaginary parts*, respectively, of r_1. Similarly, λ and $-\omega$ are the real and imaginary parts of r_2.

As in Example 3.2.3, it is reasonable to expect that the solutions of $ay'' + by' + cy = 0$ are linear combinations of $e^{(\lambda+i\omega)x}$ and $e^{(\lambda-i\omega)x}$. Again, the exponential notation suggests that

$$e^{(\lambda+i\omega)x} = e^{\lambda x}e^{i\omega x} \quad \text{and} \quad e^{(\lambda-i\omega)x} = e^{\lambda x}e^{-i\omega x},$$

so even though we have not defined $e^{i\omega x}$ and $e^{-i\omega x}$, it is reasonable to expect that every linear combination of $e^{(\lambda+i\omega)x}$ and $e^{(\lambda-i\omega)x}$ can be written as $y = ue^{\lambda x}$, where u depends upon x. To determine u, we first observe that since $r_1 = \lambda + i\omega$ and $r_2 = \lambda - i\omega$

are the roots of the characteristic equation, p must be of the form

$$\begin{aligned} p(r) &= a(r - r_1)(r - r_2) \\ &= a(r - \lambda - i\omega)(r - \lambda + i\omega) \\ &= a\left[(r - \lambda)^2 + \omega^2\right] \\ &= a(r^2 - 2\lambda r + \lambda^2 + \omega^2). \end{aligned}$$

Therefore $ay'' + by' + cy = 0$ can be written as

$$a\left[y'' - 2\lambda y' + (\lambda^2 + \omega^2)y\right] = 0.$$

Since $a \neq 0$ this equation has the same solutions as

$$y'' - 2\lambda y' + (\lambda^2 + \omega^2)y = 0. \tag{3.2.18}$$

To determine u we note that if $y = ue^{\lambda x}$ then

$$y' = u'e^{\lambda x} + \lambda ue^{\lambda x} \quad \text{and} \quad y'' = u''e^{\lambda x} + 2\lambda u'e^{\lambda x} + \lambda^2 ue^{\lambda x}.$$

Substituting these expressions into (3.2.18) and factoring out $e^{\lambda x}$ leaves

$$(u'' + 2\lambda u' + \lambda^2 u) - 2\lambda(u' + \lambda u) + (\lambda^2 + \omega^2)u = 0,$$

which simplifies to

$$u'' + \omega^2 u = 0.$$

From Example 3.1.2, the general solution of this equation is

$$u = c_1 \cos \omega x + c_2 \sin \omega x.$$

Therefore any function of the form

$$y = e^{\lambda x}(c_1 \cos \omega x + c_2 \sin \omega x) \tag{3.2.19}$$

is a solution of (3.2.18). Letting $c_1 = 1$ and $c_2 = 0$ here yields the solution $y_1 = e^{\lambda x} \cos \omega x$. Letting $c_1 = 0$ and $c_2 = 1$ yields a second solution $y_2 = e^{\lambda x} \sin \omega x$. Since $y_2/y_1 = \tan \omega x$ is nonconstant, Theorem 3.1.6 implies that $\{y_1, y_2\}$ is a fundamental set of solutions of (3.2.18), and (3.2.19) is the general solution.

The next theorem compiles the results of the three examples just discussed.

Theorem 3.2.1 *Let $p(r) = ar^2 + br + c$ be the characteristic polynomial of*

$$ay'' + by' + cy = 0. \tag{3.2.20}$$

Then:

(a) *If $p(r) = 0$ has distinct real roots r_1 and r_2, then the general solution of* (3.2.20) *is*

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}.$$

(b) *If* $p(r) = 0$ *has a repeated root* r_1, *then the general solution of* (3.2.20) *is*

$$y = e^{r_1 x}(c_1 + c_2 x).$$

(c) *If* $p(r) = 0$ *has complex conjugate roots* $r_1 = \lambda + i\omega$ *and* $r_2 = \lambda - i\omega$ (*where* $\omega > 0$), *then the general solution of* (3.2.20) *is*

$$y = e^{\lambda x}(c_1 \cos \omega x + c_2 \sin \omega x).$$

Equations of Order Three or Higher

If a_n, $a_{n-1}, \dots, a_0$ are constants, then

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = 0. \tag{3.2.21}$$

can be classified as a *constant coefficient, homogeneous, differential equation of order n*.

Suppose we are able to solve the corresponding characteristic equation

$$a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0 = 0 \tag{3.2.22}$$

and find that there are n distinct, real roots $r_1, r_2, \dots, r_n$. In this case, the general solution of (3.2.21) is

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \cdots + c_n e^{r_n x},$$

as might be expected based on part (a) of Theorem 3.2.1.

Unfortunately, parts (b) and (c) of Theorem 3.2.1 are more difficult to generalize. While it is true that the solutions of (3.2.21) are determined by the zeros of the characteristic polynomial, there are many different combinations of roots that may occur. For example, a polynomial equation of degree three with real coefficients could have three distinct real roots; two distinct real roots, one of multiplicity one and the other of multiplicity two; one real root of multiplicity three; or one real root and one pair of complex conjugate roots. (Recall that complex conjugate roots of a polynomial equation correspond to the presence of an irreducible quadratic factor in the polynomial and therefore always appear in pairs.) Finally, while the quadratic formula can be used to solve any polynomial equation of degree two, it may be difficult or impossible to find roots of a polynomial equation of degree three or higher. The good news is that finding the general solution of a higher order equation uses the same concept of linear combinations just discussed, and the fundamental set of solutions used in the combinations are much as would be expected.

The next theorem is analogous to Theorem 3.1.6. As with the definitions of a fundamental set of solutions and the general solution, the definitions of a linearly independent set and the Wronskian extend to higher dimensions as expected.

Theorem 3.2.2 *Suppose the homogeneous linear* n*-th order equation*

$$P_n(x) y^{(n)} + P_{n-1}(x) y^{(n-1)} + \cdots + P_0(x) y = 0, \tag{3.2.23}$$

is such that all coefficients $P_n, P_{n-1}, \ldots, P_0$ *are continuous on* (a, b) *and* P_n *has no zeros on* (a, b)*. Let* $y_1, y_2, \ldots, y_n$ *be* n *solutions of* (3.2.23) *on* (a, b)*. Then the following statements are equivalent; that is, they are either all true or all false:*

(a) *The general solution of* (3.2.23) *on* (a, b) *is* $y = c_1y_1 + c_2y_2 + \cdots + c_ny_n$.

(b) $\{y_1, y_2, \ldots, y_n\}$ *is a fundamental set of solutions of* (3.2.23) *on* (a, b).

(c) $\{y_1, y_2, \ldots, y_n\}$ *is linearly independent on* (a, b).

(d) *The Wronskian of* $\{y_1, y_2, \ldots, y_n\}$ *is nonzero at some point in* (a, b).

(e) *The Wronskian of* $\{y_1, y_2, \ldots, y_n\}$ *is nonzero at all points in* (a, b).

Since we are currently interested in constant coefficient equations and constants are continuous on $(-\infty, \infty)$, the domain of the solution will be $(a, b) = (-\infty, \infty)$. Therefore, we choose to omit continued reference to the domain of the solution in this section.

Although we omit the proof of Theorem 3.2.2, we will demonstrate the use of the Wronskian to verify the linear independence of the solutions in the following examples.

Example 3.2.4

(a) Find the general solution of

$$y''' - 6y'' + 11y' - 6y = 0. \tag{3.2.24}$$

(b) Solve the initial value problem

$$y''' - 6y'' + 11y' - 6y = 0, \quad y(0) = 4, \quad y'(0) = 5, \quad y''(0) = 9. \tag{3.2.25}$$

(Notice that the number of initial conditions must match the order of the differential equation.)

Solution The characteristic polynomial $p(r)$ of (3.2.24) is

$$r^3 - 6r^2 + 11r - 6 = (r-1)(r-2)(r-3).$$

(By inspection, $r = 1$ is a root; then use polynomial division to find a quadratic equation that is easily factored.) Therefore $\{e^x, e^{2x}, e^{3x}\}$ is a set of solutions of (3.2.24). To verify that this is a fundamental set of solutions, we evaluate the Wronskian $W(x)$ to confirm that it is nonzero.

$$\begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix} = e^{6x} \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix}$$

The value of the 3x3 determinant is 2, and $2e^{6x}$ is never zero, so this is a fundamental set of solutions. Therefore the general solution of (3.2.24) is

$$y = c_1e^x + c_2e^{2x} + c_3e^{3x}. \tag{3.2.26}$$

(b) We must determine c_1, c_2 and c_3 in (3.2.26) so that y satisfies the initial conditions in (3.2.25). Differentiating (3.2.26) twice yields

$$\begin{aligned} y' &= c_1e^x + 2c_2e^{2x} + 3c_3e^{3x} \\ y'' &= c_1e^x + 4c_2e^{2x} + 9c_3e^{3x}. \end{aligned} \tag{3.2.27}$$

Setting $x = 0$ in (3.2.26) and (3.2.27) and imposing the initial conditions yields

$$\begin{aligned} c_1 + c_2 + c_3 &= 4 \\ c_1 + 2c_2 + 3c_3 &= 5 \\ c_1 + 4c_2 + 9c_3 &= 9. \end{aligned}$$

The solution of this system is $c_1 = 4$, $c_2 = -1$, $c_3 = 1$. Therefore the solution of (3.2.25) is

$$y = 4e^x - e^{2x} + e^{3x}$$

■

It is helpful to understand that there is no need to obtain a formula for the Wronskian. Theorem 3.2.2 tells us that the Wronskian either has no zeros on (a, b) or is zero everywhere. This means we can simply evaluate the Wronskian at some convenient point in (a, b). This is demonstrated in the next two examples.

Example 3.2.5 Find the general solution of

$$y^{(4)} - 16y = 0. \tag{3.2.28}$$

Solution The characteristic polynomial of (3.2.28) is $p(r) = r^4 - 16$ which factors as

$$(r^2 - 4)(r^2 + 4) = (r - 2)(r + 2)(r^2 + 4).$$

Based on Theorem 3.2.1, it is reasonable to expect that $\{e^{2x}, e^{-2x}, \cos 2x, \sin 2x\}$ is a fundamental set of solutions of (3.2.28). The Wronskian of this set is

$$W(x) = \begin{vmatrix} e^{2x} & e^{-2x} & \cos 2x & \sin 2x \\ 2e^{2x} & -2e^{-2x} & -2\sin 2x & 2\cos 2x \\ 4e^{2x} & 4e^{-2x} & -4\cos 2x & -4\sin 2x \\ 8e^{2x} & -8e^{-2x} & 8\sin 2x & -8\cos 2x \end{vmatrix}.$$

Rather than finding a formula for the Wronskian, we test the convenient point $x = 0$:

$$W(0) = \begin{vmatrix} 1 & 1 & 1 & 0 \\ 2 & -2 & 0 & 2 \\ 4 & 4 & -4 & 0 \\ 8 & -8 & 0 & -8 \end{vmatrix}$$

Using technology, the value of the $4x4$ determinant is found to be -512. Therefore, $\{e^{2x}, e^{-2x}, \cos 2x, \sin 2x\}$ is linearly independent, and

$$y_1 = c_1e^{2x} + c_2e^{-2x} + c_3\cos 2x + c_4\sin 2x$$

is the general solution of (3.2.28). ■

Example 3.2.6 Find the general solution of

$$y''' - y'' + y' - y = 0. \tag{3.2.29}$$

Solution The characteristic polynomial $p(r)$ of (3.2.29) is

$$r^3 - r^2 + r - 1 = (r-1)(r^2+1).$$

Based on Theorem 3.2.1, it is reasonable to expect that $\{e^x, \cos x, \sin x\}$ is a fundamental set of solutions of (3.2.29). The Wronskian of this set is

$$W(x) = \begin{vmatrix} \cos x & \sin x & e^x \\ -\sin x & \cos x & e^x \\ -\cos x & -\sin x & e^x \end{vmatrix}.$$

For convenience, we evaluate $W(0)$ to get

$$\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{vmatrix} = 2,$$

which verifies that $\{\cos x, \sin x, e^x\}$ is linearly independent and therefore

$$y = c_1 \cos x + c_2 \sin x + c_3 e^x$$

is the general solution of (3.2.29). ■

The concept of repeated roots can be extended to higher order equations as well. For example, if (3.2.22) has a single real root r of multiplicity m, then

$$\{e^{rx}, xe^{rx}, \dots, x^{m-1}e^{rx}\}$$

is a fundamental set of solutions and

$$y = c_1 e^{rx} + c_2 x e^{rx} + \cdots + c_m x^{m-1} e^{rx}$$

is the general solution of (3.2.21).

Example 3.2.7 Find the general solution of

$$y''' + 3y'' + 3y' + y = 0. \tag{3.2.30}$$

Solution The characteristic polynomial $p(r)$ of (3.2.30) is

$$r^3 + 3r^2 + 3r + 1 = (r+1)^3.$$

Therefore the general solution of (3.2.30) is

$$y = e^{-x}(c_1 + c_2 x + c_3 x^2).$$

(We leave it to you to verify linear independence by inspection of the Wronskian.) ■

3.2 Exercises

In Exercises 1–18 find the general solution.

1. $y'' + 5y' - 6y = 0$

2. $y'' - 4y' + 5y = 0$

3. $y'' + 8y' + 7y = 0$

4. $y'' - 4y' + 4y = 0$

5. $y'' + 2y' + 10y = 0$

6. $y'' + 6y' + 10y = 0$

7. $y'' - 8y' + 16y = 0$

8. $y'' + y' = 0$

9. $y'' - 2y' + 3y = 0$

10. $y'' + 6y' + 13y = 0$

11. $4y'' + 4y' + 10y = 0$

12. $10y'' - 3y' - y = 0$

13. $y''' - 3y'' + 3y' - y = 0$

14. $y^{(4)} + 8y'' - 9y = 0$

15. $y''' - y'' + 16y' - 16y = 0$

16. $2y''' + 3y'' - 2y' - 3y = 0$

17. $y^{(4)} - 16y = 0$

18. $y^{(4)} + 12y'' + 36y = 0$

In Exercises 19–28 *solve the initial value problem.*

19. $y'' + 14y' + 50y = 0, \quad y(0) = 2, \quad y'(0) = -17$

20. $6y'' - y' - y = 0, \quad y(0) = 10, \quad y'(0) = 0$

21. $6y'' + y' - y = 0, \quad y(0) = -1, \quad y'(0) = 3$

22. $4y'' - 4y' - 3y = 0, \quad y(0) = \frac{13}{12}, \quad y'(0) = \frac{23}{24}$

23. $4y'' - 12y' + 9y = 0, \quad y(0) = 3, \quad y'(0) = \frac{5}{2}$

24. $y''' - 2y'' + 4y' - 8y = 0, \quad y(0) = 2, \quad y'(0) = -2, \; y''(0) = 0$

25. $y''' + 3y'' - y' - 3y = 0, \quad y(0) = 0, \quad y'(0) = 14, \quad y''(0) = -40$

26. $8y''' - 4y'' - 2y' + y = 0, \quad y(0) = 4, \quad y'(0) = -3, \quad y''(0) = -1$

27. $y^{(4)} - 16y = 0, \quad y(0) = 2, \; y'(0) = 2, \; y''(0) = -2, \; y'''(0) = 0$

28. $4y^{(4)} - 13y'' + 9y = 0, \quad y(0) = 1, \quad y'(0) = 3, \quad y''(0) = 1, \quad y'''(0) = 3$

In Exercises 29–34 *solve the initial value problem and graph the solution.*

29. $y'' + 7y' + 12y = 0, \quad y(0) = -1, \quad y'(0) = 0$

30. $y'' - 6y' + 9y = 0, \quad y(0) = 0, \quad y'(0) = 2$

31. $36y'' - 12y' + y = 0, \quad y(0) = 3, \quad y'(0) = \frac{5}{2}$

32. $y'' + 4y' + 10y = 0, \quad y(0) = 3, \quad y'(0) = -2$

33. $y''' - y'' - y' + y = 0, \quad y(0) = -2, \quad y'(0) = 9, \quad y''(0) = 4$

34. $3y''' - y'' - 7y' + 5y = 0, \quad y(0) = \frac{14}{5}, \quad y'(0) = 0, \quad y''(0) = 10$

In Exercises 35–36 *solve the boundary value problem.*

35. $y'' - 10y' + 25y = 0, \quad y(0) = 1, \quad y(1) = 0$

36. $y'' + y = 0, \quad y'(0) = 0, \quad y'(\pi/2) = 0$

3.3 NONHOMOGENEOUS LINEAR EQUATIONS

We will now consider the nonhomogeneous linear second order equation

$$y'' + p(x)y' + q(x)y = f(x), \tag{3.3.1}$$

where the function f is not identically zero. The next theorem, an extension of Theorem 3.1.1, gives sufficient conditions for existence and uniqueness of solutions of initial value problems for (3.3.1). We omit the proof, which is beyond the scope of this book.

Theorem 3.3.1 *Suppose* p, , q *and* f *are continuous on an open interval* (a, b), *let* x_0 *be any point in* (a, b), *and let* k_0 *and* k_1 *be arbitrary real numbers. Then the initial value problem*

$$y'' + p(x)y' + q(x)y = f(x), \quad y(x_0) = k_0, \quad y'(x_0) = k_1$$

has a unique solution on (a, b).

To find the general solution of (3.3.1) on an interval (a, b) where p, q, and f are continuous, it is necessary to find the general solution of the associated homogeneous equation

$$y'' + p(x)y' + q(x)y = 0 \tag{3.3.2}$$

on (a, b). We call (3.3.2) the *complementary equation* for (3.3.1).

The next theorem shows how to find the general solution of (3.3.1) if we know one solution y_p of (3.3.1) and a fundamental set of solutions of (3.3.2). We call y_p a *particular solution* of (3.3.1). The particular solution may be found by observation, by guessing (then checking), or by some other means. In this section, we will limit ourselves to applications of Theorem 3.3.2 where we can guess at the form of the particular solution.

Theorem 3.3.2 *Suppose* p, q, *and* f *are continuous on* (a,b). *Let* y_p *be a particular solution of*

$$y''+p(x)y'+q(x)y=f(x) \tag{3.3.3}$$

on (a,b), *and let* $\{y_1,y_2\}$ *be a fundamental set of solutions of the complementary equation*

$$y''+p(x)y'+q(x)y=0 \tag{3.3.4}$$

on (a,b). *Then* y *is a solution of* (3.3.3) *on* (a,b) *if and only if*

$$y=y_p+c_1y_1+c_2y_2, \tag{3.3.5}$$

where c_1 *and* c_2 *are constants.*

Proof We first show that y in (3.3.5) is a solution of (3.3.3) for any choice of the constants c_1 and c_2. Differentiating (3.3.5) twice yields

$$y'=y_p'+c_1y_1'+c_2y_2' \quad \text{and} \quad y''=y_p''+c_1y_1''+c_2y_2'',$$

so

$$\begin{aligned}
y''+p(x)y'+q(x)y &= (y_p''+c_1y_1''+c_2y_2'')+p(x)(y_p'+c_1y_1'+c_2y_2')\\
&\quad +q(x)(y_p+c_1y_1+c_2y_2)\\
&= (y_p''+p(x)y_p'+q(x)y_p)+c_1(y_1''+p(x)y_1'+q(x)y_1)\\
&\quad +c_2(y_2''+p(x)y_2'+q(x)y_2)\\
&= f+c_1\cdot 0+c_2\cdot 0\\
&= f,
\end{aligned}$$

since y_p satisfies (3.3.3) and y_1 and y_2 satisfy (3.3.4).

Now we will show that every solution of (3.3.3) has the form (3.3.5) for some choice of the constants c_1 and c_2. Suppose y is a solution of (3.3.3). We will show that $y-y_p$ is a solution of (3.3.4), and therefore of the form $y-y_p=c_1y_1+c_2y_2$, which implies (3.3.5). To see this, we compute

$$\begin{aligned}
(y-y_p)''+p(x)(y-y_p)'+q(x)(y-y_p) &= (y''-y_p'')+p(x)(y'-y_p')\\
&\quad +q(x)(y-y_p)\\
&= (y''+p(x)y'+q(x)y)\\
&\quad -(y_p''+p(x)y_p'+q(x)y_p)\\
&= f(x)-f(x)\\
&= 0,
\end{aligned}$$

since y and y_p both satisfy (3.3.3). ■

We say that (3.3.5) is the *general solution of* (3.3.3) *on* (a,b).

If P_2, P_1, and F are continuous and P_2 has no zeros on (a, b), then Theorem 3.3.2 implies that the general solution of

$$P_2(x)y'' + P_1(x)y' + P_0(x)y = F(x) \tag{3.3.6}$$

on (a, b) is $y = y_p + c_1y_1 + c_2y_2$, where y_p is a particular solution of (3.3.6) on (a, b) and $\{y_1, y_2\}$ is a fundamental set of solutions of

$$P_2(x)y'' + P_1(x)y' + P_0(x)y = 0$$

on (a, b). To see this, we rewrite (3.3.6) as

$$y'' + \frac{P_1(x)}{P_2(x)}y' + \frac{P_0(x)}{P_2(x)}y = \frac{F(x)}{P_2(x)}$$

and apply Theorem 3.3.2 with $p = P_1/P_2$, $q = P_0/P_2$, and $f = F/P_2$.

To avoid awkward wording in examples and exercises, we will not specify the interval (a, b) when we ask for the general solution of a specific linear second order equation, or for a fundamental set of solutions of a homogeneous linear second order equation. Let us agree that this always means that we want the general solution (or a fundamental set of solutions, as the case may be) on every open interval on which p, q, and f are continuous if the equation is of the form (3.3.3), or on which P_2, P_1, P_0, and F are continuous and P_2 has no zeros, if the equation is of the form (3.3.6). We leave it to you to identify these intervals in specific examples and exercises.

For completeness, we point out that if P_2, P_1, P_0, and F are all continuous on an open interval (a, b), but P_2 does have a zero in (a, b), then (3.3.6) may fail to have a general solution on (a, b) in the sense just defined.

Example 3.3.1

(a) Find the general solution of

$$y'' + y = 1. \tag{3.3.7}$$

(b) Solve the initial value problem

$$y'' + y = 1, \quad y(0) = 2, \quad y'(0) = 7. \tag{3.3.8}$$

Solution (a) We can apply Theorem 3.3.2 with $(a, b) = (-\infty, \infty)$, since the functions $p \equiv 0$, $q \equiv 1$, and $f \equiv 1$ in (**??**) are continuous on $(-\infty, \infty)$. By inspection we see that $y_p \equiv 1$ is a particular solution of (**??**). Since $y_1 = \cos x$ and $y_2 = \sin x$ form a fundamental set of solutions of the complementary equation $y'' + y = 0$, the general solution of (**??**) is

$$y = 1 + c_1 \cos x + c_2 \sin x. \tag{3.3.9}$$

(b) Imposing the initial condition $y(0) = 2$ in (3.3.9) yields $2 = 1 + c_1$, so $c_1 = 1$. Differentiating (3.3.9) yields

$$y' = -c_1 \sin x + c_2 \cos x.$$

Imposing the initial condition $y'(0) = 7$ here yields $c_2 = 7$, so the solution of (**??**) is

$$y = 1 + \cos x + 7 \sin x.$$

■

Figure 3.1 is a graph of this function.

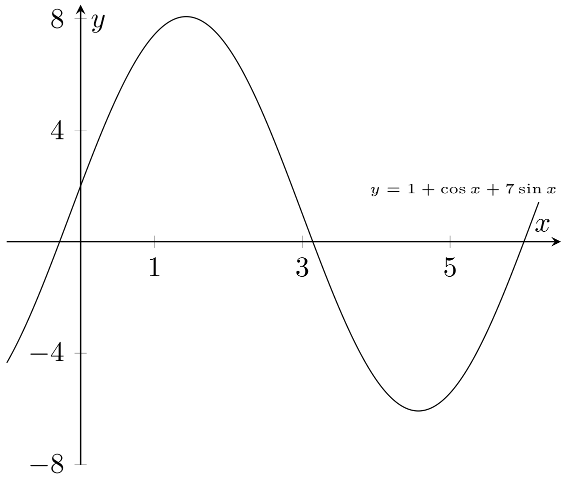


Figure 3.1 $y = 1 + \cos x + 7 \sin x$

Example 3.3.2

(a) Find the general solution of

$$y'' - 2y' + y = -3 - x + x^2. \tag{3.3.10}$$

(b) Solve the initial value problem

$$y'' - 2y' + y = -3 - x + x^2, \quad y(0) = -2, \quad y'(0) = 1. \tag{3.3.11}$$

Solution (a) The characteristic polynomial of the complementary equation

$$y'' - 2y' + y = 0$$

is $r^2 - 2r + 1 = (r - 1)^2$, so $y_1 = e^x$ and $y_2 = xe^x$ form a fundamental set of solutions of the complementary equation. To guess a form for a particular solution of (3.3.10), we note that substituting a second degree polynomial $y_p = A + Bx + Cx^2$ into the left side of (3.3.10) will produce another second degree polynomial with coefficients that depend upon A, B, and C. The trick is to choose A, B, and C so the polynomials on the two sides of (3.3.10) have the same coefficients; thus, if

$$y_p = A + Bx + Cx^2 \quad \text{then} \quad y_p' = B + 2Cx \quad \text{and} \quad y_p'' = 2C,$$

so

$$\begin{aligned} y_p'' - 2y_p' + y_p &= 2C - 2(B + 2Cx) + (A + Bx + Cx^2) \\ &= (2C - 2B + A) + (-4C + B)x + Cx^2 \\ &= -3 - x + x^2. \end{aligned}$$

Equating coefficients of like powers of x on the two sides of the last equality yields

$$\begin{aligned} C &= 1 \\ B - 4C &= -1 \\ A - 2B + 2C &= -3, \end{aligned}$$

so $C = 1$, $B = -1 + 4C$, and $A = -3 - 2C + 2B$. Substitution then shows that $B = 3$ and $A = 1$. Therefore $y_p = 1 + 3x + x^2$ is a particular solution of (3.3.10) and Theorem 3.3.2 implies that

$$y = 1 + 3x + x^2 + e^x(c_1 + c_2x) \tag{3.3.12}$$

is the general solution of (3.3.10).

(b) Imposing the initial condition $y(0) = -2$ in (3.3.12) yields $-2 = 1 + c_1$, so $c_1 = -3$. Differentiating (3.3.12) yields

$$y' = 3 + 2x + e^x(c_1 + c_2x) + c_2e^x,$$

and imposing the initial condition $y'(0) = 1$ yields $1 = 3 + c_1 + c_2$, so $c_2 = 1$. Therefore the solution of (3.3.11) is

$$y = 1 + 3x + x^2 - e^x(3 - x).$$

■

Figure 3.2 is a graph of this solution.

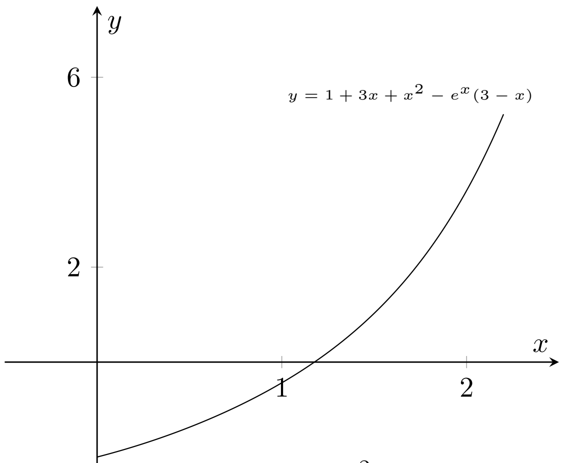


Figure 3.2 $y = 1 + 3x + x^2 - e^x(3 - x)$

Example 3.3.3 Find the general solution of

$$x^2y'' + xy' - 4y = 2x^4 \tag{3.3.13}$$

on $(-\infty, 0)$ and $(0, \infty)$.

Solution In Example 3.1.3, we verified that $y_1 = x^2$ and $y_2 = 1/x^2$ form a fundamental set of solutions of the complementary equation

$$x^2y'' + xy' - 4y = 0$$

on $(-\infty, 0)$ and $(0, \infty)$. To find a particular solution of (3.3.13), we note that if $y_p = Ax^4$, where A is a constant, then both sides of (3.3.13) will be constant multiples of x^4 and we may be able to choose A so the two sides are equal. This is true in this example, since if $y_p = Ax^4$ then substituting derivatives gives

$$x^2y_p'' + xy_p' - 4y_p = x^2(12Ax^2) + x(4Ax^3) - 4Ax^4.$$

Algebraic simplification of (3.3.13) then yields $12Ax^4 = 2x^4$, so we can choose $A = 1/6$ to make the equation true. Therefore, $y_p = x^4/6$ is a particular solution of (3.3.13) on $(-\infty, \infty)$. Theorem 3.3.2 implies that the general solution of (3.3.13) on $(-\infty, 0)$ and $(0, \infty)$ is

$$y = \frac{x^4}{6} + c_1x^2 + \frac{c_2}{x^2}.$$

■

The Principle of Superposition

The next theorem enables us to break a nonhomogeneous equation into simpler parts, find a particular solution for each part, and then combine their solutions to obtain a particular solution of the original problem.

Theorem 3.3.3 *[The Principle of Superposition] Suppose y_{p_1} is a particular solution of*

$$y'' + p(x)y' + q(x)y = f_1(x)$$

on (a, b) and y_{p_2} is a particular solution of

$$y'' + p(x)y' + q(x)y = f_2(x)$$

on (a, b). Then

$$y_p = y_{p_1} + y_{p_2}$$

is a particular solution of

$$y'' + p(x)y' + q(x)y = f_1(x) + f_2(x)$$

on (a, b).

Proof If $y_p = y_{p_1} + y_{p_2}$ then

$$\begin{aligned} y_p'' + p(x)y_p' + q(x)y_p &= (y_{p_1} + y_{p_2})'' + p(x)(y_{p_1} + y_{p_2})' + q(x)(y_{p_1} + y_{p_2}) \\ &= \left(y_{p_1}'' + p(x)y_{p_1}' + q(x)y_{p_1}\right) + \left(y_{p_2}'' + p(x)y_{p_2}' + q(x)y_{p_2}\right) \\ &= f_1(x) + f_2(x). \ \blacksquare \end{aligned}$$

It is easy to generalize Theorem 3.3.3 to the equation

$$y'' + p(x)y' + q(x)y = f(x) \tag{3.3.14}$$

where

$$f = f_1 + f_2 + \cdots + f_k;$$

thus, if y_{p_i} is a particular solution of

$$y'' + p(x)y' + q(x)y = f_i(x)$$

on (a, b) for $i = 1, 2, \ldots, k$, then $y_{p_1} + y_{p_2} + \cdots + y_{p_k}$ is a particular solution of (3.3.14) on (a, b). Moreover, by a proof similar to the proof of Theorem 3.3.3 we can formulate the principle of superposition in terms of a linear equation written in the form

$$P_2(x)y'' + P_1(x)y' + P_0(x)y = F(x);$$

that is, if y_{p_1} is a particular solution of

$$P_2(x)y'' + P_1(x)y' + P_0(x)y = F_1(x)$$

on (a, b) and y_{p_2} is a particular solution of

$$P_2(x)y'' + P_1(x)y' + P_0(x)y = F_2(x)$$

on (a, b), then $y_{p_1} + y_{p_2}$ is a solution of

$$P_2(x)y'' + P_1(x)y' + P_0(x)y = F_1(x) + F_2(x)$$

on (a, b).

Example 3.3.4 The function $y_{p_1} = x^4/15$ is a particular solution of

$$x^2y'' + 4xy' + 2y = 2x^4 \tag{3.3.15}$$

on $(-\infty, \infty)$ and $y_{p_2} = x^2/3$ is a particular solution of

$$x^2y'' + 4xy' + 2y = 4x^2 \tag{3.3.16}$$

on $(-\infty, \infty)$. Use the principle of superposition to find a particular solution of

$$x^2y'' + 4xy' + 2y = 2x^4 + 4x^2 \tag{3.3.17}$$

on $(-\infty, \infty)$.

Solution The right side $F(x) = 2x^4 + 4x^2$ in (3.3.17) is the sum of the right sides

$$F_1(x) = 2x^4 \quad \text{and} \quad F_2(x) = 4x^2.$$

in (3.3.15) and (3.3.16). Therefore the principle of superposition implies that

$$y_p = y_{p_1} + y_{p_2} = \frac{x^4}{15} + \frac{x^2}{3}$$

is a particular solution of (3.3.17). ■

3.3 Exercises

In Exercises 1–6 find a particular solution by the method used in Example 3.3.2. Then find the general solution and, where indicated, solve the initial value problem and graph the solution.

1. $y'' + 5y' - 6y = 22 + 18x - 18x^2$

2. $y'' - 4y' + 5y = 1 + 5x$

3. $y'' + 8y' + 7y = -8 - x + 24x^2 + 7x^3$

4. $y'' - 4y' + 4y = 2 + 8x - 4x^2$

5. $y'' + 2y' + 10y = 4 + 26x + 6x^2 + 10x^3, \quad y(0) = 2, \quad y'(0) = 9$

6. $y'' + 6y' + 10y = 22 + 20x, \quad y(0) = 2, \; y'(0) = -2$

7. Show that the method used in Example 3.3.2 will not yield a particular solution of

$$y'' + y' = 1 + 2x + x^2; \qquad \text{(A)}$$

that is, (A) does not have a particular solution of the form $y_p = A + Bx + Cx^2$, where A, B, and C are constants.

In Exercises 8–13 *find a particular solution by the method used in Example* 3.3.3.

8. $x^2y'' + 7xy' + 8y = \dfrac{6}{x}$

9. $x^2y'' - 7xy' + 7y = 13x^{1/2}$

10. $x^2y'' - xy' + y = 2x^3$

11. $x^2y'' + 5xy' + 4y = \dfrac{1}{x^3}$

12. $x^2y'' + xy' + y = 10x^{1/3}$

13. $x^2y'' - 3xy' + 13y = 2x^4$

14. Show that the method suggested for finding a particular solution in Exercises 8-13 will not yield a particular solution of

$$x^2y'' + 3xy' - 3y = \frac{1}{x^3}; \qquad \text{(A)}$$

that is, (A) does not have a particular solution of the form $y_p = A/x^3$.

15. Prove: If a, b, c, α, and M are constants and $M \neq 0$ then

$$ax^2y'' + bxy' + cy = Mx^\alpha$$

has a particular solution $y_p = Ax^\alpha$ ($A =$ constant) if and only if $a\alpha(\alpha-1)+b\alpha+c \neq 0$.

If a, b, c, and α are constants, then

$$a(e^{\alpha x})'' + b(e^{\alpha x})' + ce^{\alpha x} = (a\alpha^2 + b\alpha + c)e^{\alpha x}.$$

Use this in Exercises 16–21 to find a particular solution . Then find the general solution and, where indicated, solve the initial value problem and graph the solution.

16. $y'' + 5y' - 6y = 6e^{3x}$ **17.** $y'' - 4y' + 5y = e^{2x}$

18. $y'' + 8y' + 7y = 10e^{-2x}, \quad y(0) = -2, \ y'(0) = 10$

19. $y'' - 4y' + 4y = e^x, \quad y(0) = 2, \quad y'(0) = 0$

20. $y'' + 2y' + 10y = e^{x/2}$ **21.** $y'' + 6y' + 10y = e^{-3x}$

22. Show that the method suggested for finding a particular solution in Exercises 16-21 will not yield a particular solution of

$$y'' - 7y' + 12y = 5e^{4x}; \qquad (A)$$

that is, (A) does not have a particular solution of the form $y_p = Ae^{4x}$.

23. Prove: If α and M are constants and $M \neq 0$ then constant coefficient equation

$$ay'' + by' + cy = Me^{\alpha x}$$

has a particular solution $y_p = Ae^{\alpha x}$ ($A =$ constant) if and only if $e^{\alpha x}$ isn't a solution of the complementary equation.

If ω is a constant, differentiating a linear combination of $\cos\omega x$ and $\sin\omega x$ with respect to x yields another linear combination of $\cos\omega x$ and $\sin\omega x$. In Exercises 24–29 use this to find a particular solution of the equation. Then find the general solution and, where indicated, solve the initial value problem and graph the solution.

24. $y'' - 8y' + 16y = 23\cos x - 7\sin x$

25. $y'' + y' = -8\cos 2x + 6\sin 2x$

26. $y'' - 2y' + 3y = -6\cos 3x + 6\sin 3x$

27. $y'' + 6y' + 13y = 18\cos x + 6\sin x$

28. $y'' + 7y' + 12y = -2\cos 2x + 36\sin 2x, \quad y(0) = -3, \quad y'(0) = 3$

29. $y'' - 6y' + 9y = 18\cos 3x + 18\sin 3x, \quad y(0) = 2, \quad y'(0) = 2$

30. Find the general solution of

$$y'' + \omega_0^2 y = M\cos\omega x + N\sin\omega x,$$

where M and N are constants and ω and ω_0 are distinct positive numbers.

31. Show that the method suggested for finding a particular solution in Exercises **??-??** will not yield a particular solution of

$$y'' + y = \cos x + \sin x; \tag{A}$$

that is, (A) does not have a particular solution of the form $y_p = A\cos x + B\sin x$.

32. Prove: If M, N are constants (not both zero) and $\omega > 0$, the constant coefficient equation

$$ay'' + by' + cy = M\cos\omega x + N\sin\omega x \tag{A}$$

has a particular solution that is a linear combination of $\cos\omega x$ and $\sin\omega x$ if and only if the left side of (A) is not of the form $a(y'' + \omega^2 y)$, so that $\cos\omega x$ and $\sin\omega x$ are solutions of the complementary equation.

In Exercises 33–38 *refer to the cited exercises and use the principal of superposition to find a particular solution. Then find the general solution.*

33. $y'' + 5y' - 6y = 22 + 18x - 18x^2 + 6e^{3x}$ (See Exercises 1 and 16.)

34. $y'' - 4y' + 5y = 1 + 5x + e^{2x}$ (See Exercises 2 and 17.)

35. $y'' + 8y' + 7y = -8 - x + 24x^2 + 7x^3 + 10e^{-2x}$ (See Exercises 3 and 18.)

36. $y'' - 4y' + 4y = 2 + 8x - 4x^2 + e^x$ (See Exercises 4 and 19.)

37. $y'' + 2y' + 10y = 4 + 26x + 6x^2 + 10x^3 + e^{x/2}$ (See Exercises 5 and 20.)

38. $y'' + 6y' + 10y = 22 + 20x + e^{-3x}$ (See Exercises 6 and 21.)

3.4 THE METHOD OF UNDETERMINED COEFFICIENTS I

In this section we consider the constant coefficient equation

$$ay'' + by' + cy = e^{\alpha x}G(x), \tag{3.4.1}$$

where α is a constant and G is a polynomial.

From Theorem 3.3.2, the general solution of (3.4.1) is $y = y_p + c_1y_1 + c_2y_2$, where y_p is a particular solution of (3.4.1) and $\{y_1, y_2\}$ is a fundamental set of solutions of the complementary equation

$$ay'' + by' + cy = 0.$$

In Section 3.2 we showed how to find $\{y_1, y_2\}$. In this section we will show how to find y_p. The procedure that we will use is called *the method of undetermined coefficients*.

Our first example is similar to Exercises 16–21 from Section 3.3.

Example 3.4.1 Find a particular solution of

$$y'' - 7y' + 12y = 4e^{2x}. \tag{3.4.2}$$

Then find the general solution.

Solution Substituting $y_p = Ae^{2x}$ for y in (3.4.2) will produce a constant multiple of Ae^{2x} on the left side of (3.4.2), so it may be possible to choose A so that y_p is a solution of (3.4.2). To try this, let $y_p = Ae^{2x}$ so that

$$y_p'' - 7y_p' + 12y_p = 4Ae^{2x} - 14Ae^{2x} + 12Ae^{2x}.$$

Algebraic simplification of (3.4.2) then yields $2Ae^{2x} = 4e^{2x}$, so we can choose $A = 2$ to make the equation true. Therefore $y_p = 2e^{2x}$ is a particular solution of (3.4.2). To find the general solution, we note that the characteristic polynomial $p(r)$ of the complementary equation

$$y'' - 7y' + 12y = 0 \tag{3.4.3}$$

is $r^2 - 7r + 12 = (r-3)(r-4)$, so $\{e^{3x}, e^{4x}\}$ is a fundamental set of solutions of (3.4.3). Therefore the general solution of (3.4.2) is

$$y = 2e^{2x} + c_1e^{3x} + c_2e^{4x}.$$

■

Example 3.4.2 Find a particular solution of

$$y'' - 7y' + 12y = 5e^{4x}. \tag{3.4.4}$$

Then find the general solution.

Solution Fresh from our success in finding a particular solution of (3.4.2) — where we chose $y_p = Ae^{2x}$ because the right side of (3.4.2) is a constant multiple of e^{2x} — it may seem reasonable to try $y_p = Ae^{4x}$ as a particular solution of (3.4.4). However, this will not work, since we saw in Example 3.4.1 that e^{4x} is a solution of the complementary equation (3.4.3), so substituting $y_p = Ae^{4x}$ into the left side of (3.4.4) produces zero on the left, no matter how we choose A. To discover a suitable form for y_p, we use the same approach that we used in Section 3.2 to find a second solution of

$$ay'' + by' + cy = 0$$

in the case where the characteristic equation has a repeated real root: we look for solutions of (3.4.4) in the form $y = ue^{4x}$, where u is a function to be determined. Substituting

$$y = ue^{4x}, \quad y' = u'e^{4x} + 4ue^{4x}, \quad \text{and} \quad y'' = u''e^{4x} + 8u'e^{4x} + 16ue^{4x} \tag{3.4.5}$$

into (3.4.4) and then multiplying by the reciprocal of the common factor e^{4x} yields

$$(u'' + 8u' + 16u) - 7(u' + 4u) + 12u = 5,$$

which reduces to

$$u'' + u' = 5.$$

By inspection we see that $u_p = 5x$ is a particular solution of this equation, so $y_p = 5xe^{4x}$ is a particular solution of (3.4.4). Therefore

$$y = 5xe^{4x} + c_1e^{3x} + c_2e^{4x}$$

is the general solution. ■

Example 3.4.3 Find a particular solution of

$$y'' - 8y' + 16y = 2e^{4x}. \tag{3.4.6}$$

Solution Since the characteristic polynomial $p(r)$ of the complementary equation

$$y'' - 8y' + 16y = 0 \tag{3.4.7}$$

is $r^2 - 8r + 16 = (r - 4)^2$, both $y_1 = e^{4x}$ and $y_2 = xe^{4x}$ are solutions of (3.4.7). Therefore (3.4.6) cannot have a particular solution of the form $y_p = Ae^{4x}$ or $y_p = Axe^{4x}$. As in Example 3.4.2, we look for solutions of (3.4.6) in the form $y = ue^{4x}$, where u is a function to be determined. Substituting from (3.4.5) into (3.4.6) and then multiplying by the reciprocal of the common factor e^{4x} yields

$$(u'' + 8u' + 16u) - 8(u' + 4u) + 16u = 2,$$

which reduces to

$$u'' = 2.$$

Integrating twice and taking the constants of integration to be zero shows that $u_p = x^2$ is a particular solution of this equation, so $y_p = x^2e^{4x}$ is a particular solution of (3.4.4). Therefore

$$y = e^{4x}(x^2 + c_1 + c_2x)$$

is the general solution. ■

The preceding examples illustrate the following facts concerning the form of a particular solution y_p of a constant coefficent equation

$$ay'' + by' + cy = ke^{\alpha x},$$

where k is a nonzero constant:

(a) If $e^{\alpha x}$ is not a solution of the complementary equation

$$ay'' + by' + cy = 0, \tag{3.4.8}$$

then $y_p = Ae^{\alpha x}$, where A is a constant. (See Example 3.4.1).

(b) If $e^{\alpha x}$ is a solution of (3.4.8) but $xe^{\alpha x}$ is not, then $y_p = Axe^{\alpha x}$, where A is a constant. (See Example 3.4.2.)

(c) If both $e^{\alpha x}$ and $xe^{\alpha x}$ are solutions of (3.4.8), then $y_p = Ax^2e^{\alpha x}$, where A is a constant. (See Example 3.4.3.)

In all three cases you can just substitute the appropriate form for y_p and its derivatives directly into

$$ay_p'' + by_p' + cy_p = ke^{\alpha x},$$

and solve for the constant A, as we did in Example 3.4.1. However, if the equation is

$$ay'' + by' + cy = ke^{\alpha x}G(x),$$

where G is a polynomial of degree greater than zero, we recommend that you use the substitution $y = ue^{\alpha x}$ as we did in Examples 3.4.2 and 3.4.3. The equation for u will turn out to be

$$au'' + p'(\alpha)u' + p(\alpha)u = G(x), \tag{3.4.9}$$

where $p(r) = ar^2 + br + c$ is the characteristic polynomial of the complementary equation and $p'(r) = 2ar + b$; however, you should not memorize this since it is easy to derive the equation for u in any particular case. Note, however, that if $e^{\alpha x}$ is a solution of the complementary equation then $p(\alpha) = 0$, so (3.4.9) reduces to

$$au'' + p'(\alpha)u' = G(x),$$

while if both $e^{\alpha x}$ and $xe^{\alpha x}$ are solutions of the complementary equation then $p(r) = a(r-\alpha)^2$ and $p'(r) = 2a(r-\alpha)$, so both $p(\alpha)$ and $p'(\alpha)$ are zero in which case (3.4.9) reduces to

$$au'' = G(x).$$

Example 3.4.4 Find a particular solution of

$$y'' - 3y' + 2y = e^{3x}(-1 + 2x + x^2). \tag{3.4.10}$$

Solution Substituting

$$y = ue^{3x}, \quad y' = u'e^{3x} + 3ue^{3x}, \quad \text{and } y'' = u''e^{3x} + 6u'e^{3x} + 9ue^{3x}$$

into (3.4.10) and then multiplying by the reciprocal of the common factor e^{3x} yields

$$(u'' + 6u' + 9u) - 3(u' + 3u) + 2u = -1 + 2x + x^2,$$

which reduces to

$$u'' + 3u' + 2u = -1 + 2x + x^2. \tag{3.4.11}$$

As in Example 2, in order to guess a form for a particular solution of (3.4.11), we note that substituting a second degree polynomial $u_p = A + Bx + Cx^2$ for u in the left side of (3.4.11) produces another second degree polynomial with coefficients that depend upon A, B, and C; thus,

$$\text{if} \quad u_p = A + Bx + Cx^2 \quad \text{then} \quad u_p' = B + 2Cx \quad \text{and} \quad u_p'' = 2C.$$

If u_p is to satisfy (3.4.11), we must have

$$\begin{aligned} u_p'' + 3u_p' + 2u_p &= 2C + 3(B + 2Cx) + 2(A + Bx + Cx^2) \\ &= (2C + 3B + 2A) + (6C + 2B)x + 2Cx^2 \\ &= -1 + 2x + x^2. \end{aligned}$$

Equating coefficients of like powers of x on the two sides of the last equality yields

$$\begin{aligned} 2C &= 1 \\ 2B + 6C &= 2 \\ 2A + 3B + 2C &= -1. \end{aligned}$$

Solving these equations for C, B, and A (in that order) yields $C = 1/2, B = -1/2$, and $A = -1/4$. Therefore

$$u_p = -\frac{1}{4}(1 + 2x - 2x^2)$$

is a particular solution of (3.4.11), and

$$y_p = u_p e^{3x} = -\frac{e^{3x}}{4}(1 + 2x - 2x^2)$$

is a particular solution of (3.4.10). ■

Example 3.4.5 Find a particular solution of

$$y'' - 4y' + 3y = e^{3x}(6 + 8x + 12x^2). \tag{3.4.12}$$

Solution Substituting

$$y = ue^{3x}, \quad y' = u'e^{3x} + 3ue^{3x}, \quad \text{and } y'' = u''e^{3x} + 6u'e^{3x} + 9ue^{3x}$$

into (3.4.12) and then multiplying by the reciprocal of the common factor e^{3x} yields

$$(u'' + 6u' + 9u) - 4(u' + 3u) + 3u = 6 + 8x + 12x^2,$$

which reduces to

$$u'' + 2u' = 6 + 8x + 12x^2. \tag{3.4.13}$$

There is no u term in this equation, since e^{3x} is a solution of the complementary equation for (3.4.12). Therefore (3.4.13) cannot have a particular solution of the form $u_p = A + Bx + Cx^2$ that we used successfully in Example 3.4.4, since with this choice of u_p,

$$u_p'' + 2u_p' = 2C + (B + 2Cx)$$

cannot contain the last term ($12x^2$) on the right side of (3.4.13). Instead, let us try $u_p = Ax + Bx^2 + Cx^3$ on the grounds that

$$u_p' = A + 2Bx + 3Cx^2 \quad \text{and} \quad u_p'' = 2B + 6Cx$$

together contain all the powers of x that appear on the right side of (3.4.13).

Substituting these expressions in place of u' and u'' in the left side of (3.4.13) yields

$$(2B + 6Cx) + 2(A + 2Bx + 3Cx^2) = (2B + 2A) + (6C + 4B)x + 6Cx^2.$$

Comparing coefficients of like powers of x on the two sides of (3.4.13) shows that u_p is a particular solution if

$$\begin{array}{rcr} 6C &=& 12 \\ 4B + 6C &=& 8 \\ 2A + 2B \qquad &=& 6. \end{array}$$

Solving these equations successively yields $C = 2$, $B = -1$, and $A = 4$. Therefore

$$u_p = x(4 - x + 2x^2)$$

is a particular solution of (3.4.13), and

$$y_p = u_p e^{3x} = xe^{3x}(4 - x + 2x^2)$$

is a particular solution of (3.4.12). ■

Example 3.4.6 Find a particular solution of

$$4y'' + 4y' + y = e^{-x/2}(-8 + 48x + 144x^2). \qquad (3.4.14)$$

Solution Substituting

$$y = ue^{-x/2}, \quad y' = u'e^{-x/2} - \frac{1}{2}ue^{-x/2}, \quad \text{and} \quad y'' = u''e^{-x/2} - u'e^{-x/2} + \frac{1}{4}ue^{-x/2}$$

into (3.4.14) and then multiplying by the reciprocal of the common factor $e^{-x/2}$ yields

$$4\left(u'' - u' + \frac{u}{4}\right) + 4\left(u' - \frac{u}{2}\right) + u = -8 + 48x + 144x^2,$$

which reduces to

$$u'' = -2 + 12x + 36x^2, \qquad (3.4.15)$$

which does not contain u or u' because $e^{-x/2}$ and $xe^{-x/2}$ are both solutions of the complementary equation. To obtain a particular solution of (3.4.15) we integrate twice, taking the constant of integration to be zero each time; thus,

$$u_p' = -2x + 6x^2 + 12x^3 \quad \text{and} \quad u_p = -x^2 + 2x^3 + 3x^4.$$

Therefore, y_p is

$$u_p e^{-x/2} = x^2 e^{-x/2}(-1 + 2x + 3x^2)$$

and is a particular solution of (3.4.14). ■

Summary

The preceding examples illustrate the following facts concerning particular solutions of a constant coefficent equation of the form

$$ay'' + by' + cy = e^{\alpha x}G(x),$$

where G is a polynomial:

(a) If $e^{\alpha x}$ is not a solution of the complementary equation

$$ay'' + by' + cy = 0, \tag{3.4.16}$$

then $y_p = e^{\alpha x}Q(x)$, where Q is a polynomial of the same degree as G. (See Example 3.4.4).

(b) If $e^{\alpha x}$ is a solution of (3.4.16) but $xe^{\alpha x}$ is not, then $y_p = xe^{\alpha x}Q(x)$, where Q is a polynomial of the same degree as G. (See Example 3.4.5.)

(c) If both $e^{\alpha x}$ and $xe^{\alpha x}$ are solutions of (3.4.16), then $y_p = x^2e^{\alpha x}Q(x)$, where Q is a polynomial of the same degree as G. (See Example 3.4.6.)

In all three cases, you can just substitute the appropriate form for y_p and its derivatives directly into

$$ay_p'' + by_p' + cy_p = e^{\alpha x}G(x),$$

and solve for the coefficients of the polynomial Q. However, if you try this you will see that the computations are more tedious than those that you encounter by making the substitution $y = ue^{\alpha x}$ and finding a particular solution of the resulting equation for u. In Case **(a)** the equation for u will be of the form

$$au'' + p'(\alpha)u' + p(\alpha)u = G(x),$$

with a particular solution of the form $u_p = Q(x)$, a polynomial of the same degree as G, whose coefficients can be found by the method used in Example 3.4.4. In Case **(b)** the equation for u will be of the form

$$au'' + p'(\alpha)u' = G(x)$$

(no u term on the left), with a particular solution of the form $u_p = xQ(x)$, where Q is a polynomial of the same degree as G whose coeffcents can be found by the method used in Example 3.4.5. In Case **(c)** the equation for u will be of the form

$$au'' = G(x)$$

with a particular solution of the form $u_p = x^2Q(x)$ that can be obtained by integrating $G(x)/a$ twice and taking the constants of integration to be zero, as in Example 3.4.6.

Using the Principle of Superposition

The next example shows how to combine the method of undetermined coefficients and Theorem 3.3.3, the principle of superposition.

Example 3.4.7 Find a particular solution of

$$y'' - 7y' + 12y = 4e^{2x} + 5e^{4x}. \tag{3.4.17}$$

Solution In Example 3.4.1 we found that $y_{p_1} = 2e^{2x}$ is a particular solution of

$$y'' - 7y' + 12y = 4e^{2x},$$

and in Example 3.4.2 we found that $y_{p_2} = 5xe^{4x}$ is a particular solution of

$$y'' - 7y' + 12y = 5e^{4x}.$$

Therefore the principle of superposition implies that $y_p = 2e^{2x} + 5xe^{4x}$ is a particular solution of (3.4.17). ■

3.4 Exercises

In Exercises 1–14 find a particular solution.

1. $y'' - 3y' + 2y = e^{3x}(1 + x)$ **2.** $y'' - 6y' + 5y = e^{-3x}(35 - 8x)$

3. $y'' - 2y' - 3y = e^{x}(-8 + 3x)$ **4.** $y'' + 2y' + y = e^{2x}(-7 - 15x + 9x^2)$

5. $y'' + 4y = e^{-x}(7 - 4x + 5x^2)$ **6.** $y'' - y' - 2y = e^{x}(9 + 2x - 4x^2)$

7. $y'' - 4y' - 5y = -6xe^{-x}$ **8.** $y'' - 3y' + 2y = e^{x}(3 - 4x)$

9. $y'' + y' - 12y = e^{3x}(-6 + 7x)$ **10.** $2y'' - 3y' - 2y = e^{2x}(-6 + 10x)$

11. $y'' + 2y' + y = e^{-x}(2 + 3x)$ **12.** $y'' - 2y' + y = e^{x}(1 - 6x)$

13. $y'' - 4y' + 4y = e^{2x}(1 - 3x + 6x^2)$

14. $9y'' + 6y' + y = e^{-x/3}(2 - 4x + 4x^2)$

In Exercises 15–19 *find the general solution.*

15. $y'' - 3y' + 2y = e^{3x}(1 + x)$

16. $y'' - 6y' + 8y = e^{x}(11 - 6x)$

17. $y'' + 6y' + 9y = e^{2x}(3 - 5x)$

18. $y'' + 2y' - 3y = -16xe^{x}$

19. $y'' - 2y' + y = e^{x}(2 - 12x)$

In Exercises 20–23 *solve the initial value problem and plot the solution.*

20. $y'' - 4y' - 5y = 9e^{2x}(1 + x), \quad y(0) = 0, \quad y'(0) = -10$

21. $y'' + 3y' - 4y = e^{2x}(7 + 6x), \quad y(0) = 2, \quad y'(0) = 8$

22. $y'' + 4y' + 3y = -e^{-x}(2 + 8x), \quad y(0) = 1, \quad y'(0) = 2$

23. $y'' - 3y' - 10y = 7e^{-2x}, \quad y(0) = 1, \quad y'(0) = -17$

In Exercises 24–29 *use the principle of superposition to find a particular solution.*

24. $y'' + y' + y = xe^{x} + e^{-x}(1 + 2x)$

25. $y'' - 7y' + 12y = -e^{x}(17 - 42x) - e^{3x}$

26. $y'' - 8y' + 16y = 6xe^{4x} + 2 + 16x + 16x^2$

27. $y'' - 3y' + 2y = -e^{2x}(3 + 4x) - e^{x}$

28. $y'' - 2y' + 2y = e^{x}(1 + x) + e^{-x}(2 - 8x + 5x^2)$

29. $y'' + y = e^{-x}(2 - 4x + 2x^2) + e^{3x}(8 - 12x - 10x^2)$

Exercises 30–35 *treat the equations considered in Examples* 3.4.1–3.4.6. *Substitute the suggested form of* y_p *into the equation and equate the resulting coefficients of like functions on the two sides of the resulting equation to derive a set of simultaneous equations for the coefficients in* y_p. *Then solve for the coefficients to obtain* y_p. *Compare the work you've done with the work required to obtain the same results in Examples* 3.4.1–3.4.6.

30. Compare with Example 3.4.1:

$$y'' - 7y' + 12y = 4e^{2x}; \quad y_p = Ae^{2x}$$

31. Compare with Example 3.4.2:

$$y'' - 7y' + 12y = 5e^{4x}; \quad y_p = Axe^{4x}$$

32. Compare with Example 3.4.3.

$$y'' - 8y' + 16y = 2e^{4x}; \quad y_p = Ax^2e^{4x}$$

33. Compare with Example 3.4.4:

$$y'' - 3y' + 2y = e^{3x}(-1 + 2x + x^2), \quad y_p = e^{3x}(A + Bx + Cx^2)$$

34. Compare with Example 3.4.5:

$$y'' - 4y' + 3y = e^{3x}(6 + 8x + 12x^2), \quad y_p = e^{3x}(Ax + Bx^2 + Cx^3)$$

35. Compare with Example 3.4.6:

$$4y'' + 4y' + y = e^{-x/2}(-8 + 48x + 144x^2), \quad y_p = e^{-x/2}(Ax^2 + Bx^3 + Cx^4)$$

36. Write $y = ue^{\alpha x}$ to find the general solution.

(a) $y'' + 2y' + y = \dfrac{e^{-x}}{\sqrt{x}}$ **(b)** $y'' + 6y' + 9y = e^{-3x}\ln x$

(c) $y'' - 4y' + 4y = \dfrac{e^{2x}}{1+x}$ **(d)** $4y''+4y'+y = 4e^{-x/2}\left(\dfrac{1}{x} + x\right)$

3.5 THE METHOD OF UNDETERMINED COEFFICIENTS II

In this section we consider the constant coefficient equation

$$ay'' + by' + cy = e^{\lambda x}\left(P(x)\cos\omega x + Q(x)\sin\omega x\right) \tag{3.5.1}$$

where λ and ω are real numbers, $\omega \neq 0$, and P and Q are polynomials. The function f on the right is called a *forcing function*, since in physical applications it is often related to a force acting on some system modeled by the differential equation. We want to find a particular solution of (3.5.1). As in Section 3.4, the procedure that we will use is called *the method of undetermined coefficients*.

Forcing Functions Without Exponential Factors

We begin with the case where $\lambda = 0$ in (3.5.1); that is, we want to find a particular solution of

$$ay'' + by' + cy = P(x)\cos\omega x + Q(x)\sin\omega x. \tag{3.5.2}$$

Differentiating $x^r\cos\omega x$ and $x^r\sin\omega x$ yields

$$\frac{d}{dx}x^r\cos\omega x = -\omega x^r\sin\omega x + rx^{r-1}\cos\omega x$$

and

$$\frac{d}{dx}x^r\sin\omega x = \omega x^r\cos\omega x + rx^{r-1}\sin\omega x.$$

This implies that if

$$y_p = A(x)\cos\omega x + B(x)\sin\omega x$$

where A and B are polynomials, then

$$ay_p'' + by_p' + cy_p = F(x)\cos\omega x + G(x)\sin\omega x,$$

where F and G are polynomials with coefficients that can be expressed in terms of the coefficients of A and B. This suggests that we try to choose A and B so that $F = P$ and $G = Q$, respectively. Then y_p will be a particular solution of (3.5.2). The next theorem tells us how to choose the proper form for y_p. We omit the proof.

Theorem 3.5.1 *Suppose ω is a positive number and P and Q are polynomials. Let k be the larger of the degrees of P and Q. Then the equation*

$$ay'' + by' + cy = P(x)\cos\omega x + Q(x)\sin\omega x$$

has a particular solution

$$y_p = A(x)\cos\omega x + B(x)\sin\omega x, \tag{3.5.3}$$

where

$$A(x) = A_0 + A_1x + \cdots + A_kx^k \quad \textit{and} \quad B(x) = B_0 + B_1x + \cdots + B_kx^k,$$

provided that $\cos\omega x$ and $\sin\omega x$ are not solutions of the complementary equation. In the case where $\cos\omega x$ and $\sin\omega x$ are solutions of the complementary equation, then there exists a particular solution of the form (3.5.3), *where*

$$A(x) = A_0x + A_1x^2 + \cdots + A_kx^{k+1} \quad \textit{and} \quad B(x) = B_0x + B_1x^2 + \cdots + B_kx^{k+1}.$$

Example 3.5.1 Find a particular solution of

$$y'' - 2y' + y = 5\cos 2x + 10\sin 2x. \tag{3.5.4}$$

Solution In (3.5.4) the coefficients of $\cos 2x$ and $\sin 2x$ are both zero degree polynomials (constants). Therefore Theorem 3.5.1 implies that (3.5.4) has a particular solution

$$y_p = A\cos 2x + B\sin 2x.$$

Since

$$y_p' = -2A\sin 2x + 2B\cos 2x \quad \text{and} \quad y_p'' = -4(A\cos 2x + B\sin 2x),$$

replacing y by y_p in (3.5.4) yields

$$\begin{aligned} y_p'' - 2y_p' + y_p &= -4(A\cos 2x + B\sin 2x) - 4(-A\sin 2x + B\cos 2x) \\ &\quad +(A\cos 2x + B\sin 2x) \\ &= (-3A - 4B)\cos 2x + (4A - 3B)\sin 2x. \end{aligned}$$

Equating the coefficients of $\cos 2x$ and $\sin 2x$ here with the corresponding coefficients on the right side of (3.5.4) shows that y_p is a solution of (3.5.4) if

$$\begin{aligned} -3A - 4B &= 5 \\ 4A - 3B &= 10. \end{aligned}$$

Solving these equations yields $A = 1$, $B = -2$. Therefore

$$y_p = \cos 2x - 2\sin 2x$$

is a particular solution of (3.5.4). ■

Example 3.5.2 Find a particular solution of

$$y'' + 4y = 8\cos 2x + 12\sin 2x. \tag{3.5.5}$$

Solution The procedure used in Example 3.5.1 does not work here. To see why, notice that substituting $y_p = A\cos 2x + B\sin 2x$ for y in (3.5.5) yields

$$y_p'' + 4y_p = -4(A\cos 2x + B\sin 2x) + 4(A\cos 2x + B\sin 2x),$$

which reduces to zero for any choice of A and B. This is due to the fact that both $\cos 2x$ and $\sin 2x$ are solutions of the complementary equation for (3.5.5). For example if $y = \cos 2x$, then

$$y'' + 4y = -4\cos 2x + 4\cos 2x,$$

which reduces to zero. (You should verify that $\sin 2x$ is also a solution.) We should therefore try a particular solution of the form

$$y_p = x(A\cos 2x + B\sin 2x). \tag{3.5.6}$$

Then

$$y_p' = A\cos 2x + B\sin 2x + 2x(-A\sin 2x + B\cos 2x)$$

and

$$\begin{aligned} y_p'' &= -4A\sin 2x + 4B\cos 2x - 4x(A\cos 2x + B\sin 2x) \\ &= -4A\sin 2x + 4B\cos 2x - 4y_p, \end{aligned}$$

so

$$y_p'' + 4y_p = -4A\sin 2x + 4B\cos 2x.$$

Therefore y_p is a solution of (3.5.5) if

$$-4A\sin 2x + 4B\cos 2x = 8\cos 2x + 12\sin 2x,$$

which holds if $A = -3$ and $B = 2$. Therefore

$$y_p = -x(3\cos 2x - 2\sin 2x)$$

is a particular solution of (3.5.5). ■

Example 3.5.3 Find a particular solution of

$$y'' + 3y' + 2y = (16 + 20x)\cos x + 10\sin x. \tag{3.5.7}$$

Solution The coefficients of $\cos x$ and $\sin x$ in (3.5.7) are polynomials of degree one and zero, respectively. Therefore Theorem 3.5.1 tells us to look for a particular solution of (3.5.7) of the form

$$y_p = (A_0 + A_1x)\cos x + (B_0 + B_1x)\sin x. \tag{3.5.8}$$

Then

$$y_p' = (A_1 + B_0 + B_1x)\cos x + (B_1 - A_0 - A_1x)\sin x \tag{3.5.9}$$

and

$$y_p'' = (2B_1 - A_0 - A_1x)\cos x - (2A_1 + B_0 + B_1x)\sin x, \tag{3.5.10}$$

so

$$\begin{aligned} y_p'' + 3y_p' + 2y_p &= [A_0 + 3A_1 + 3B_0 + 2B_1 + (A_1 + 3B_1)x]\cos x \\ &\quad + [B_0 + 3B_1 - 3A_0 - 2A_1 + (B_1 - 3A_1)x]\sin x. \end{aligned} \tag{3.5.11}$$

(You should verify this.) Comparing the coefficients of $x\cos x$, $x\sin x$, $\cos x$, and $\sin x$ here with the corresponding coefficients in (3.5.7) shows that y_p is a solution of (3.5.7) if

$$\begin{aligned} A_1 + 3B_1 &= 20 \\ -3A_1 + B_1 &= 0 \\ A_0 + 3B_0 + 3A_1 + 2B_1 &= 16 \\ -3A_0 + B_0 - 2A_1 + 3B_1 &= 10. \end{aligned}$$

Solving the first two equations yields $A_1 = 2$, $B_1 = 6$. Rearranging terms in the last two equations yields

$$\begin{aligned} A_0 + 3B_0 &= 16 - 3A_1 - 2B_1 \\ -3A_0 + B_0 &= 10 + 2A_1 - 3B_1, \end{aligned}$$

so that substituting the known values for A_1 and B_1 gives

$$\begin{aligned} A_0 + 3B_0 &= -2 \\ -3A_0 + B_0 &= -4. \end{aligned}$$

Solving this system of two equations yields $A_0 = 1$, $B_0 = -1$. Substituting $A_0 = 1$, $A_1 = 2$, $B_0 = -1$, $B_1 = 6$ into (3.5.8) shows that

$$y_p = (1 + 2x)\cos x - (1 - 6x)\sin x$$

is a particular solution of (3.5.7). ■

A Useful Observation

In (3.5.9), (3.5.10), and (3.5.11) the polynomials multiplying $\sin x$ can be obtained by replacing A_0, A_1, B_0, and B_1 by B_0, B_1, $-A_0$, and $-A_1$, respectively, in the polynomials mutiplying $\cos x$. An analogous result applies in general, as follows.

Theorem 3.5.2 *If*

$$y_p = A(x)\cos\omega x + B(x)\sin\omega x,$$

where $A(x)$ *and* $B(x)$ *are polynomials with coefficients* $A_0 \dots, A_k$ *and* $B_0, \dots, B_k$*, then the polynomials multiplying* $\sin\omega x$ *in*

$$y_p', \quad y_p'', \quad \textit{and} \quad ay_p'' + by_p' + cy_p$$

can be obtained by replacing $A_0, \dots, A_k$ *by* $B_0, \dots, B_k$ *and* $B_0, \dots, B_k$ *by* $-A_0, \dots, -A_k$ *in the corresponding polynomials multiplying* $\cos\omega x$*.*

We will not use this theorem in our examples, but we recommend that you use it to check your manipulations when you work the exercises.

Example 3.5.4 Find a particular solution of

$$y'' + y = (8 - 4x)\cos x - (8 + 8x)\sin x. \tag{3.5.12}$$

Solution According to Theorem 3.5.1, we should look for a particular solution of the form

$$y_p = (A_0x + A_1x^2)\cos x + (B_0x + B_1x^2)\sin x, \tag{3.5.13}$$

since $\cos x$ and $\sin x$ are solutions of the complementary equation. However, let us try

$$y_p = (A_0 + A_1x)\cos x + (B_0 + B_1x)\sin x \tag{3.5.14}$$

first, so you can see why it does not work. From (3.5.10),

$$y_p'' = (2B_1 - A_0 - A_1x)\cos x - (2A_1 + B_0 + B_1x)\sin x,$$

which together with (3.5.14) implies that

$$y_p'' + y_p = 2B_1\cos x - 2A_1\sin x.$$

Since the right side of this equation does not contain $x\cos x$ or $x\sin x$, (3.5.14) canot satisfy (3.5.12) no matter how we choose A_0, A_1, B_0, and B_1.

Now let y_p be as in (3.5.13). Then

$$\begin{aligned} y_p' &= \left[A_0 + (2A_1 + B_0)x + B_1x^2\right]\cos x \\ &\quad + \left[B_0 + (2B_1 - A_0)x - A_1x^2\right]\sin x \end{aligned}$$

and

$$\begin{aligned} y_p'' &= \left[2A_1 + 2B_0 - (A_0 - 4B_1)x - A_1x^2\right]\cos x \\ &\quad + \left[2B_1 - 2A_0 - (B_0 + 4A_1)x - B_1x^2\right]\sin x, \end{aligned}$$

so that

$$y_p'' + y_p = (2A_1 + 2B_0 + 4B_1x)\cos x + (2B_1 - 2A_0 - 4A_1x)\sin x.$$

Comparing the coefficients of $\cos x$ and $\sin x$ here with the corresponding coefficients in (3.5.12) shows that y_p is a solution of (3.5.12) if

$$\begin{aligned} 4B_1 &= -4 \\ -4A_1 &= -8 \\ 2B_0 + 2A_1 &= 8 \\ -2A_0 + 2B_1 &= -8. \end{aligned}$$

The solution of this system is $A_1 = 2$, $B_1 = -1$, $A_0 = 3$, $B_0 = 2$. Therefore

$$y_p = x\left[(3 + 2x)\cos x + (2 - x)\sin x\right]$$

is a particular solution of (3.5.12). ■

Forcing Functions with Exponential Factors

To find a particular solution of

$$ay'' + by' + cy = e^{\lambda x}\left(P(x)\cos\omega x + Q(x)\sin\omega x\right) \tag{3.5.15}$$

when $\lambda \neq 0$, we recall from Section 3.4 that substituting $y = ue^{\lambda x}$ into (3.5.15) will produce a constant coefficient equation for u with the forcing function $P(x)\cos\omega x + Q(x)\sin\omega x$. We can find a particular solution u_p of this equation by the procedure that we used in Examples 3.5.1–3.5.4. Then $y_p = u_p e^{\lambda x}$ is a particular solution of (3.5.15).

Example 3.5.5 Find a particular solution of

$$y'' - 3y' + 2y = e^{-2x}\left[2\cos 3x - (34 - 150x)\sin 3x\right]. \tag{3.5.16}$$

Solution Let $y = ue^{-2x}$. Then

$$\begin{aligned} y'' - 3y' + 2y &= e^{-2x}\left[(u'' - 4u' + 4u) - 3(u' - 2u) + 2u\right] \\ &= e^{-2x}(u'' - 7u' + 12u. \end{aligned}$$

Comparing this to (3.5.16) reveals that we need to solve the equation

$$u'' - 7u' + 12u = 2\cos 3x - (34 - 150x)\sin 3x. \tag{3.5.17}$$

Since $\cos 3x$ and $\sin 3x$ are not solutions of the complementary equation

$$u'' - 7u' + 12u = 0,$$

Theorem 3.5.1 tells us to look for a particular solution of (3.5.17) of the form

$$u_p = (A_0 + A_1 x)\cos 3x + (B_0 + B_1 x)\sin 3x. \tag{3.5.18}$$

In this case,

$$\begin{aligned} u_p' &= (A_1 + 3B_0 + 3B_1x)\cos 3x + (B_1 - 3A_0 - 3A_1x)\sin 3x \\ \text{and} \quad u_p'' &= (-9A_0 + 6B_1 - 9A_1x)\cos 3x - (9B_0 + 6A_1 + 9B_1x)\sin 3x, \end{aligned}$$

so that

$$\begin{aligned} u_p'' - 7u_p' + 12u_p &= [3A_0 - 21B_0 - 7A_1 + 6B_1 + (3A_1 - 21B_1)x]\cos 3x \\ &\quad + [21A_0 + 3B_0 - 6A_1 - 7B_1 + (21A_1 + 3B_1)x]\sin 3x. \end{aligned}$$

Comparing the coefficients of $x\cos 3x$, $x\sin 3x$, $\cos 3x$, and $\sin 3x$ here with the corresponding coefficients on the right side of (3.5.17) shows that u_p is a solution of (3.5.17) if

$$\begin{aligned} 3A_1 - 21B_1 &= 0 \\ 21A_1 + 3B_1 &= 150 \\ 3A_0 - 21B_0 - 7A_1 + 6B_1 &= 2 \\ 21A_0 + 3B_0 - 6A_1 - 7B_1 &= -34. \end{aligned} \tag{3.5.19}$$

Solving the first two equations yields $A_1 = 7$, $B_1 = 1$. Substituting these values into the last two equations of (3.5.19) and rearranging terms gives

$$\begin{aligned} 3A_0 - 21B_0 &= 45 \\ 21A_0 + 3B_0 &= 15. \end{aligned}$$

Solving this system yields $A_0 = 1$, $B_0 = -2$. Substituting $A_0 = 1$, $A_1 = 7$, $B_0 = -2$, and $B_1 = 1$ into (**??**) shows that

$$u_p = (1 + 7x)\cos 3x - (2 - x)\sin 3x$$

is a particular solution of (3.5.17). Therefore

$$y_p = e^{-2x}\left[(1 + 7x)\cos 3x - (2 - x)\sin 3x\right]$$

is a particular solution of (3.5.16). ■

Example 3.5.6 Find a particular solution of

$$y'' + 2y' + 5y = e^{-x}\left[(6 - 16x)\cos 2x - (8 + 8x)\sin 2x\right]. \tag{3.5.20}$$

Solution Let $y = ue^{-x}$. Then

$$\begin{aligned} y'' + 2y' + 5y &= e^{-x}\left[(u'' - 2u' + u) + 2(u' - u) + 5u\right] \\ &= e^{-x}(u'' + 4u). \end{aligned}$$

Comparing this to (3.5.20) reveals that we need to solve the equation

$$u'' + 4u = (6 - 16x)\cos 2x - (8 + 8x)\sin 2x. \tag{3.5.21}$$

Notice that $(\cos 2x)'' = -4\cos 2x$ and $(\sin 2x)'' = -4\sin 2x$ so that both satisfy the equation $u'' + 4u = 0$. Since $\cos 2x$ and $\sin 2x$ are solutions of

$$u'' + 4u = 0,$$

they must also be solutions of the complementary equation $y'' + 2y' + 5y = 0$ (because e^{-x} is never zero). Therefore, Theorem 3.5.1 tells us to look for a particular solution of (3.5.21) of the form

$$u_p = (A_0x + A_1x^2)\cos 2x + (B_0x + B_1x^2)\sin 2x.$$

In this case,

$$\begin{aligned} u_p' &= \left[A_0 + (2A_1 + 2B_0)x + 2B_1x^2\right]\cos 2x \\ &\quad + \left[B_0 + (2B_1 - 2A_0)x - 2A_1x^2\right]\sin 2x \end{aligned}$$

and

$$\begin{aligned} u_p'' &= \left[2A_1 + 4B_0 - (4A_0 - 8B_1)x - 4A_1x^2\right]\cos 2x \\ &\quad + \left[2B_1 - 4A_0 - (4B_0 + 8A_1)x - 4B_1x^2\right]\sin 2x, \end{aligned}$$

so that

$$u_p'' + 4u_p = (2A_1 + 4B_0 + 8B_1x)\cos 2x + (2B_1 - 4A_0 - 8A_1x)\sin 2x.$$

Equating the coefficients of $x\cos 2x$, $x\sin 2x$, $\cos 2x$, and $\sin 2x$ here with the corresponding coefficients on the right side of (3.5.21) shows that u_p is a solution of (3.5.21) if

$$\begin{aligned} 8B_1 &= -16 \\ -8A_1 &= -\ 8 \\ 4B_0 + 2A_1 &= \quad 6 \\ -4A_0 + 2B_1 &= \ -8. \end{aligned} \tag{3.5.22}$$

The solution of this system is $A_1 = 1$, $B_1 = -2$, $B_0 = 1$, $A_0 = 1$. Therefore

$$u_p = x[(1 + x)\cos 2x + (1 - 2x)\sin 2x]$$

is a particular solution of (3.5.21), and

$$y_p = xe^{-x}\left[(1 + x)\cos 2x + (1 - 2x)\sin 2x\right]$$

is a particular solution of (3.5.20). ■

You can also find a particular solution of (3.5.20) by substituting

$$y_p = xe^{-x}\left[(A_0 + A_1x)\cos 2x + (B_0 + B_1x)\sin 2x\right]$$

for y in (3.5.20) and equating the coefficients of $xe^{-x}\cos 2x$, $xe^{-x}\sin 2x$, $e^{-x}\cos 2x$, and $e^{-x}\sin 2x$ in the resulting expression for

$$y_p'' + 2y_p' + 5y_p$$

with the corresponding coefficients on the right side of (3.5.20). This leads to the same system (3.5.22) of equations for A_0, A_1, B_0, and B_1 that we obtained in Example 3.5.6. However, if you try this approach you will see that deriving (3.5.22) this way is much more tedious than the way we did it in Example 3.5.6.

3.5 Exercises

In Exercises 1–17 find a particular solution.

1. $y'' + 3y' + 2y = 7\cos x - \sin x$

2. $y'' + 3y' + y = (2 - 6x)\cos x - 9\sin x$

3. $y'' + 2y' + y = e^x(6\cos x + 17\sin x)$

4. $y'' + 3y' - 2y = -e^{2x}(5\cos 2x + 9\sin 2x)$

5. $y'' - y' + y = e^x(2 + x)\sin x$

6. $y'' + 3y' - 2y = e^{-2x}\left[(4 + 20x)\cos 3x + (26 - 32x)\sin 3x\right]$

7. $y'' + 4y = -12\cos 2x - 4\sin 2x$

8. $y'' + y = (-4 + 8x)\cos x + (8 - 4x)\sin x$

9. $4y'' + y = -4\cos x/2 - 8x\sin x/2$

10. $y'' + 2y' + 2y = e^{-x}(8\cos x - 6\sin x)$

11. $y'' - 2y' + 5y = e^x\left[(6 + 8x)\cos 2x + (6 - 8x)\sin 2x\right]$

12. $y'' + 2y' + y = 8x^2\cos x - 4x\sin x$

13. $y'' + 3y' + 2y = (12 + 20x + 10x^2)\cos x + 8x\sin x$

14. $y'' + 3y' + 2y = (1 - x - 4x^2)\cos 2x - (1 + 7x + 2x^2)\sin 2x$

15. $y'' - 5y' + 6y = -e^x\left[(4 + 6x - x^2)\cos x - (2 - 4x + 3x^2)\sin x\right]$

16. $y'' - 2y' + y = -e^x\left[(3 + 4x - x^2)\cos x + (3 - 4x - x^2)\sin x\right]$

17. $y'' - 2y' + 2y = e^x\left[(2 - 2x - 6x^2)\cos x + (2 - 10x + 6x^2)\sin x\right]$

In Exercises 18–21 *find a particular solution and graph it.*

18. $y'' + 2y' + y = e^{-x}\left[(5 - 2x)\cos x - (3 + 3x)\sin x\right]$

19. $y'' + 9y = -6\cos 3x - 12\sin 3x$

20. $y'' + 3y' + 2y = (1 - x - 4x^2)\cos 2x - (1 + 7x + 2x^2)\sin 2x$

21. $y'' + 4y' + 3y = e^{-x}\left[(2 + x + x^2)\cos x + (5 + 4x + 2x^2)\sin x\right]$

In Exercises 22–26 *solve the initial value problem.*

22. $y'' - 7y' + 6y = -e^x(17\cos x - 7\sin x), \quad y(0) = 4,\ y'(0) = 2$

23. $y'' - 2y' + 2y = -e^x(6\cos x + 4\sin x), \quad y(0) = 1,\ y'(0) = 4$

24. $y'' + 6y' + 10y = -40e^x \sin x, \quad y(0) = 2, \quad y'(0) = -3$

25. $y'' - 6y' + 10y = -e^{3x}(6\cos x + 4\sin x), \quad y(0) = 2, \quad y'(0) = 7$

26. $y'' - 3y' + 2y = e^{3x}\left[21\cos x - (11 + 10x)\sin x\right], \; y(0) = 0, \quad y'(0) = 6$

In Exercises 27–32 *use the principle of superposition to find a particular solution. Where indicated, solve the initial value problem.*

27. $y'' - 2y' - 3y = 4e^{3x} + e^x(\cos x - 2\sin x)$

28. $y'' + y = 4\cos x - 2\sin x + xe^x + e^{-x}$

29. $y'' - 3y' + 2y = xe^x + 2e^{2x} + \sin x$

30. $y'' - 2y' + 2y = 4xe^x \cos x + xe^{-x} + 1 + x^2$

31. $y'' - 4y' + 4y = e^{2x}(1 + x) + e^{2x}(\cos x - \sin x) + 3e^{3x} + 1 + x$

32. $y'' - 4y' + 4y = 6e^{2x} + 25\sin x, \quad y(0) = 5, \; y'(0) = 3$

In Exercises 33–35 *solve the initial value problem and graph the solution.*

33. $y'' + 4y = -e^{-2x}\left[(4 - 7x)\cos x + (2 - 4x)\sin x\right], \; y(0) = 3, \quad y'(0) = 1$

34. $y'' + 4y' + 4y = 2\cos 2x + 3\sin 2x + e^{-x}, \quad y(0) = -1, \; y'(0) = 2$

35. $y'' + 4y = e^x(11 + 15x) + 8\cos 2x - 12\sin 2x, \quad y(0) = 3, \; y'(0) = 5$

3.6 REDUCTION OF ORDER

In this section we give a method for finding the general solution of

$$P_2(x)y'' + P_1(x)y' + P_0(x)y = F(x) \tag{3.6.1}$$

if we know a nontrivial solution y_1 of the complementary equation

$$P_2(x)y'' + P_1(x)y' + P_0(x)y = 0. \tag{3.6.2}$$

The method is called *reduction of order* because it reduces the task of solving (3.6.1) to solving a first order equation. Unlike the method of undetermined coefficients, it does not require P_2, P_1, and P_0 to be constants, or F to be of any special form.

By now you should not be surprised that we look for solutions of (3.6.1) in the form

$$y = uy_1 \tag{3.6.3}$$

where u is to be determined so that y satisfies (3.6.1). Substituting (3.6.3) and

$$\begin{aligned} y' &= u'y_1 + uy_1' \\ y'' &= u''y_1 + 2u'y_1' + uy_1'' \end{aligned}$$

into (3.6.1) yields

$$P_2(x)(u''y_1 + 2u'y_1' + uy_1'') + P_1(x)(u'y_1 + uy_1') + P_0(x)uy_1 = F(x).$$

Collecting the coefficients of u, u', and u'' yields

$$(P_2y_1)u'' + (2P_2y_1' + P_1y_1)u' + (P_2y_1'' + P_1y_1' + P_2y_1)u = F. \tag{3.6.4}$$

However, the coefficient of u is zero, since y_1 satisfies (3.6.2). Therefore (3.6.4) reduces to

$$Q_2(x)u'' + Q_1(x)u' = F, \tag{3.6.5}$$

with

$$Q_2 = P_2y_1 \quad \text{and} \quad Q_1 = 2P_2y_1' + P_1y_1.$$

(It is not worthwhile to memorize the formulas for Q_2 and Q_1!) Since (3.6.5) is a linear first order equation in u', we can solve it for u' by variation of parameters as we did in the introductory chapter, integrate the solution to obtain u, and then obtain y from (3.6.3).

Example 3.6.1

(a) Find the general solution of

$$xy'' - (2x+1)y' + (x+1)y = x^2, \tag{3.6.6}$$

given that $y_1 = e^x$ is a solution of the complementary equation

$$xy'' - (2x+1)y' + (x+1)y = 0. \tag{3.6.7}$$

(b) Using the results from part **(a)**, find a fundamental set of solutions of (3.6.7).

Solution (a) If $y = ue^x$, then $y' = u'e^x + ue^x$ and $y'' = u''e^x + 2u'e^x + ue^x$, so

$$\begin{aligned} xy'' - (2x+1)y' + (x+1)y &= x(u''e^x + 2u'e^x + ue^x) \\ &\quad -(2x+1)(u'e^x + ue^x) + (x+1)ue^x \\ &= (xu'' - u')e^x. \end{aligned}$$

Therefore $y = ue^x$ is a solution of (3.6.6) if and only if

$$(xu'' - u')e^x = x^2,$$

which is a first order equation in u'. We rewrite it as

$$u'' - \frac{u'}{x} = xe^{-x}. \tag{3.6.8}$$

To focus on how we apply variation of parameters to this equation, we temporarily write $z = u'$, so that (3.6.8) becomes

$$z' - \frac{z}{x} = xe^{-x}. \tag{3.6.9}$$

We leave it to you to show (by separation of variables) that $z_1 = x$ is a solution of the complementary equation

$$z' - \frac{z}{x} = 0$$

for (3.6.9). By applying variation of parameters, we can now see that every solution of (3.6.9) is of the form

$$z = vx \quad \text{where} \quad v'x = xe^{-x}, \quad \text{so} \quad v' = e^{-x} \quad \text{and} \quad v = -e^{-x} + C_1.$$

Since $u' = z = vx$, u is a solution of (3.6.8) if and only if

$$u' = vx = -xe^{-x} + C_1x.$$

Integrating this yields

$$u = (x+1)e^{-x} + \frac{C_1}{2}x^2 + C_2.$$

Therefore the general solution of (3.6.6) is

$$y = ue^x = x + 1 + \frac{C_1}{2}x^2e^x + C_2e^x. \tag{3.6.10}$$

(b) By letting $C_1 = C_2 = 0$ in (3.6.10), we see that $y_{p_1} = x+1$ is a solution of (3.6.6). By letting $C_1 = 2$ and $C_2 = 0$, we see that $y_{p_2} = x+1+x^2e^x$ is also a solution of (3.6.6). Since the difference of two solutions of (3.6.6) is a solution of (3.6.7), $y_2 = y_{p_1} - y_{p_2} = x^2e^x$ is a solution of (3.6.7). Since y_2/y_1 is nonconstant and we already know that $y_1 = e^x$ is a solution of (3.6.6), Theorem 3.1.6 implies that $\{e^x, x^2e^x\}$ is a fundamental set of solutions of (3.6.7). ■

Although (3.6.10) is a correct form for the general solution of (3.6.6), it is silly to leave the arbitrary coefficient of x^2e^x as $C_1/2$ where C_1 is an arbitrary constant. Moreover, it is sensible to make the subscripts of the coefficients of $y_1 = e^x$ and $y_2 = x^2e^x$ consistent with the subscripts of the functions themselves. Therefore we rewrite (3.6.10) as

$$y = x + 1 + c_1e^x + c_2x^2e^x$$

by simply renaming the arbitrary constants. We will also do this in the next two examples, and in the answers to the exercises.

Example 3.6.2

(a) Find the general solution of

$$x^2y'' + xy' - y = x^2 + 1,$$

given that $y_1 = x$ is a solution of the complementary equation

$$x^2y'' + xy' - y = 0. \tag{3.6.11}$$

Using this result, find a fundamental set of solutions of (3.6.11).

(b) Solve the initial value problem

$$x^2y'' + xy' - y = x^2 + 1, \quad y(1) = 2,\ y'(1) = -3. \tag{3.6.12}$$

Solution (a) If $y = ux$, then $y' = u'x + u$ and $y'' = u''x + 2u'$, so

$$\begin{aligned} x^2y'' + xy' - y &= x^2(u''x + 2u') + x(u'x + u) - ux \\ &= x^3u'' + 3x^2u'. \end{aligned}$$

Therefore $y = ux$ is a solution of (3.6.12) if and only if

$$x^3u'' + 3x^2u' = x^2 + 1,$$

which is a first order equation in u'. We rewrite it as

$$u'' + \frac{3}{x}u' = \frac{1}{x} + \frac{1}{x^3}. \tag{3.6.13}$$

To focus on how we apply variation of parameters to this equation, we temporarily write $z = u'$, so that (3.6.13) becomes

$$z' + \frac{3}{x}z = \frac{1}{x} + \frac{1}{x^3}. \tag{3.6.14}$$

We leave it to you to show by separation of variables that $z_1 = 1/x^3$ is a solution of the complementary equation

$$z' + \frac{3}{x}z = 0$$

for (3.6.14). By variation of parameters, every solution of (3.6.14) is of the form

$$z = \frac{v}{x^3} \quad \text{where} \quad \frac{v'}{x^3} = \frac{1}{x} + \frac{1}{x^3}, \quad \text{so} \quad v' = x^2 + 1 \quad \text{and} \quad v = \frac{x^3}{3} + x + C_1.$$

Since $u' = z = v/x^3$, u is a solution of (3.6.14) if and only if

$$u' = \frac{v}{x^3} = \frac{1}{3} + \frac{1}{x^2} + \frac{C_1}{x^3}.$$

Integrating this yields

$$u = \frac{x}{3} - \frac{1}{x} - \frac{C_1}{2x^2} + C_2.$$

Therefore the general solution of (3.6.12) is

$$y = ux = \frac{x^2}{3} - 1 - \frac{C_1}{2x} + C_2 x. \tag{3.6.15}$$

Reasoning as in the solution of Example 3.6.1**(a)**, we conclude that $y_1 = x$ and $y_2 = 1/x$ form a fundamental set of solutions for (3.6.11).

As before, we rename the constants in (3.6.15) and rewrite it as

$$y = \frac{x^2}{3} - 1 + c_1 x + \frac{c_2}{x}. \tag{3.6.16}$$

(b) Differentiating (3.6.16) yields

$$y' = \frac{2x}{3} + c_1 - \frac{c_2}{x^2}. \tag{3.6.17}$$

Setting $x = 1$ in (3.6.16) and (3.6.17) and imposing the initial conditions $y(1) = 2$ and $y'(1) = -3$ yields

$$\begin{aligned} c_1 + c_2 &= \frac{8}{3} \\ c_1 - c_2 &= -\frac{11}{3}. \end{aligned}$$

Solving these equations yields $c_1 = -1/2$, $c_2 = 19/6$. Therefore the solution of (3.6.12) is

$$y = \frac{x^2}{3} - 1 - \frac{x}{2} + \frac{19}{6x}.$$

■

As expected, using reduction of order to find the general solution of a homogeneous linear second order equation leads to a homogeneous linear first order equation in u' that can be solved by separation of variables. The next example illustrates this.

Example 3.6.3 Find the general solution and a fundamental set of solutions of

$$x^2 y'' - 3xy' + 3y = 0, \tag{3.6.18}$$

given that $y_1 = x$ is a solution.

Solution If $y = ux$ then $y' = u'x + u$ and $y'' = u''x + 2u'$, so

$$\begin{aligned} x^2 y'' - 3xy' + 3y &= x^2(u''x + 2u') - 3x(u'x + u) + 3ux \\ &= x^3 u'' - x^2 u'. \end{aligned}$$

Therefore $y = ux$ is a solution of (3.6.18) if and only if

$$x^3 u'' - x^2 u' = 0.$$

Separating the variables u' and x yields

$$\frac{u''}{u'} = \frac{1}{x},$$

so

$$\ln|u'| = \ln|x| + k, \quad \text{or, equivalently,} \quad u' = C_1 x.$$

Therefore

$$u = \frac{C_1}{2}x^2 + C_2,$$

so the general solution of (3.6.18) is

$$y = ux = \frac{C_1}{2}x^3 + C_2 x,$$

which we rewrite as

$$y = c_1 x + c_2 x^3.$$

Therefore $\{x, x^3\}$ is a fundamental set of solutions of (3.6.18). ■

3.6 Exercises

In Exercises 1–17 find the general solution, given that y_1 satisfies the complementary equation. Then use the result to find a fundamental set of solutions of the complementary equation.

1. $(2x+1)y'' - 2y' - (2x+3)y = (2x+1)^2; \quad y_1 = e^{-x}$

2. $x^2y'' + xy' - y = \dfrac{4}{x^2}; \quad y_1 = x$

3. $x^2y'' - xy' + y = x; \quad y_1 = x$

4. $y'' - 3y' + 2y = \dfrac{1}{1+e^{-x}}; \quad y_1 = e^{2x}$

5. $y'' - 2y' + y = 7x^{3/2}e^x; \quad y_1 = e^x$

6. $4x^2y'' + (4x - 8x^2)y' + (4x^2 - 4x - 1)y = 4x^{1/2}e^x(1+4x); \quad y_1 = x^{1/2}e^x$

7. $y'' - 2y' + 2y = e^x \sec x; \quad y_1 = e^x \cos x$

8. $y'' + 4xy' + (4x^2+2)y = 8e^{-x(x+2)}; \quad y_1 = e^{-x^2}$

9. $x^2y'' + xy' - 4y = -6x - 4; \quad y_1 = x^2$

10. $x^2y'' + 2x(x-1)y' + (x^2 - 2x + 2)y = x^3e^{2x}; \quad y_1 = xe^{-x}$

11. $x^2y'' - x(2x-1)y' + (x^2 - x - 1)y = x^2e^x; \quad y_1 = xe^x$

12. $(1-2x)y'' + 2y' + (2x-3)y = (1 - 4x + 4x^2)e^x; \quad y_1 = e^x$

13. $x^2y'' - 3xy' + 4y = 4x^4; \quad y_1 = x^2$

14. $2xy'' + (4x+1)y' + (2x+1)y = 3x^{1/2}e^{-x}; \quad y_1 = e^{-x}$

15. $xy'' - (2x+1)y' + (x+1)y = -e^x; \quad y_1 = e^x$

16. $4x^2y'' - 4x(x+1)y' + (2x+3)y = 4x^{5/2}e^{2x}; \quad y_1 = x^{1/2}$

17. $x^2y'' - 5xy' + 8y = 4x^2; \quad y_1 = x^2$

In Exercises 18–30 *find a fundamental set of solutions, given that* y_1 *is a solution.*

18. $xy'' + (2-2x)y' + (x-2)y = 0; \quad y_1 = e^x$

19. $x^2y'' - 4xy' + 6y = 0; \quad y_1 = x^2$

20. $x^2(\ln|x|)^2y'' - (2x\ln|x|)y' + (2+\ln|x|)y = 0; \quad y_1 = \ln|x|$

21. $4xy'' + 2y' + y = 0; \quad y_1 = \sin\sqrt{x}$

22. $xy'' - (2x+2)y' + (x+2)y = 0; \quad y_1 = e^x$

23. $x^2y'' - (2a-1)xy' + a^2y = 0; \quad y_1 = x^a$

24. $x^2y'' - 2xy' + (x^2+2)y = 0; \quad y_1 = x\sin x$

25. $xy'' - (4x+1)y' + (4x+2)y = 0; \quad y_1 = e^{2x}$

26. $4x^2(\sin x)y'' - 4x(x\cos x + \sin x)y' + (2x\cos x + 3\sin x)y = 0; \quad y_1 = x^{1/2}$

27. $4x^2y'' - 4xy' + (3-16x^2)y = 0; \quad y_1 = x^{1/2}e^{2x}$

28. $(2x+1)xy'' - 2(2x^2-1)y' - 4(x+1)y = 0; \quad y_1 = 1/x$

29. $(x^2-2x)y'' + (2-x^2)y' + (2x-2)y = 0; \quad y_1 = e^x$

30. $xy'' - (4x+1)y' + (4x+2)y = 0; \quad y_1 = e^{2x}$

In Exercises 31–33 *solve the initial value problem, given that* y_1 *satisfies the complementary equation.*

31. $x^2y'' - 3xy' + 4y = 4x^4, \quad y(-1) = 7, \quad y'(-1) = -8; \quad y_1 = x^2$

32. $(3x-1)y'' - (3x+2)y' - (6x-8)y = 0, \quad y(0) = 2,\ y'(0) = 3; \quad y_1 = e^{2x}$

33. $(x+1)^2y'' - 2(x+1)y' - (x^2+2x-1)y = (x+1)^3e^x, \quad y(0) = 1, \quad y'(0) = -1;$
$y_1 = (x+1)e^x$

In Exercises 34 *and* 35 *solve the initial value problem and graph the solution, given that* y_1 *satisfies the complementary equation.*

34. $x^2y'' + 2xy' - 2y = x^2, \quad y(1) = \frac{5}{4},\ y'(1) = \frac{3}{2}; \quad y_1 = x$

35. $(x^2-4)y'' + 4xy' + 2y = x + 2, \quad y(0) = -\frac{1}{3}, \quad y'(0) = -1; \quad y_1 = \frac{1}{x-2}$

3.7 VARIATION OF PARAMETERS

In this section we give a method called *variation of parameters* for finding a particular solution of

$$P_2(x)y'' + P_1(x)y' + P_0(x)y = F(x) \tag{3.7.1}$$

if we know a fundamental set $\{y_1, y_2\}$ of solutions of the complementary equation

$$P_2(x)y'' + P_1(x)y' + P_0(x)y = 0. \tag{3.7.2}$$

Having found a particular solution y_p by this method, we can write the general solution of (3.7.1) as

$$y = y_p + c_1y_1 + c_2y_2.$$

Since we need only one nontrivial solution of (3.7.2) to find the general solution of (3.7.1) by reduction of order, it is natural to ask why we are interested in variation of parameters, which requires two linearly independent solutions of (3.7.2) to achieve the same goal. Here are two answers:

- If we already know two linearly independent solutions of (3.7.2), then variation of parameters will probably be simpler than reduction of order.
- Variation of parameters generalizes naturally to a method for finding particular solutions of linear systems of equations (which we will study later), while reduction of order does not.

We will now derive the method. As usual, we consider solutions of (3.7.1) and (3.7.2) on an interval (a, b) where P_2, P_1, P_0, and F are continuous and P_2 has no zeros. Suppose that $\{y_1, y_2\}$ is a fundamental set of solutions of the complementary equation (3.7.2). We look for a particular solution of (3.7.1) in the form

$$y_p = u_1y_1 + u_2y_2 \tag{3.7.3}$$

where u_1 and u_2 are functions to be determined so that y_p satisfies (3.7.1). You may not think this is a good idea, since there are now two unknown functions to be determined, rather than one. However, since u_1 and u_2 have to satisfy only one condition (that y_p is a solution of (3.7.1)), we can impose a second condition that produces a convenient simplification, as follows.

Differentiating (3.7.3) yields

$$y_p' = u_1y_1' + u_2y_2' + u_1'y_1 + u_2'y_2. \tag{3.7.4}$$

As our second condition on u_1 and u_2 we require that

$$u_1'y_1 + u_2'y_2 = 0. \tag{3.7.5}$$

Then (3.7.4) becomes

$$y_p' = u_1y_1' + u_2y_2'; \tag{3.7.6}$$

that is, (3.7.5) permits us to differentiate y_p (once!) as if u_1 and u_2 are constants. Differentiating (3.7.4) yields

$$y_p'' = u_1y_1'' + u_2y_2'' + u_1'y_1' + u_2'y_2'. \tag{3.7.7}$$

(There are no terms involving u_1'' and u_2'' here, as there would be if we had not required (3.7.5).) Substituting (3.7.3), (3.7.6), and (3.7.7) into (3.7.1) and collecting the coefficients of u_1 and u_2 yields

$$u_1(P_2y_1'' + P_1y_1' + P_0y_1) + u_2(P_2y_2'' + P_1y_2' + P_0y_2) + P_2(u_1'y_1' + u_2'y_2') = F.$$

As in the derivation of the method of reduction of order, the coefficients of u_1 and u_2 here are both zero because y_1 and y_2 satisfy the complementary equation. Hence, we can rewrite the last equation as

$$P_2(u_1'y_1' + u_2'y_2') = F. \tag{3.7.8}$$

Therefore y_p in (3.7.3) satisfies (3.7.1) if

$$\begin{aligned} u_1'y_1 + u_2'y_2 &= 0 \\ u_1'y_1' + u_2'y_2' &= \frac{F}{P_2}, \end{aligned} \tag{3.7.9}$$

where the first equation is the same as (3.7.5) and the second is from (3.7.8).

We will now show that you can always solve (3.7.9) for u_1' and u_2'. (The method that we use here will always work, but simpler methods usually work when you are dealing with specific equations.) To obtain u_1', multiply the first equation in (3.7.9) by y_2' and the second equation by y_2. This yields

$$\begin{aligned} u_1'y_1y_2' + u_2'y_2y_2' &= 0 \\ u_1'y_1'y_2 + u_2'y_2'y_2 &= \frac{Fy_2}{P_2}. \end{aligned}$$

Subtracting the second equation from the first yields

$$u_1'(y_1y_2' - y_1'y_2) = -\frac{Fy_2}{P_2}. \tag{3.7.10}$$

Since $\{y_1, y_2\}$ is a fundamental set of solutions of (3.7.2) on (a, b), Theorem 3.1.6 implies that the Wronskian $y_1y_2' - y_1'y_2$ has no zeros on (a, b). Therefore we can solve (3.7.10) for u_1', to obtain

$$u_1' = -\frac{Fy_2}{P_2(y_1y_2' - y_1'y_2)}. \tag{3.7.11}$$

We leave it to you to start from (3.7.9) and show by a similar argument that

$$u_2' = \frac{Fy_1}{P_2(y_1y_2' - y_1'y_2)}. \tag{3.7.12}$$

We can now obtain u_1 and u_2 by integrating u_1' and u_2'. The constants of integration can be taken to be zero, since any choice of u_1 and u_2 in (3.7.3) will suffice.

You should not memorize (3.7.11) and (3.7.12). On the other hand, you do not want to derive the whole procedure for every specific problem. We recommend a compromise:

(a) Write

$$y_p = u_1y_1 + u_2y_2 \tag{3.7.13}$$

to remind yourself of what you are doing.

(b) Write the system

$$\begin{aligned} u_1'y_1 + u_2'y_2 &= 0 \\ u_1'y_1' + u_2'y_2' &= \frac{F}{P_2} \end{aligned} \tag{3.7.14}$$

for the specific problem you are trying to solve.

(c) Solve (3.7.14) for u_1' and u_2' by any convenient method.

(d) Obtain u_1 and u_2 by integrating u_1' and u_2', taking the constants of integration to be zero.

(e) Substitute u_1 and u_2 into (3.7.13) to obtain y_p.

Example 3.7.1 Find a particular solution y_p of

$$x^2y'' - 2xy' + 2y = x^{9/2}, \tag{3.7.15}$$

given that $y_1 = x$ and $y_2 = x^2$ are solutions of the complementary equation

$$x^2y'' - 2xy' + 2y = 0.$$

Then find the general solution of (3.7.15).

Solution We set

$$y_p = u_1x + u_2x^2,$$

where

$$\begin{aligned} u_1'x + \; u_2'x^2 &= 0 \\ u_1' \; + 2u_2'x &= \frac{x^{9/2}}{x^2}. \end{aligned}$$

From the first equation, $u_1' = -u_2'x$. Substituting this into the second equation yields $u_2'x = x^{5/2}$, so $u_2' = x^{3/2}$ and therefore $u_1' = -u_2'x = -x^{5/2}$. Integrating and taking the constants of integration to be zero yields

$$u_1 = -\frac{2}{7}x^{7/2} \quad \text{and} \quad u_2 = \frac{2}{5}x^{5/2}.$$

Therefore $y_p = u_1x + u_2x^2$ yields

$$-\frac{2}{7}x^{7/2}x + \frac{2}{5}x^{5/2}x^2 = \frac{4}{35}x^{9/2},$$

and the general solution of (3.7.15) is

$$y = \frac{4}{35}x^{9/2} + c_1x + c_2x^2.$$

■

Example 3.7.2 Find a particular solution y_p of

$$(x-1)y'' - xy' + y = (x-1)^2, \tag{3.7.16}$$

given that $y_1 = x$ and $y_2 = e^x$ are solutions of the complementary equation

$$(x-1)y'' - xy' + y = 0.$$

Then find the general solution of (3.7.16).

Solution We set

$$y_p = u_1x + u_2e^x,$$

where

$$\begin{aligned} u_1'x + u_2'e^x &= 0 \\ u_1' \;\; + u_2'e^x &= \frac{(x-1)^2}{x-1} = x-1. \end{aligned}$$

Subtracting the first equation from the second yields $-u_1'(x-1) = x-1$, so $u_1' = -1$. From this and the first equation, $u_2' = xe^{-x}$. Integrating and taking the constants of integration to be zero yields

$$u_1 = -x \quad \text{and} \quad u_2 = -(x+1)e^{-x}.$$

Therefore $y_p = u_1x + u_2e^x$ yields

$$(-x)x + (-(x+1)e^{-x})e^x = -x^2 - x - 1,$$

so the general solution $y = y_p + c_1x + c_2e^x$ of (3.7.16) is

$$-x^2 - x - 1 + c_1x + c_2e^x = -x^2 - 1 + (c_1-1)x + c_2e^x. \tag{3.7.17}$$

However, since c_1 is an arbitrary constant, so is $c_1 - 1$; therefore, we improve the appearance of this result by renaming the constant and writing the general solution as

$$y = -x^2 - 1 + c_1x + c_2e^x. \; ■ \tag{3.7.18}$$

There is nothing wrong with leaving the general solution of (3.7.16) in the form (3.7.17); however, we think You will agree that (3.7.18) is preferable. We can also view the transition from (3.7.17) to (3.7.18) differently. In this example the particular solution $y_p = -x^2 - x - 1$ contained the term $-x$, which satisfies the complementary equation. We can drop this term and redefine $y_p = -x^2 - 1$, since $-x^2 - x - 1$ is a solution of (3.7.16) and x is a solution of the complementary equation; hence, $-x^2 - 1 = (-x^2 - x - 1) + x$ is also a solution of (3.7.16). In general, it is always legitimate to drop linear combinations of $\{y_1, y_2\}$ from particular solutions obtained by variation of parameters. We will do this in the following examples and in the answers to exercises that ask for a particular solution. Therefore, do not be concerned if your answer to such an exercise differs from ours only by a solution of the complementary equation.

Example 3.7.3 Find a particular solution of

$$y'' + 3y' + 2y = \frac{1}{1+e^x}. \tag{3.7.19}$$

Then find the general solution.

Solution

The characteristic polynomial $p(r)$ of the complementary equation

$$y'' + 3y' + 2y = 0 \tag{3.7.20}$$

is $r^2 + 3r + 2 = (r+1)(r+2)$, so $y_1 = e^{-x}$ and $y_2 = e^{-2x}$ form a fundamental set of solutions of (3.7.20). We look for a particular solution of (3.7.19) in the form

$$y_p = u_1 e^{-x} + u_2 e^{-2x},$$

where

$$\begin{aligned} u_1' e^{-x} + \ \ u_2' e^{-2x} &= 0 \\ -u_1' e^{-x} - 2u_2' e^{-2x} &= \frac{1}{1+e^x}. \end{aligned}$$

Adding the two equations in the system yields an equation in x for u_2':

$$-u_2' e^{-2x} = \frac{1}{1+e^x}, \quad \text{so} \quad u_2' = -\frac{e^{2x}}{1+e^x}.$$

From the first equation in the system, we find an expression for u_1' and then substitute $u_2'(x)$:

$$-u_2' e^{-x} = \frac{e^x}{1+e^x}.$$

Finally, to solve for u_1, integrate by means of the substitution $v = e^x$ and take the constant of integration to be zero so that

$$\int \frac{e^x}{1+e^x}\, dx = \int \frac{dv}{1+v}.$$

Replacing v with e^x gives $u_1 = \ln(1 + e^x)$.

The expression for u_2 can be transformed with the same substitution of $v = e^x$:

$$-\int \frac{e^{2x}}{1 + e^x}\, dx = -\int \frac{v}{1 + v}\, dv.$$

(Notice that $e^{2x} = e^x e^x$.) Next, use long division to rewrite the improper integral as

$$\int \left[\frac{1}{1 + v} - 1\right] dv.$$

Finally, we integrate (using zero as the constant of integration) and replace v with e^x to find that u_2 is

$$\ln(1 + v) - v = \ln(1 + e^x) - e^x.$$

Therefore

$$\begin{aligned} y_p &= u_1 e^{-x} + u_2 e^{-2x} \\ &= [\ln(1 + e^x)]e^{-x} + [\ln(1 + e^x) - e^x]\, e^{-2x}, \end{aligned}$$

so

$$y_p = \left(e^{-x} + e^{-2x}\right) \ln(1 + e^x) - e^{-x}.$$

Since the last term on the right satisfies the complementary equation, we drop it and redefine

$$y_p = \left(e^{-x} + e^{-2x}\right) \ln(1 + e^x).$$

The general solution y of (3.7.19) is

$$y_p + c_1 e^{-x} + c_2 e^{-2x} = \left(e^{-x} + e^{-2x}\right) \ln(1 + e^x) + c_1 e^{-x} + c_2 e^{-2x}.$$

■

Example 3.7.4 Solve the initial value problem

$$(x^2 - 1)y'' + 4xy' + 2y = \frac{2}{x + 1}, \quad y(0) = -1, \quad y'(0) = -5, \tag{3.7.21}$$

given that

$$y_1 = \frac{1}{x - 1} \quad \text{and} \quad y_2 = \frac{1}{x + 1}$$

are solutions of the complementary equation

$$(x^2 - 1)y'' + 4xy' + 2y = 0.$$

Solution We first use variation of parameters to find a particular solution of

$$(x^2-1)y''+4xy'+2y=\frac{2}{x+1}$$

on $(-1,1)$ in the form

$$y_p=\frac{u_1}{x-1}+\frac{u_2}{x+1},$$

where

$$\begin{aligned}\frac{u_1'}{x-1}+\frac{u_2'}{x+1} &= 0 \\ -\frac{u_1'}{(x-1)^2}-\frac{u_2'}{(x+1)^2} &= \frac{2}{(x+1)(x^2-1)}.\end{aligned} \tag{3.7.22}$$

Multiplying the first equation by $1/(x-1)$ and adding the result to the second equation yields

$$\left[\frac{1}{x^2-1}-\frac{1}{(x+1)^2}\right]u_2'=\frac{2}{(x+1)(x^2-1)}. \tag{3.7.23}$$

Now we use algebra to rewrite the rational expression in x on the left side of (3.7.23) as

$$\frac{(x+1)-(x-1)}{(x+1)(x^2-1)}=\frac{2}{(x+1)(x^2-1)},$$

which implies that $u_2'=1$. Therefore,

$$u_2=\int dx=x.$$

From the first equation of the system (3.7.22), u_1' has the form

$$-\frac{x-1}{x+1}u_2'=-\frac{x-1}{x+1},$$

since we have shown that $u_2'=1$. Now apply some clever algebra to rewrite the rational expression before integrating:

$$-\int\frac{x+1-2}{x+1}\,dx=\int\left[\frac{2}{x+1}-1\right]\,dx$$

Finally, integrate (taking the constants of integration to be zero) To see that

$$u_1=2\ln(x+1)-x.$$

Therefore, the particular solution we seek is

$$\frac{u_1}{x-1}+\frac{u_2}{x+1}=[2\ln(x+1)-x]\,\frac{1}{x-1}+x\frac{1}{x+1}.$$

We can use algebra to rewrite the solution as

$$\frac{2\ln(x+1)}{x-1} + x\left[\frac{1}{x+1} - \frac{1}{x-1}\right] = \frac{2\ln(x+1)}{x-1} - \frac{2x}{(x+1)(x-1)},$$

which allows us to see that

$$\frac{2x}{(x+1)(x-1)} = \left[\frac{1}{x+1} + \frac{1}{x-1}\right]$$

is a solution of the complementary equation. Therefore we use the particular solution

$$y_p = \frac{2\ln(x+1)}{x-1}$$

in the general solution of (3.7.24) to get

$$y = \frac{2\ln(x+1)}{x-1} + \frac{c_1}{x-1} + \frac{c_2}{x+1}. \qquad (3.7.24)$$

Differentiating this yields

$$y' = \frac{2}{x^2-1} - \frac{2\ln(x+1)}{(x-1)^2} - \frac{c_1}{(x-1)^2} - \frac{c_2}{(x+1)^2}.$$

Setting $x = 0$ in the last two equations and imposing the initial conditions $y(0) = -1$ and $y'(0) = -5$ yields the system

$$\begin{aligned} -c_1 + c_2 &= -1 \\ -2 - c_1 - c_2 &= -5. \end{aligned}$$

The solution of this system is $c_1 = 2,\ c_2 = 1$. Substituting these into (3.7.24) yields

$$\begin{aligned} y &= \frac{2\ln(x+1)}{x-1} + \frac{2}{x-1} + \frac{1}{x+1} \\ &= \frac{2\ln(x+1)}{x-1} + \frac{3x+1}{x^2-1} \end{aligned}$$

as the solution of (3.7.21). Figure 3.1 is a graph of the solution. ■

Comparison of Methods

We have now considered three methods for solving nonhomogeneous linear equations: undetermined coefficients, reduction of order, and variation of parameters. It is natural to ask which method is best for a given problem. The method of undetermined coefficients should be used for constant coefficient equations with forcing functions that are linear combinations of polynomials multiplied by functions of the form $e^{\alpha x}$, $e^{\lambda x}\cos\omega x$, or $e^{\lambda x}\sin\omega x$. Although the other two methods can be used to solve such problems, they will be more difficult except in the most trivial cases, because of the integrations involved.

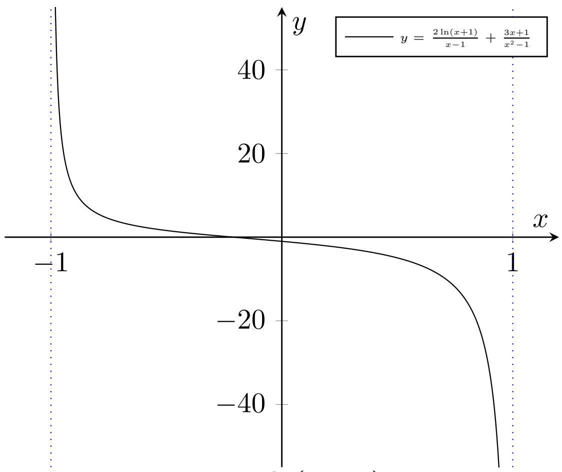


Figure 3.1 $y = \dfrac{2\ln(x+1)}{x-1} + \dfrac{3x+1}{x^2-1}$

If the equation is not a constant coefficient equation or the forcing function is not of the form just specified, the method of undetermined coefficients does not apply and the choice is necessarily between the other two methods. The case could be made that reduction of order is better because it requires only one solution of the complementary equation while variation of parameters requires two. However, variation of parameters will probably be easier if you already know a fundamental set of solutions of the complementary equation.

3.7 Exercises

In Exercises 1–6 use variation of parameters to find a particular solution.

1. $y'' + 9y = \tan 3x$

2. $y'' + 4y = \sin 2x \sec^2 2x$

3. $y'' - 3y' + 2y = \dfrac{4}{1+e^{-x}}$

4. $y'' - 2y' + 2y = 3e^x \sec x$

5. $y'' - 2y' + y = 14x^{3/2}e^x$

6. $y'' - y = \dfrac{4e^{-x}}{1-e^{-2x}}$

In Exercises 7–29 use variation of parameters to find a particular solution, given the solutions y_1, y_2 of the complementary equation.

7. $x^2y'' + xy' - y = 2x^2 + 2; \quad y_1 = x, \quad y_2 = \dfrac{1}{x}$

8. $xy'' + (2-2x)y' + (x-2)y = e^{2x}; \quad y_1 = e^x, \quad y_2 = \dfrac{e^x}{x}$

9. $4x^2y'' + (4x - 8x^2)y' + (4x^2 - 4x - 1)y = 4x^{1/2}e^x, \quad x > 0;$
$y_1 = x^{1/2}e^x, \; y_2 = x^{-1/2}e^x$

10. $y'' + 4xy' + (4x^2+2)y = 4e^{-x(x+2)}; \quad y_1 = e^{-x^2}, \quad y_2 = xe^{-x^2}$

11. $x^2y'' - 4xy' + 6y = x^{5/2}, \; x > 0; \quad y_1 = x^2, \; y_2 = x^3$

12. $x^2y'' - 3xy' + 3y = 2x^4 \sin x; \quad y_1 = x, \; y_2 = x^3$

13. $(2x+1)y'' - 2y' - (2x+3)y = (2x+1)^2e^{-x}; \quad y_1 = e^{-x}, \quad y_2 = xe^x$

14. $4xy'' + 2y' + y = \sin\sqrt{x}; \quad y_1 = \cos\sqrt{x}, \quad y_2 = \sin\sqrt{x}$

15. $xy'' - (2x+2)y' + (x+2)y = 6x^3e^x; \quad y_1 = e^x, \quad y_2 = x^3e^x$

16. $x^2y'' - (2a-1)xy' + a^2y = x^{a+1}; \quad y_1 = x^a, \quad y_2 = x^a \ln x$

17. $x^2y'' - 2xy' + (x^2+2)y = x^3 \cos x; \quad y_1 = x\cos x, \quad y_2 = x \sin x$

18. $xy'' - y' - 4x^3y = 8x^5; \quad y_1 = e^{x^2}, \; y_2 = e^{-x^2}$

19. $(\sin x)y'' + (2\sin x - \cos x)y' + (\sin x - \cos x)y = e^{-x}; \quad y_1 = e^{-x}, \quad y_2 = e^{-x}\cos x$

20. $4x^2y'' - 4xy' + (3 - 16x^2)y = 8x^{5/2}; \quad y_1 = \sqrt{x}e^{2x}, \; y_2 = \sqrt{x}e^{-2x}$

21. $4x^2y'' - 4xy' + (4x^2+3)y = x^{7/2}; \quad y_1 = \sqrt{x}\sin x, \; y_2 = \sqrt{x}\cos x$

22. $x^2y'' - 2xy' - (x^2-2)y = 3x^4; \quad y_1 = xe^x, \; y_2 = xe^{-x}$

23. $x^2y'' - 2x(x+1)y' + (x^2+2x+2)y = x^3e^x; \quad y_1 = xe^x, \quad y_2 = x^2e^x$

24. $x^2y'' - xy' - 3y = x^{3/2}; \quad y_1 = 1/x, \quad y_2 = x^3$

25. $x^2y'' - x(x+4)y' + 2(x+3)y = x^4e^x; \quad y_1 = x^2, \quad y_2 = x^2e^x$

26. $x^2y'' - 2x(x+2)y' + (x^2+4x+6)y = 2xe^x; \quad y_1 = x^2e^x, \quad y_2 = x^3e^x$

27. $x^2y'' - 4xy' + (x^2+6)y = x^4; \quad y_1 = x^2\cos x, \quad y_2 = x^2 \sin x$

28. $(x-1)y'' - xy' + y = 2(x-1)^2e^x; \quad y_1 = x, \quad y_2 = e^x$

29. $4x^2y'' - 4x(x+1)y' + (2x+3)y = x^{5/2}e^x; \quad y_1 = \sqrt{x}, \quad y_2 = \sqrt{x}e^x$

In Exercises 30–32 use variation of parameters to solve the initial value problem, given y_1, y_2 are solutions of the complementary equation.

30. $(3x-1)y'' - (3x+2)y' - (6x-8)y = (3x-1)^2e^{2x}, \quad y(0) = 1, \; y'(0) = 2;$
$y_1 = e^{2x}, \; y_2 = xe^{-x}$

31. $(x-1)^2y'' - 2(x-1)y' + 2y = (x-1)^2, \quad y(0) = 3, \quad y'(0) = -6;$
$y_1 = x - 1, \; y_2 = x^2 - 1$

Figure 3.1 **(a)** $y > 0$ **(b)** $y = 0$, **(c)** $y < 0$

Figure 3.2 A spring – mass system with damping

32. $(x-1)^2y'' - (x^2-1)y' + (x+1)y = (x-1)^3e^x, \quad y(0) = 4, \quad y'(0) = -6;$
$y_1 = (x-1)e^x, \quad y_2 = x-1$

In Exercises 33–35 use variation of parameters to solve the initial value problem and graph the solution, given that y_1, y_2 *are solutions of the complementary equation.*

33. $(x^2-1)y'' + 4xy' + 2y = 2x, \quad y(0) = 0,\ y'(0) = -2; \quad y_1 = \dfrac{1}{x-1},\ y_2 = \dfrac{1}{x+1}$

34. $x^2y'' + 2xy' - 2y = -2x^2, \quad y(1) = 1,\ y'(1) = -1; \quad y_1 = x,\ y_2 = \dfrac{1}{x^2}$

35. $(x+1)(2x+3)y'' + 2(x+2)y' - 2y = (2x+3)^2, \quad y(0) = 0, \quad y'(0) = 0;$
$y_1 = x+2, \quad y_2 = \dfrac{1}{x+1}$

3.8 APPLICATIONS TO SPRINGS

We consider the motion of an object of mass m, suspended from a spring of negligible mass. We say that the spring–mass system is in *equilibrium* when the object is at rest and the forces acting on it sum to zero. The position of the object in this case is the *equilibrium position*. We define y to be the displacement of the object from its equilibrium position (Figure 3.1), measured positive upward.

Our model accounts for several kinds of forces acting on the object:

- The force due to gravity is represented by $-mg$. This force is also known as weight.
- Another force F_s is exerted by the spring resisting change in its length. The *natural length* of the spring is its length with no mass attached. We assume that the spring obeys *Hooke's law*: If the length of the spring is changed by an amount ΔL from its natural length, then the spring exerts a force $F_s = k\Delta L$, where k is a positive number called the *spring constant*. If the spring is stretched, then $\Delta L > 0$ and $F_s > 0$, so the spring force is upward; if the spring is compressed, then $\Delta L < 0$ and $F_s < 0$, so the spring force is downward.
- In some models, there may be a *damping force* $F_d = -cy'$ that resists the motion with a force proportional to the velocity of the object. It may be due to air resistance or friction in the spring. However, a convenient way to visualize a damping force is to assume that the object is rigidly attached to a piston with negligible mass immersed in a cylinder filled with a viscous liquid (Figure 3.2). As the piston moves, the liquid exerts a damping force. We say that the motion is *undamped* if $c = 0$, or *damped* if $c > 0$.

- In some models, there may be an external force F, other than the force due to gravity, that may vary with t, but is independent of displacement and velocity. We say that the motion is *free* if $F \equiv 0$, or *forced* if $F \not\equiv 0$.

From Newton's second law of motion, whenever the net force acting on an object is not zero, the net force must be proportional to its acceleration. Thus, in our model, my'' must equal the sum of the forces acting on the object. More precisely,

$$my'' = -mg + F_s - cy' + F. \tag{3.8.1}$$

We now relate F_s to y. In the absence of external forces, the object stretches the spring by an amount $\Delta\lambda$ to assume its equilibrium position. Since the sum of the forces acting on an object in equilibrium is zero, Hooke's Law implies that $mg = k\Delta\lambda$. However, if the object is displaced y units from its equilibrium position, the total change in the length of the spring becomes $\Delta L = \Delta\lambda - y$, and Hooke's law now implies that

$$F_s = k\Delta L = k\Delta\lambda - ky.$$

Substituting this into (3.8.1) yields

$$my'' = -mg + k\Delta L - ky - cy' + F.$$

Since $mg = k\Delta\lambda$ this can be written as

$$my'' + cy' + ky = F. \tag{3.8.2}$$

We call this *the equation of motion*.

Simple Harmonic Motion

We first consider spring–mass systems without damping where the motion is also free; that is, both $c = 0$ and F=0. We begin with an initial value problem.

Example 3.8.1 An object stretches a spring 6 inches in equilibrium.

(a) Set up the equation of motion and find its general solution.

(b) Find the displacement of the object for $t > 0$ if it's initially displaced 18 inches above equilibrium and given a downward velocity of 3 ft/s.

Solution (a) Setting $c = 0$ and $F = 0$ in (3.8.2) yields the equation of motion

$$my'' + ky = 0,$$

which we rewrite as

$$y'' + \frac{k}{m}y = 0. \tag{3.8.3}$$

Although we would need the weight of the object to obtain k from the equation $mg = k\Delta\lambda$ we can determine the coefficient k/m using $\Delta\lambda$ because we know the acceleration

Figure 3.3 $y = \frac{3}{2}\cos 8t - \frac{3}{8}\sin 8t$

due to gravity. Consistent with the units used in the problem statement, we take $g = 32$ ft/s^2. Although $\Delta\lambda$ is stated in inches, we must convert it to feet to be consistent with this choice of g; that is, $\Delta\lambda = 1/2$ ft. Using $k/m = g/\Delta\lambda$, we see that

$$\frac{k}{m} = \frac{32}{1/2} = 64$$

and (3.8.3) becomes

$$y'' + 64y = 0. \tag{3.8.4}$$

The characteristic equation of (3.8.4) is

$$r^2 + 64 = 0,$$

which has the zeros $r = \pm 8i$. Therefore the general solution of (3.8.4) is

$$y = c_1 \cos 8t + c_2 \sin 8t. \tag{3.8.5}$$

(b)The initial upward displacement of 18 inches is positive and must be expressed in feet. The initial downward velocity is negative; thus,

$$y(0) = \frac{3}{2} \quad \text{and} \quad y'(0) = -3.$$

Differentiating (3.8.5) yields

$$y' = -8c_1 \sin 8t + 8c_2 \cos 8t. \tag{3.8.6}$$

Setting $t = 0$ in (3.8.5) and (3.8.6) and imposing the initial conditions shows that $c_1 = 3/2$ and $c_2 = -3/8$. Therefore

$$y = \frac{3}{2}\cos 8t - \frac{3}{8}\sin 8t,$$

where y is in feet (Figure 3.3). ■

We now consider the equation

$$my'' + ky = 0$$

where m and k are arbitrary positive numbers. Dividing through by m and defining $\omega_0 = \sqrt{k/m}$ yields

$$y'' + \omega_0^2 y = 0.$$

The general solution of this equation is

$$y = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t. \tag{3.8.7}$$

We can rewrite this in a more useful form by defining

$$R = \sqrt{c_1^2 + c_2^2}, \tag{3.8.8}$$

Figure 3.4 $R = \sqrt{c_1^2 + c_2^2}$; $c_1 = R\cos\phi$; $c_2 = R\sin\phi$

and

$$c_1 = R\cos\phi \quad \text{and} \quad c_2 = R\sin\phi. \tag{3.8.9}$$

Substituting from (3.8.9) into (3.8.7) and applying the identity

$$\cos\omega_0 t\cos\phi + \sin\omega_0 t\sin\phi = \cos(\omega_0 t - \phi)$$

yields

$$y = R\cos(\omega_0 t - \phi). \tag{3.8.10}$$

From (3.8.8) and (3.8.9) we see that the R and ϕ can be interpreted as polar coordinates of the point with rectangular coordinates (c_1, c_2) (Figure 3.4). Given c_1 and c_2, we can compute R from (3.8.8) and find ϕ by noting that

$$\tan\phi = \frac{c_2}{c_1}.$$

There are infinitely many angles ϕ, differing by integer multiples of 2π, that satisfy this equation. We will always choose ϕ so that $-\pi \leqslant \phi < \pi$.

The motion described by (3.8.7) or (3.8.10) is *simple harmonic motion*. We see from either of these equations that the motion is periodic, with period

$$T = 2\pi/\omega_0.$$

This is the time required for the object to complete one full cycle of oscillation (for example, to move from its highest position to its lowest position and back to its highest position). Since the highest and lowest positions of the object are $y = R$ and $y = -R$, we say that R is the *amplitude* of the oscillation. The angle ϕ in (3.8.10) is the *phase angle*, measured in radians. Equation (3.8.10) is the *amplitude–phase form* of the displacement. If t is in seconds then ω_0 is in radians per second (rad/s); this is the *frequency* of the motion. It is also called the *natural frequency* of the spring–mass system without damping.

Example 3.8.2 We found the displacement of the object in Example 3.8.1 to be

$$y = \frac{3}{2}\cos 8t - \frac{3}{8}\sin 8t.$$

Find the frequency, period, amplitude, and phase angle of the motion.

Solution The frequency is $\omega_0 = 8$ rad/s, and the period is $T = 2\pi/\omega_0 = \pi/4$ s. Since $c_1 = 3/2$ and $c_2 = -3/8$, the amplitude is

$$R = \sqrt{c_1^2 + c_2^2} = \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{3}{8}\right)^2} = \frac{3}{8}\sqrt{17}.$$

The phase angle is determined by

$$\tan\phi = \frac{-\frac{3}{8}}{\frac{3}{2}}. \tag{3.8.11}$$

Using a calculator, we find from (3.8.11) that

$$\phi \approx -.245 \text{ rad}.$$

Since $\cos\phi > 0$ and $\sin\phi < 0$, the angle is in the fourth quadrant and that the calculated value of the phase angle is correct. ■

Example 3.8.3 The natural length of a spring is 1 m. An object is attached to it and the length of the spring increases to 102 cm when the object is in equilibrium. Then the object is initially displaced downward 1 cm and given an upward velocity of 14 cm/s. Find the displacement for $t > 0$. Also, find the natural frequency, period, amplitude, and phase angle of the resulting motion. Express the answer in terms of centimeters.

Solution To use centimeters, we convert gravity to $g = 980$ cm/s^2. Since $\Delta\lambda = 2$ cm, $\omega_0^2 = g/\Delta\lambda = 490$. Therefore

$$y'' + 490y = 0, \quad y(0) = -1, \quad y'(0) = 14.$$

The general solution of the differential equation is

$$y = c_1 \cos 7\sqrt{10}t + c_2 \sin 7\sqrt{10}t,$$

so

$$y' = 7\sqrt{10}\left(-c_1 \sin 7\sqrt{10}t + c_2 \cos 7\sqrt{10}t\right).$$

Substituting the initial conditions into the last two equations yields $c_1 = -1$ and $c_2 = 2/\sqrt{10}$. Hence,

$$y = -\cos 7\sqrt{10}t + \frac{2}{\sqrt{10}} \sin 7\sqrt{10}t.$$

The frequency is $7\sqrt{10}$ rad/s, and the period is $T = 2\pi/(7\sqrt{10})$ s. The amplitude is

$$R = \sqrt{c_1^2 + c_2^2} = \sqrt{(-1)^2 + \left(\frac{2}{\sqrt{10}}\right)^2} = \sqrt{\frac{7}{5}} \text{ cm}.$$

The phase angle is determined by

$$\tan\phi = \frac{\frac{2}{\sqrt{10}}}{-1}.$$

Here it is important to notice that since $\cos\phi < 0$ and $\sin\phi > 0$, the phase angle is in the second quadrant. This means that we must add π to the calculated value of the angle provided by the definition of the inverse tangent function. With the aid of a calculator, we find that

$$\phi \approx 2.58 \text{ rad}.$$

■

Undamped Forced Oscillation

In many mechanical problems a device is subjected to periodic external forces. For example, soldiers marching in cadence on a bridge cause periodic disturbances in the bridge, and the engines of a propeller–driven aircraft cause periodic disturbances in its wings. In the absence of sufficient damping forces, such disturbances – even if small in magnitude – can cause structural breakdown if they are at certain critical frequencies. To illustrate, this we consider the motion of an object in a spring–mass system without damping, subject to an external force

$$F(t) = F_0 \cos \omega t$$

where F_0 is a constant. In this case the equation of motion (3.8.2) is

$$my'' + ky = F_0 \cos \omega t,$$

which we rewrite as

$$y'' + \omega_0^2 y = \frac{F_0}{m} \cos \omega t \tag{3.8.12}$$

with $\omega_0 = \sqrt{k/m}$. We will see from the next two examples that the solutions of (3.8.12) with $\omega \neq \omega_0$ behave very differently from the solutions with $\omega = \omega_0$.

Example 3.8.4 Solve the initial value problem

$$y'' + \omega_0^2 y = \frac{F_0}{m} \cos \omega t, \quad y(0) = 0, \quad y'(0) = 0, \tag{3.8.13}$$

given that $\omega \neq \omega_0$.

Solution We first obtain a particular solution of (3.8.12) by the method of undetermined coefficients. Since $\omega \neq \omega_0$, $\cos \omega t$ is not a solution of the complementary equation

$$y'' + \omega_0^2 y = 0.$$

Therefore (3.8.12) has a particular solution of the form

$$y_p = A \cos \omega t + B \sin \omega t.$$

Since

$$y_p'' = -\omega^2 (A \cos \omega t + B \sin \omega t),$$

$$y_p'' + \omega_0^2 y_p = \frac{F_0}{m} \cos \omega t$$

if and only if

$$(\omega_0^2 - \omega^2)\,(A\cos\omega t + B\sin\omega t) = \frac{F_0}{m}\cos\omega t.$$

This holds if and only if

$$A = \frac{F_0}{m(\omega_0^2 - \omega^2)} \quad \text{and} \quad B = 0,$$

so

$$y_p = \frac{F_0}{m(\omega_0^2 - \omega^2)}\cos\omega t.$$

The general solution of (3.8.12) is

$$y = \frac{F_0}{m(\omega_0^2 - \omega^2)}\cos\omega t + c_1\cos\omega_0 t + c_2\sin\omega_0 t, \tag{3.8.14}$$

so

$$y' = \frac{-\omega F_0}{m(\omega_0^2 - \omega^2)}\sin\omega t + \omega_0(-c_1\sin\omega_0 t + c_2\cos\omega_0 t).$$

The initial conditions $y(0) = 0$ and $y'(0) = 0$ in (3.8.13) imply that

$$c_1 = -\frac{F_0}{m(\omega_0^2 - \omega^2)} \quad \text{and} \quad c_2 = 0.$$

Substituting these into (3.8.14) yields

$$y = \frac{F_0}{m(\omega_0^2 - \omega^2)}(\cos\omega t - \cos\omega_0 t). \tag{3.8.15}$$

■

It is revealing to write this solution in a different form. We start with the trigonometric identities

$$\begin{aligned}\cos(\alpha - \beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta\\ \cos(\alpha + \beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta.\end{aligned}$$

Subtracting the second identity from the first yields

$$\cos(\alpha - \beta) - \cos(\alpha + \beta) = 2\sin\alpha\sin\beta \tag{3.8.16}$$

Now let

$$\alpha - \beta = \omega t \quad \text{and} \quad \alpha + \beta = \omega_0 t, \tag{3.8.17}$$

so that

$$\alpha = \frac{(\omega_0 + \omega)t}{2} \quad \text{and} \quad \beta = \frac{(\omega_0 - \omega)t}{2}. \tag{3.8.18}$$

Figure 3.5 Undamped oscillation with beats

Substituting (3.8.18) and (3.8.17) into (3.8.16) yields

$$\cos\omega t - \cos\omega_0 t = 2\sin\frac{(\omega_0-\omega)t}{2}\sin\frac{(\omega_0+\omega)t}{2},$$

and substituting this into (3.8.15) yields

$$y = R(t)\sin\frac{(\omega_0+\omega)t}{2}, \tag{3.8.19}$$

where

$$R(t) = \frac{2F_0}{m(\omega_0^2-\omega^2)}\sin\frac{(\omega_0-\omega)t}{2}. \tag{3.8.20}$$

From (3.8.19) we can regard y as a sinusoidal variation with frequency $(\omega_0+\omega)/2$ and variable amplitude $|R(t)|$. In Figure 3.5 the dashed curve above the t axis is $y = |R(t)|$, the dashed curve below the t axis is $y = -|R(t)|$, and the displacement y appears as an oscillation bounded by them. The oscillation of y for t on an interval between successive zeros of $R(t)$ is called a *beat*.

You can see from (3.8.20) and (3.8.19) that

$$|y(t)| \leqslant \frac{2|F_0|}{m|\omega_0^2-\omega^2|};$$

moreover, if $\omega+\omega_0$ is sufficiently large compared with $\omega-\omega_0$, then $|y|$ assumes values close to (perhaps equal to) this upper bound during each beat. However, the oscillation remains bounded for all t. (This assumes that the spring can withstand deflections of this size and continue to obey Hooke's law.) The next example shows that this is not the case if $\omega = \omega_0$.

Example 3.8.5 Find the general solution of

$$y'' + \omega_0^2 y = \frac{F_0}{m}\cos\omega_0 t. \tag{3.8.21}$$

Solution We first obtain a particular solution y_p of (3.8.21). Since $\cos\omega_0 t$ is a solution of the complementary equation, the form for y_p is

$$y_p = t(A\cos\omega_0 t + B\sin\omega_0 t). \tag{3.8.22}$$

Then

$$y_p' = A\cos\omega_0 t + B\sin\omega_0 t + \omega_0 t(-A\sin\omega_0 t + B\cos\omega_0 t)$$

and

$$y_p'' = 2\omega_0(-A\sin\omega_0 t + B\cos\omega_0 t) - \omega_0^2 t(A\cos\omega_0 t + B\sin\omega_0 t). \tag{3.8.23}$$

Figure 3.6 Unbounded displacement due to resonance

From (3.8.23) and (3.8.22), we see that y_p satisfies (3.8.21) if

$$-2A\omega_0 \sin \omega_0 t + 2B\omega_0 \cos \omega_0 t = \frac{F_0}{m} \cos \omega_0 t;$$

that is, if

$$A = 0 \quad \text{and} \quad B = \frac{F_0}{2m\omega_0}.$$

Therefore

$$y_p = \frac{F_0 t}{2m\omega_0} \sin \omega_0 t$$

is a particular solution of (3.8.21). The general solution of (3.8.21) is

$$y = \frac{F_0 t}{2m\omega_0} \sin \omega_0 t + c_1 \cos \omega_0 t + c_2 \sin \omega_0 t.$$

The graph of y_p is shown in Figure 3.6, where it can be seen that y_p oscillates between the dashed lines

$$y = \frac{F_0 t}{2m\omega_0} \quad \text{and} \quad y = -\frac{F_0 t}{2m\omega_0}$$

with increasing amplitude that approaches ∞ as $t \to \infty$. Of course, this means that the spring must eventually fail to obey Hooke's law or break. ■

This phenomenon of unbounded displacements of a spring–mass system in response to a periodic forcing function at its natural frequency is called *resonance*. More complicated mechanical structures can also exhibit resonance–like phenomena. For example, rhythmic oscillations of a suspension bridge by wind forces or of an airplane wing by periodic vibrations of reciprocating engines can cause damage or even failure if the frequencies of the disturbances are close to critical frequencies determined by the parameters of the mechanical system in question.

Free Vibrations With Damping

We now consider the motion of an object in a spring–mass system with damping but with unforced motion. In this case, the equation of motion is

$$my'' + cy' + ky = 0. \tag{3.8.24}$$

Now suppose the object is displaced from equilibrium and given an initial velocity. Intuition suggests that if the damping force is sufficiently weak, the resulting motion will be oscillatory, as in the undamped case. On the other hand, if the damping force is sufficiently strong, the object may just move slowly toward the equilibrium position without ever reaching it. We will confirm these intuitive ideas mathematically. The characteristic equation of (3.8.24) is

$$mr^2 + cr + k = 0.$$

Figure 3.7 Underdamped motion

The roots of this equation are

$$r_1 = \frac{-c - \sqrt{c^2 - 4mk}}{2m} \quad \text{and} \quad r_2 = \frac{-c + \sqrt{c^2 - 4mk}}{2m}. \tag{3.8.25}$$

We have seen that the form of the solution of (3.8.24) depends upon whether $c^2 - 4mk$ is positive, negative, or zero. We now consider these three cases.

Underdamped Motion

We say the motion is *underdamped* if $c < \sqrt{4mk}$. In this case r_1 and r_2 in (3.8.25) are complex conjugates, which we write as

$$r_1 = -\frac{c}{2m} - i\omega_1 \quad \text{and} \quad r_2 = -\frac{c}{2m} + i\omega_1,$$

where

$$\omega_1 = \frac{\sqrt{4mk - c^2}}{2m}.$$

The general solution of (3.8.24) in this case is

$$y = e^{-ct/2m}(c_1 \cos \omega_1 t + c_2 \sin \omega_1 t).$$

By the method used to derive the amplitude–phase form of the displacement of an object in simple harmonic motion, we can rewrite this equation as

$$y = Re^{-ct/2m} \cos(\omega_1 t - \phi), \tag{3.8.26}$$

where

$$R = \sqrt{c_1^2 + c_2^2}.$$

The factor $Re^{-ct/2m}$ in (3.8.26) is called the *time–varying amplitude* of the motion. A typical graph of (3.8.26) is shown in Figure 3.7. As illustrated in that figure, the graph of y oscillates between the dashed exponential curves $y = \pm Re^{-ct/2m}$.

Overdamped Motion

We say the motion is *overdamped* if $c > \sqrt{4mk}$. In this case the zeros r_1 and r_2 of the characteristic polynomial are real, with $r_1 < r_2 < 0$ (see (3.8.25)), and the general solution of (3.8.24) is

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

Again $\lim_{t\to\infty} y(t) = 0$ as in the underdamped case, but the motion is not oscillatory, since y cannot equal zero for more than one value of t unless $c_1 = c_2 = 0$.

Critically Damped Motion

We say the motion is *critically damped* if $c = \sqrt{4mk}$. In this case $r_1 = r_2 = -c/2m$ and the general solution of (3.8.24) is

$$y = e^{-ct/2m}(c_1 + c_2 t).$$

Again $\lim_{t\to\infty} y(t) = 0$ and the motion is nonoscillatory, since y cannot equal zero for more than one value of t unless $c_1 = c_2 = 0$.

Example 3.8.6 Suppose a 64 lb weight stretches a spring 6 inches in equilibrium and experiences a damping force of c lb for each ft/sec of velocity.

(a) Write the equation of motion of the object and determine the value of c for which the motion is critically damped.

(b) Find the displacement y for $t > 0$ if the motion is critically damped and the initial conditions are $y(0) = 1$ and $y'(0) = 20$.

(c) Find the displacement y for $t > 0$ if the motion is critically damped and the initial conditions are $y(0) = 1$ and $y'(0) = -20$.

Solution (a) Here $m = 2$ (since the force of weight is $-mg = 64$) and $k = 64/.5 = 128$ lb/ft. Therefore the equation of motion (3.8.24) becomes

$$2y'' + cy' + 128y = 0. \tag{3.8.27}$$

The characteristic equation is

$$2r^2 + cr + 128 = 0,$$

which has roots

$$r = \frac{-c \pm \sqrt{c^2 - 8 \cdot 128}}{4}.$$

Therefore the damping is critical if

$$c = \sqrt{8 \cdot 128} = 32 \text{ lb–sec/ft}.$$

(b) Setting $c = 32$ in (3.8.27) and cancelling the common factor 2 yields

$$y'' + 16y + 64y = 0.$$

Figure 3.8 **(a)** $y = e^{-8t}(1 + 28t)$ **(b)** $y = e^{-8t}(1 - 12t)$

The characteristic equation is

$$r^2 + 16r + 64y = (r + 8)^2 = 0.$$

Hence, the general solution is

$$y = e^{-8t}(c_1 + c_2 t). \tag{3.8.28}$$

Differentiating this yields

$$y' = -8y + c_2 e^{-8t}. \tag{3.8.29}$$

Imposing the initial conditions $y(0) = 1$ and $y'(0) = 20$ in the last two equations shows that $1 = c_1$ and $20 = -8 + c_2$. Hence, the solution of the initial value problem is

$$y = e^{-8t}(1 + 28t).$$

Therefore the object approaches equilibrium from above as $t \to \infty$. There's no oscillation.

(c)Imposing the initial conditions $y(0) = 1$ and $y'(0) = -20$ in (3.8.28) and (3.8.29) yields $1 = c_1$ and $-20 = -8 + c_2$. Hence, the solution of this initial value problem is

$$y = e^{-8t}(1 - 12t).$$

Therefore the object moves downward through equilibrium just once, and then approaches equilibrium from below as $t \to \infty$. Again, there is no oscillation. The solutions of these two initial value problems are graphed in Figure 3.8.

Example 3.8.7 Find the displacement of the object in Example 3.8.6 if the damping constant is $c = 4$ lb–sec/ft and the initial conditions are $y(0) = 1.5$ ft and $y'(0) = -3$ ft/sec.

Solution With $c = 4$, the equation of motion (3.8.4) becomes

$$y'' + 2y' + 64y = 0 \tag{3.8.30}$$

after cancelling the common factor 2. The characteristic equation

$$r^2 + 2r + 64 = 0$$

has complex conjugate roots

$$r = \frac{-2 \pm \sqrt{4 - 4 \cdot 64}}{2} = -1 \pm 3\sqrt{7}i.$$

Therefore the motion is underdamped and the general solution of (3.8.30) is

$$y = e^{-t}(c_1 \cos 3\sqrt{7}t + c_2 \sin 3\sqrt{7}t).$$

Differentiating this yields

$$y' = -y + 3\sqrt{7}e^{-t}(-c_1 \sin 3\sqrt{7}t + c_2 \cos 3\sqrt{7}t).$$

Imposing the initial conditions $y(0) = 1.5$ and $y'(0) = -3$ in the last two equations yields $1.5 = c_1$ and $-3 = -1.5 + 3\sqrt{7}c_2$. Hence, the solution of the initial value problem is

$$y = e^{-t}\left(\frac{3}{2}\cos 3\sqrt{7}t - \frac{1}{2\sqrt{7}}\sin 3\sqrt{7}t\right). \tag{3.8.31}$$

The amplitude of the function in parentheses is

$$R = \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{1}{2\sqrt{7}}\right)^2} = \sqrt{\frac{9}{4} + \frac{1}{4\cdot 7}} = \sqrt{\frac{64}{4\cdot 7}} = \frac{4}{\sqrt{7}}.$$

Therefore we can rewrite (3.8.31) as

$$y = \frac{4}{\sqrt{7}}e^{-t}\cos(3\sqrt{7}t - \phi).$$

■

Example 3.8.8 Let the damping constant in Example 1 be $c = 40$ lb–sec/ft. Find the displacement y for $t > 0$ if $y(0) = 1$ and $y'(0) = 1$.

Solution With $c = 40$, the equation of motion (3.8.27) reduces to

$$y'' + 20y' + 64y = 0 \tag{3.8.32}$$

after cancelling the common factor 2. The characteristic equation

$$r^2 + 20r + 64 = (r + 16)(r + 4) = 0$$

has the roots $r_1 = -4$ and $r_2 = -16$. Therefore the general solution of (3.8.32) is

$$y = c_1e^{-4t} + c_2e^{-16t}. \tag{3.8.33}$$

Differentiating this yields

$$y' = -4e^{-4t} - 16c_2e^{-16t}.$$

The last two equations and the initial conditions $y(0) = 1$ and $y'(0) = 1$ imply that

$$\begin{array}{rcrl} c_1 & + & c_2 & = 1 \\ -4c_1 & - & 16c_2 & = 1. \end{array}$$

The solution of this system is $c_1 = 17/12$, $c_2 = -5/12$. Substituting these into (3.8.33) yields

$$y = \frac{17}{12}e^{-4t} - \frac{5}{12}e^{-16t}$$

as the solution of the given initial value problem. ■

3.8 Exercises

In Exercises 1–12, assume that there is no damping.

1. An object stretches a spring 4 inches in equilibrium. Find and graph its displacement for $t > 0$ if it's initially displaced 36 inches above equilibrium and given a downward velocity of 2 ft/s.

2. An object stretches a string 1.2 inches in equilibrium. Find its displacement for $t > 0$ if it's initially displaced 3 inches below equilibrium and given a downward velocity of 2 ft/s.

3. A spring with natural length .5 m has length 50.5 cm with a mass of 2 gm suspended from it. The mass is initially displaced 1.5 cm below equilibrium and released with zero velocity. Find its displacement for $t > 0$.

4. An object stretches a spring 6 inches in equilibrium. Find its displacement for $t > 0$ if it's initially displaced 3 inches above equilibrium and given a downward velocity of 6 inches/s. Find the frequency, period, amplitude and phase angle of the motion.

5. An object stretches a spring 5 cm in equilibrium. It is initially displaced 10 cm above equilibrium and given an upward velocity of .25 m/s. Find and graph its displacement for $t > 0$. Find the frequency, period, amplitude, and phase angle of the motion.

6. A 10 kg mass stretches a spring 70 cm in equilibrium. Suppose a 2 kg mass is attached to the spring, initially displaced 25 cm below equilibrium, and given an upward velocity of 2 m/s. Find its displacement for $t > 0$. Find the frequency, period, amplitude, and phase angle of the motion.

7. A weight stretches a spring 1.5 inches in equilibrium. The weight is initially displaced 8 inches above equilibrium and given a downward velocity of 4 ft/s. Find its displacement for $t > 0$.

8. A weight stretches a spring 6 inches in equilibrium. The weight is initially displaced 6 inches above equilibrium and given a downward velocity of 3 ft/s. Find its displacement for $t > 0$.

9. A 64 lb weight is attached to a spring with constant $k = 8$ lb/ft and subjected to an external force $F(t) = 2\sin t$. The weight is initially displaced 6 inches above equilibrium and given an upward velocity of 2 ft/s. Find its displacement for $t > 0$.

10. A unit mass hangs in equilibrium from a spring with constant $k = 1/16$. Starting at $t = 0$, a force $F(t) = 3\sin t$ is applied to the mass. Find its displacement for $t > 0$.

11. A 4 lb weight stretches a spring 1 ft in equilibrium. An external force $F(t) = .25 \sin 8t$ lb is applied to the weight, which is initially displaced 4 inches above equilibrium and given a downward velocity of 1 ft/s. Find and graph its displacement for $t > 0$.

12. A 2 lb weight stretches a spring 6 inches in equilibrium. An external force $F(t) = \sin 8t$ lb is applied to the weight, which is released from rest 2 inches below equilibrium. Find its displacement for $t > 0$.

13. A 64 lb object stretches a spring 4 ft in equilibrium. A damping force is exerted with damping constant $c = 8$ lb-sec/ft. The object is initially displaced 18 inches above equilibrium and given a downward velocity of 4 ft/sec. Find its displacement and time–varying amplitude for $t > 0$.

14. A 16 lb weight is attached to a spring with natural length 5 ft. With the weight attached, the spring measures 8.2 ft. The weight is initially displaced 3 ft below equilibrium and given an upward velocity of 2 ft/sec. Find and graph its displacement for $t > 0$ if the medium resists the motion with a force of one lb for each ft/sec of velocity. Also, find its time–varying amplitude.

15. An 8 lb weight stretches a spring 1.5 inches. A damping force is exerted with damping constant c=8 lb-sec/ft. The weight is initially displaced 3 inches above equilibrium and given an upward velocity of 6 ft/sec. Find and graph its displacement for $t > 0$.

16. A 96 lb weight stretches a spring 3.2 ft in equilibrium. A damping force is exerted with damping constant c=18 lb-sec/ft. The weight is initially displaced 15 inches below equilibrium and given a downward velocity of 12 ft/sec. Find its displacement for $t > 0$.

17. An 8 lb weight stretches a spring .32 ft. The weight is initially displaced 6 inches above equilibrium and given an upward velocity of 4 ft/sec. Find its displacement for $t > 0$ if the medium exerts a damping force of 1.5 lb for each ft/sec of velocity.

18. A 32 lb weight stretches a spring 2 ft in equilibrium. A damping force is exerted with a constant $c = 8$ lb-sec/ft. The weight is initially displaced 8 inches below equilibrium and released from rest. Find its displacement for $t > 0$.

19. A mass of 20 gm stretches a spring 5 cm. A damping force is exerted with a constant 400 dyne sec/cm. Determine the displacement for $t > 0$ if the mass is initially displaced 9 cm above equilibrium and released from rest.

20. A 64 lb weight is suspended from a spring with constant $k = 25$ lb/ft. It is initially displaced 18 inches above equilibrium and released from rest. Find its displacement for $t > 0$ if the medium resists the motion with 6 lb of force for each ft/sec of velocity.

21. An 8 lb weight stretches a spring 2 inches. A damping force is exerted with a constant c=4 lb-sec/ft. The weight is initially displaced 3 inches above equilibrium and given a downward velocity of 4 ft/sec. Find its displacement for $t > 0$.

22. A 2 lb weight stretches a spring .32 ft. The weight is initially displaced 4 inches below equilibrium and given an upward velocity of 5 ft/sec. The medium provides damping with constant $c = 1/8$ lb-sec/ft. Find and graph the displacement for $t > 0$.

CHAPTER 4

SERIES SOLUTIONS OF SECOND ORDER EQUATIONS

IN THIS CHAPTER we study a class of second order differential equations that occur in many applications but do not possess solutions in terms of elementary functions. The equations considered in this chapter have variable coefficients that can be written in the form

$$P_2(x)y'' + P_1(x)y' + P_0(x)y = 0, \tag{A}$$

where P_2, P_1, and P_0 are polynomials with no common factor. We will see that if $P_2(0) \neq 0$, then solutions of (A) can be written as power series

$$y = \sum_{n=0}^{\infty} a_n x^n$$

that converge in an open interval centered at $x = 0$. For most equations that occur in applications, these polynomials are of degree two or less, so we will impose this restriction throughout the chapter.

SECTION 4.1 reviews the properties of power series.

SECTIONS 4.2 AND 4.3 are devoted to finding power series solutions of (A) in the case where $P_2(0) \neq 0$. The situation is more complicated if $P_2(0) = 0$; however, if P_1 and P_0 satisfy assumptions that apply to most equations of interest, then we are able to use a modified series method to obtain solutions of (A).

SECTION 4.4 introduces the appropriate assumptions on P_1 and P_0 in the case where $P_2(0) = 0$, and deals with *Cauchy–Euler equation*

$$ax^2y'' + bxy' + cy = 0,$$

where a, b, and c are constants. This is the simplest equation that satisfies these assumptions.

4.1 REVIEW OF POWER SERIES

Many applications give rise to differential equations with solutions that cannot be expressed in terms of elementary functions such as polynomials, rational functions, exponential and logarithmic functions, and trigonometric functions. However, the solutions of some of the most important of these equations can be expressed in terms of power series. We will study such equations in this chapter. In this section we review relevant properties of power series but will omit proofs, which can be found in any standard calculus text.

Definition 4.1.1 An infinite series of the form

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n, \tag{4.1.1}$$

where x_0 and $a_0, a_1, \dots, a_n, \dots$ are constants, is called a *power series in* $x - x_0$. We say that the power series (4.1.1) *converges* for a given x if the limit

$$\lim_{N\to\infty} \sum_{n=0}^{N} a_n(x-x_0)^n$$

exists; otherwise, we say that the power series *diverges* for the given x.

A power series in $x - x_0$ must converge if $x = x_0$, since the positive powers of $x - x_0$ are all zero in this case. This may be the only value of x for which the power series converges. However, the next theorem shows that if the power series converges for some $x \neq x_0$ then the set of all values of x for which it converges forms an interval.

Theorem 4.1.2 *For any power series*

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n,$$

exactly one of these three statements is true:

- **(i)** *The power series converges only for* $x = x_0$.
- **(ii)** *The power series converges for all values of* x.
- **(iii)** *There's a positive number* R *such that the power series converges if* $|x - x_0| < R$ *and diverges if* $|x - x_0| > R$.

In case (iii) we say that R is the *radius of convergence* of the power series. For convenience, we include the other two cases in this definition by defining $R = 0$ in case (i) and $R = \infty$ in case (ii). We define the *open interval of convergence* of $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ to be

$$(x_0 - R, x_0 + R) \quad \text{if} \quad 0 < R < \infty, \quad \text{or} \quad (-\infty, \infty) \quad \text{if} \quad R = \infty.$$

If R is finite, no general statement can be made concerning convergence at the endpoints $x = x_0 \pm R$ of the open interval of convergence; the series may converge at one or both points, or diverge at both.

Recall from calculus that a series of constants $\sum_{n=0}^{\infty} \alpha_n$ is said to *converge absolutely* if the series of absolute values $\sum_{n=0}^{\infty} |\alpha_n|$ converges. It can be shown that a power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ with a positive radius of convergence R converges absolutely in its open interval of convergence; that is, the series

$$\sum_{n=0}^{\infty} |a_n||x-x_0|^n$$

of absolute values converges if $|x-x_0| < R$. However, if $R < \infty$, the series may fail to converge absolutely at an endpoint $x_0 \pm R$, even if it converges there.

The next theorem provides a useful method for determining the radius of convergence of a power series. It is derived in calculus by applying the ratio test to the corresponding series of absolute values.

Theorem 4.1.3 *Suppose there is an integer N such that $a_n \neq 0$ if $n \geqslant N$ and*

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L,$$

where $0 \leqslant L \leqslant \infty$. Then the radius of convergence of $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ is $R = 1/L$, which should be interpreted to mean that $R = 0$ if $L = \infty$, or $R = \infty$ if $L = 0$.

Example 4.1.1 Find the radius of convergence of the series:

$$\textbf{(a)} \sum_{n=0}^{\infty} n!x^n \quad \textbf{(b)} \sum_{n=10}^{\infty} (-1)^n \frac{x^n}{n!} \quad \textbf{(c)} \sum_{n=0}^{\infty} 2^n n^2 (x-1)^n.$$

Solution (a) Here $a_n = n!$, so

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{(n+1)!}{n!} = \lim_{n\to\infty} (n+1) = \infty.$$

Hence, $R = 0$.

(b) Here $a_n = (1)^n/n!$ for $n \geqslant N = 10$, so

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{1}{n+1} = 0.$$

Hence, $R = \infty$.

(c) Here $a_n = 2^n n^2$, so

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{2^{n+1}(n+1)^2}{2^n n^2} = 2 \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^2 = 2.$$

Hence, $R = 1/2$. ■

Taylor Series

If a function f has derivatives of all orders at a point $x = x_0$, then the *Taylor series of f about* x_0 is defined by

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n.$$

In the special case where $x_0 = 0$, this series is also called the *Maclaurin series of* f.

Taylor series for most of the common elementary functions converge to the functions on their open intervals of convergence. For example, you are probably familiar with the following Maclaurin series:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad -\infty < x < \infty, \tag{4.1.2}$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad -\infty < x < \infty, \tag{4.1.3}$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \quad -\infty < x < \infty, \tag{4.1.4}$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad -1 < x < 1. \tag{4.1.5}$$

Differentiation of Power Series

A power series with a positive radius of convergence defines a function

$$f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n$$

on its open interval of convergence. We say that the series *represents* f on the open interval of convergence. A function f represented by a power series may be a familiar elementary function as in (4.1.2)–(4.1.5); however, it often happens that f is not a familiar function, so the series actually *defines* f.

The next theorem shows that a function represented by a power series has derivatives of all orders on the open interval of convergence of the power series. The theorem also provides power series representations of the derivatives.

Theorem 4.1.4 *A power series*

$$f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n$$

with positive radius of convergence R *has derivatives of all orders in its open interval of convergence, and successive derivatives can be obtained by repeatedly differentiating term by term; that is,*

$$f'(x) = \sum_{n=1}^{\infty} na_n(x-x_0)^{n-1}, \tag{4.1.6}$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1)a_n(x-x_0)^{n-2}, \tag{4.1.7}$$

$$\vdots$$

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)a_n(x-x_0)^{n-k}. \tag{4.1.8}$$

Moreover, all of these series have the same radius of convergence R.

Example 4.1.2 Let $f(x) = \sin x$. From (4.1.3),

$$f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

From (4.1.6), the derivative of $f(x)$ is

$$\sum_{n=0}^{\infty} (-1)^n \frac{d}{dx}\left[\frac{x^{2n+1}}{(2n+1)!}\right] = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!},$$

which is the series (4.1.4) for $\cos x$.

■

Uniqueness of Power Series

The next theorem shows that if f is *defined* by a power series in $x - x_0$ with a positive radius of convergence, then the power series is the Taylor series of f about x_0.

Theorem 4.1.5 *If the power series*

$$f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n$$

has a positive radius of convergence, then

$$a_n = \frac{f^{(n)}(x_0)}{n!}; \tag{4.1.9}$$

that is, $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ *is the Taylor series of* f *about* x_0.

The next theorem lists two important properties of power series that follow from Theorem 4.1.5.

Theorem 4.1.6

(a) *If*

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n = \sum_{n=0}^{\infty} b_n(x-x_0)^n$$

for all x in an open interval that contains x_0, then $a_n = b_n$ for $n = 0, 1, 2, \ldots$.

(b) *If*

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n = 0$$

for all x in an open interval that contains x_0, then $a_n = 0$ for $n = 0, 1, 2, \ldots$.

Shifting the Summation Index

In Definition 4.1.1 of a power series in $x - x_0$, the n-th term is a constant multiple of $(x - x_0)^n$. This is not true in (4.1.6), (4.1.7), and (4.1.8), where the general terms are constant multiples of $(x - x_0)^{n-1}$, $(x - x_0)^{n-2}$, and $(x - x_0)^{n-k}$, respectively. However, these series can all be rewritten so that their n-th terms are constant multiples of $(x-x_0)^n$. For example, letting $n = k + 1$ in the series in (4.1.6) yields

$$f'(x) = \sum_{k=0}^{\infty} (k+1)a_{k+1}(x-x_0)^k, \tag{4.1.10}$$

where we start the new summation index k from zero so that the first term in (4.1.10) (obtained by setting $k = 0$) is the same as the first term in (4.1.6) (obtained by setting $n = 1$). However, the sum of a series is independent of the symbol used to denote the summation index, just as the value of a definite integral is independent of the symbol used to denote the variable of integration. Therefore we can replace k by n in (4.1.10) to obtain

$$f'(x) = \sum_{n=0}^{\infty} (n+1)a_{n+1}(x-x_0)^n, \tag{4.1.11}$$

where the general term is a constant multiple of $(x - x_0)^n$.

It is not necessary to introduce the intermediate summation index k. We can obtain (4.1.11) directly from (4.1.6) by replacing n by $n + 1$ in the general term of (4.1.6) and subtracting 1 from the lower limit of (4.1.6). More generally, we use the following procedure for shifting indices.

Shifting the Summation Index in a Power Series

For any integer k, the power series

$$\sum_{n=n_0}^{\infty} b_n(x-x_0)^{n-k}$$

can be rewritten as

$$\sum_{n=n_0-k}^{\infty} b_{n+k}(x-x_0)^{n}.$$

In words, replacing n by $n+k$ in the general term and subtracting k from the lower limit of summation leaves the series unchanged.

Example 4.1.3 Rewrite the power series from (4.1.7) and (4.1.8) so that the general term in each is a constant multiple of $(x-x_0)^n$:

(a) $\displaystyle\sum_{n=2}^{\infty} n(n-1)a_n(x-x_0)^{n-2}$ **(b)** $\displaystyle\sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)a_n(x-x_0)^{n-k}.$

Solution (a) Replacing n by $n+2$ in the general term and subtracting 2 from the lower limit of summation yields

$$\sum_{n=2}^{\infty} n(n-1)a_n(x-x_0)^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}(x-x_0)^{n}.$$

(b) Replacing n by $n+k$ in the general term and subtracting k from the lower limit of summation yields

$$\sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)a_n(x-x_0)^{n-k} = \sum_{n=0}^{\infty} (n+k)(n+k-1)\cdots(n+1)a_{n+k}(x-x_0)^{n}.$$

■

Example 4.1.4 Given that

$$f(x) = \sum_{n=0}^{\infty} a_n x^n,$$

write the function xf'' as a power series in which the general term is a constant multiple of x^n.

Solution From Theorem 4.1.4 with $x_0 = 0$,

$$f''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}.$$

Therefore

$$xf''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-1}.$$

Replacing n by $n+1$ in the general term and subtracting 1 from the lower limit of summation yields

$$xf''(x) = \sum_{n=1}^{\infty} (n+1)na_{n+1}x^n.$$

We can also write this as

$$xf''(x) = \sum_{n=0}^{\infty} (n+1)na_{n+1}x^n,$$

since the first term in this last series is zero. However, we will see later that sometimes it is useful to include zero terms at the beginning of a series. ■

Linear Combinations of Power Series

If a power series is multiplied by a constant, then the constant can be placed inside the summation; that is,

$$c\sum_{n=0}^{\infty} a_n(x-x_0)^n = \sum_{n=0}^{\infty} ca_n(x-x_0)^n.$$

Two power series

$$f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n \quad \text{and} \quad g(x) = \sum_{n=0}^{\infty} b_n(x-x_0)^n$$

with positive radii of convergence can be added term by term at points common to their open intervals of convergence; thus, if the first series converges for $|x-x_0| < R_1$ and the second converges for $|x-x_0| < R_2$, then

$$f(x) + g(x) = \sum_{n=0}^{\infty} (a_n + b_n)(x-x_0)^n$$

for $|x-x_0| < R$, where R is the smaller of R_1 and R_2. More generally, linear combinations of power series can be formed term by term; for example,

$$c_1f(x) + c_2f(x) = \sum_{n=0}^{\infty} (c_1a_n + c_2b_n)(x-x_0)^n.$$

Example 4.1.5 Find the Maclaurin series for $\cosh x$ as a linear combination of the Maclaurin series for e^x and e^{-x}.

Solution By definition,

$$\cosh x = \frac{1}{2}e^x + \frac{1}{2}e^{-x}.$$

Since

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{and} \quad e^{-x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!},$$

it follows that

$$\cosh x = \sum_{n=0}^{\infty} \frac{1}{2}[1 + (-1)^n] \frac{x^n}{n!}. \tag{4.1.12}$$

Since

$$\frac{1}{2}[1 + (-1)^n] = \begin{cases} 1 & \text{if } n = 2m, \text{ an even integer,} \\ 0 & \text{if } n = 2m+1, \text{ an odd integer,} \end{cases}$$

we can rewrite (4.1.12) more simply as

$$\cosh x = \sum_{m=0}^{\infty} \frac{x^{2m}}{(2m)!}.$$

This result is valid on $(-\infty, \infty)$, since this is the open interval of convergence of the Maclaurin series for e^x and e^{-x}. ■

Example 4.1.6 Suppose

$$y = \sum_{n=0}^{\infty} a_n x^n$$

on an open interval I that contains the origin.

(a) Express

$$(2 - x)y'' + 2y$$

as a power series in x on I.

(b) Use the result of **(a)** to find necessary and sufficient conditions on the coefficients $\{a_n\}$ for y to be a solution of the homogeneous equation

$$(2 - x)y'' + 2y = 0 \tag{4.1.13}$$

on I.

Solution (a) From (4.1.7) with $x_0 = 0$,

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}.$$

Therefore

$$\begin{aligned}(2-x)y''+2y &= 2y''-xy'+2y\\ &= \sum_{n=2}^{\infty} 2n(n-1)a_nx^{n-2}-\sum_{n=2}^{\infty} n(n-1)a_nx^{n-1}+\sum_{n=0}^{\infty} 2a_nx^n.\end{aligned} \tag{4.1.14}$$

To combine the three series we shift indices in the first two to make their general terms constant multiples of x^n; thus,

$$\sum_{n=2}^{\infty} 2n(n-1)a_nx^{n-2}=\sum_{n=0}^{\infty} 2(n+2)(n+1)a_{n+2}x^n \tag{4.1.15}$$

and

$$\sum_{n=2}^{\infty} n(n-1)a_nx^{n-1}=\sum_{n=1}^{\infty}(n+1)na_{n+1}x^n. \tag{4.1.16}$$

Notice that we can add a zero term to the series in (4.1.16) by changing the lower index of summation so that when we substitute (4.1.15) and (4.1.16) into (4.1.14), all three series will start with $n=0$. The result is then

$$(2-x)y''+2y=\sum_{n=0}^{\infty}[2(n+2)(n+1)a_{n+2}-(n+1)na_{n+1}+2a_n]x^n. \tag{4.1.17}$$

(b) From (4.1.17) we see that y satisfies (4.1.13) on I if

$$2(n+2)(n+1)a_{n+2}-(n+1)na_{n+1}+2a_n=0,\quad n=0,1,2,\dots. \tag{4.1.18}$$

Conversely, Theorem 4.1.6 **(b)** implies that if $y=\sum_{n=0}^{\infty}a_nx^n$ satisfies (4.1.13) on I, then (4.1.18) holds. ■

Example 4.1.7 Suppose

$$y=\sum_{n=0}^{\infty}a_n(x-1)^n$$

on an open interval I that contains $x_0=1$. Express the function

$$(1+x)y''+2(x-1)^2y'+3y \tag{4.1.19}$$

as a power series in $x-1$ on I.

Solution Since we want a power series in $x-1$, we rewrite the coefficient of y'' in (4.1.19) as $1+x=2+(x-1)$, so (4.1.19) becomes

$$2y''+(x-1)y''+2(x-1)^2y'+3y.$$

From (4.1.6) and (4.1.7) with $x_0 = 1$,

$$y' = \sum_{n=1}^{\infty} na_n(x-1)^{n-1} \quad \text{and} \quad y'' = \sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2}.$$

At this point, we have constructed four series.

$$\begin{aligned}
2y'' &= \sum_{n=2}^{\infty} 2n(n-1)a_n(x-1)^{n-2} \\
(x-1)y'' &= \sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-1} \\
2(x-1)^2y' &= \sum_{n=1}^{\infty} 2na_n(x-1)^{n+1} \\
3y &= \sum_{n=0}^{\infty} 3a_n(x-1)^n
\end{aligned}$$

Before adding these four series, we shift indices in the first three so that their general terms become constant multiples of $(x-1)^n$. The four series now look like this.

$$2y'' = \sum_{n=0}^{\infty} 2(n+2)(n+1)a_{n+2}(x-1)^n \tag{4.1.20}$$

$$(x-1)y'' = \sum_{n=0}^{\infty} (n+1)na_{n+1}(x-1)^n \tag{4.1.21}$$

$$2(x-1)^2y' = \sum_{n=1}^{\infty} 2(n-1)a_{n-1}(x-1)^n \tag{4.1.22}$$

$$3y = \sum_{n=0}^{\infty} 3a_n(x-1)^n \tag{4.1.23}$$

Notice that we added initial zero terms to the series in (4.1.21) and (4.1.22). Adding (4.1.20) – (4.1.23) yields

$$\begin{aligned}
(1+x)y'' + 2(x-1)^2y' + 3y &= 2y'' + (x-1)y'' + 2(x-1)^2y' + 3y \\
&= \sum_{n=0}^{\infty} b_n(x-1)^n,
\end{aligned}$$

where

$$b_0 = 4a_2 + 3a_0, \tag{4.1.24}$$

$$b_n = 2(n+2)(n+1)a_{n+2} + (n+1)na_{n+1} + 2(n-1)a_{n-1} + 3a_n, \; n \geqslant 1. \tag{4.1.25}$$

The formula (4.1.24) for b_0 cannot be obtained by setting $n = 0$ in (4.1.25), since the summation in (4.1.22) begins with $n = 1$, while those in (4.1.20), (4.1.21), and (4.1.23) begin with $n = 0$. ■

4.1 Exercises

1. For each power series, use Theorem 4.1.3 to find the radius of convergence R. If $R > 0$, find the open interval of convergence.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n}(x-1)^n$ **(b)** $\displaystyle\sum_{n=0}^{\infty} 2^n n(x-2)^n$

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{9^n}x^n$ **(d)** $\displaystyle\sum_{n=0}^{\infty} \frac{n(n+1)}{16^n}(x-2)^n$

(e) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{7^n}{n!}x^n$ **(f)** $\displaystyle\sum_{n=0}^{\infty} \frac{3^n}{4^{n+1}(n+1)^2}(x+7)^n$

In Exercises 2–6 find a power series solution $y(x) = \sum_{n=0}^{\infty} a_n x^n$.

2. $(2+x)y'' + xy' + 3y$ **3.** $(1+3x^2)y'' + 3x^2y' - 2y$

4. $(1+2x^2)y'' + (2-3x)y' + 4y$ **5.** $(1+x^2)y'' + (2-x)y' + 3y$

6. $(1+3x^2)y'' - 2xy' + 4y$

7. Suppose $y(x) = \sum_{n=0}^{\infty} a_n(x+1)^n$ on an open interval that contains $x_0 = -1$. Find a power series in $x+1$ for

$$xy'' + (4+2x)y' + (2+x)y.$$

8. Suppose $y(x) = \sum_{n=0}^{\infty} a_n(x-2)^n$ on an open interval that contains $x_0 = 2$. Find a power series in $x-2$ for

$$x^2y'' + 2xy' - 3xy.$$

9. Suppose the series $\sum_{n=0}^{\infty} a_n x^n$ converges on an open interval $(-R, R)$, let r be an arbitrary real number, and define

$$y(x) = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r}$$

on $(0, R)$. Use Theorem 4.1.4 and the rule for differentiating the product of two functions to show that

$$\begin{aligned} y'(x) &= \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1}, \\ y''(x) &= \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2}, \\ &\vdots \\ y^{(k)}(x) &= \sum_{n=0}^{\infty} (n+r)(n+r-1)\cdots(n+r-k)a_n x^{n+r-k} \end{aligned}$$

on $(0, R)$.

In Exercises 10–15 *let* y *be as defined in Exercise* 9*, and write the given expression in the form* $x^r \sum_{n=0}^{\infty} b_n x^n$.

10. $x^2(1-x)y'' + x(4+x)y' + (2-x)y$

11. $x^2(1+x)y'' + x(1+2x)y' - (4+6x)y$

12. $x^2(1+x)y'' - x(1-6x-x^2)y' + (1+6x+x^2)y$

13. $x^2(1+3x)y'' + x(2+12x+x^2)y' + 2x(3+x)y$

14. $x^2(1+2x^2)y'' + x(4+2x^2)y' + 2(1-x^2)y$

15. $x^2(2+x^2)y'' + 2x(5+x^2)y' + 2(3-x^2)y$

4.2 SERIES SOLUTIONS NEAR AN ORDINARY POINT I

Many physical applications give rise to second order homogeneous linear differential equations of the form

$$P_2(x)y'' + P_1(x)y' + P_0(x)y = 0, \tag{4.2.1}$$

where P_2, P_1, and P_0 are polynomials. Some examples are: Airy's equation,

$$y'' - xy = 0,$$

which occurs in astronomy and quantum physics; Bessel's equation,

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0,$$

which occurs in problems displaying cylindrical symmetry such as diffraction of light through a circular aperture, propagation of electromagnetic radiation through a coaxial cable, and vibrations of a circular drum head; and Legendre's equation,

$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0,$$

which occurs in problems displaying spherical symmetry (particularly in electromagnetism). Usually the solutions of these types of equations cannot be expressed in terms of familiar elementary functions. Therefore we will consider the problem of representing solutions of (4.2.1) with series.

We assume throughout that P_2, P_1 and P_0 have no common factors. Then we say that x_0 is an *ordinary point* of (4.2.1) if $P_2(x_0) \neq 0$, or a *singular point* if $P_2(x_0) = 0$. For Legendre's equation,

$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0, \tag{4.2.2}$$

$x_0 = 1$ and $x_0 = -1$ are singular points and all other points are ordinary points. For Bessel's equation,

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0,$$

$x_0 = 0$ is a singular point and all other points are ordinary points. If P_2 is a nonzero constant as in Airy's equation,

$$y'' - xy = 0, \tag{4.2.3}$$

then every point is an ordinary point.

Since polynomials are continuous everywhere, P_1/P_2 and P_0/P_2 are continuous at any point x_0 that is not a zero of P_2. Therefore, if x_0 is an ordinary point of (4.2.1) and a_0 and a_1 are arbitrary real numbers, then the initial value problem

$$P_2(x)y'' + P_1(x)y' + P_0(x)y = 0, \quad y(x_0) = a_0, \quad y'(x_0) = a_1 \tag{4.2.4}$$

has a unique solution on the largest open interval that contains x_0 and does not contain any zeros of P_2. To see this, we rewrite the differential equation in (4.2.4) as

$$y'' + \frac{P_1(x)}{P_2(x)}y' + \frac{P_0(x)}{P_2(x)}y = 0$$

and apply Theorem 3.1.1 with $p = P_1/P_2$ and $q = P_0/P_2$. In this section and the next we consider the problem of representing solutions of (4.2.1) by power series that converge for values of x near an ordinary point x_0.

We state the next theorem without proof.

Theorem 4.2.1 *Suppose P_0, P_1, and P_2 are polynomials with no common factor and P_2 is not identically zero. Let x_0 be a point such that $P_2(x_0) \neq 0$, and let ρ be the distance from x_0 to the nearest zero of P_2 in the complex plane. (If P_2 is constant, then $\rho = \infty$.) Then every solution of*

$$P_2(x)y'' + P_1(x)y' + P_0(x)y = 0 \tag{4.2.5}$$

can be represented by a power series

$$y = \sum_{n=0}^{\infty} a_n(x - x_0)^n \tag{4.2.6}$$

that converges at least on the open interval $(x_0 - \rho, x_0 + \rho)$. (If P_2 is nonconstant, so that ρ is necessarily finite, then the open interval of convergence of (4.2.6) may be larger than $(x_0 - \rho, x_0 + \rho)$. If P_2 is constant then $\rho = \infty$ and $(x_0 - \rho, x_0 + \rho) = (-\infty, \infty)$.)

We call (4.2.6) a *power series solution in $x - x_0$* of (4.2.5). We will now develop a method for finding power series solutions of (4.2.5). For this purpose we write (4.2.5) as $Ly = 0$, where

$$Ly = P_2y'' + P_1y' + P_0y. \tag{4.2.7}$$

Theorem 4.2.1 implies that every solution of $Ly = 0$ on $(x_0 - \rho, x_0 + \rho)$ can be written as

$$y = \sum_{n=0}^{\infty} a_n(x - x_0)^n.$$

Setting $x = x_0$ in this series and in the series

$$y' = \sum_{n=1}^{\infty} na_n(x - x_0)^{n-1}$$

shows that $y(x_0) = a_0$ and $y'(x_0) = a_1$. Since every initial value problem (4.2.4) has a unique solution, this means that a_0 and a_1 can be chosen arbitrarily, and $a_2, a_3, \dots$ are uniquely determined by them.

To find $a_2, a_3, \dots,$ first write P_0, P_1, and P_2 in powers of $x - x_0$, then substitute

$$y = \sum_{n=0}^{\infty} a_n(x - x_0)^n,$$

$$y' = \sum_{n=1}^{\infty} na_n(x - x_0)^{n-1},$$

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n(x - x_0)^{n-2}$$

into (4.2.7) and collect the coefficients of like powers of $x - x_0$. This yields

$$Ly = \sum_{n=0}^{\infty} b_n(x - x_0)^n, \tag{4.2.8}$$

where $\{b_0, b_1, \dots, b_n, \dots\}$ are expressed in terms of $\{a_0, a_1, \dots, a_n, \dots\}$ and the coefficients of P_0, P_1, and P_2, written in powers of $x - x_0$. Since (4.2.8) and **(a)** of Theorem 4.1.6 imply that $Ly = 0$ if and only if $b_n = 0$ for $n \geqslant 0$, all power series solutions in $x - x_0$ of $Ly = 0$ can be obtained by choosing a_0 and a_1 arbitrarily and computing $a_2, a_3, \dots,$ successively so that $b_n = 0$ for $n \geqslant 0$. For simplicity, we call the power series obtained this way *the power series in $x - x_0$ for the general solution* of $Ly = 0$, without explicitly identifying the open interval of convergence of the series.

Example 4.2.1 Let x_0 be an arbitrary real number. Find the power series in $x - x_0$ for the general solution of

$$y'' + y = 0. \tag{4.2.9}$$

Solution Here

$$Ly = y'' + y.$$

If

$$y = \sum_{n=0}^{\infty} a_n(x - x_0)^n,$$

then

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n(x - x_0)^{n-2},$$

so

$$Ly = \sum_{n=2}^{\infty} n(n-1)a_n(x-x_0)^{n-2} + \sum_{n=0}^{\infty} a_n(x-x_0)^n.$$

To collect coefficients of like powers of $x - x_0$, we shift the summation index in the first sum. This yields

$$Ly = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x-x_0)^n + \sum_{n=0}^{\infty} a_n(x-x_0)^n = \sum_{n=0}^{\infty} b_n(x-x_0)^n,$$

with

$$b_n = (n+2)(n+1)a_{n+2} + a_n.$$

Therefore $Ly = 0$ if and only if

$$a_{n+2} = \frac{-a_n}{(n+2)(n+1)}, \quad n \geqslant 0, \tag{4.2.10}$$

where a_0 and a_1 are arbitrary. Since the indices on the left and right sides of (4.2.10) differ by two, we write (4.2.10) separately for n even $(n = 2m)$ and n odd $(n = 2m+1)$. This yields

$$a_{2m+2} = \frac{-a_{2m}}{(2m+2)(2m+1)}, \quad m \geqslant 0, \tag{4.2.11}$$

and

$$a_{2m+3} = \frac{-a_{2m+1}}{(2m+3)(2m+2)}, \quad m \geqslant 0. \tag{4.2.12}$$

Computing the coefficients of the even powers of $x - x_0$ from (4.2.11) yields

$$\begin{aligned}
a_2 &= -\frac{a_0}{2\cdot 1} \\
a_4 &= -\frac{a_2}{4\cdot 3} = -\frac{1}{4\cdot 3}\left(-\frac{a_0}{2\cdot 1}\right) = \frac{a_0}{4\cdot 3\cdot 2\cdot 1}, \\
a_6 &= -\frac{a_4}{6\cdot 5} = -\frac{1}{6\cdot 5}\left(\frac{a_0}{4\cdot 3\cdot 2\cdot 1}\right) = -\frac{a_0}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1},
\end{aligned}$$

and, in general,

$$a_{2m} = (-1)^m \frac{a_0}{(2m)!}, \quad m \geqslant 0. \tag{4.2.13}$$

Computing the coefficients of the odd powers of $x - x_0$ from (4.2.12) yields

$$\begin{aligned}
a_3 &= -\frac{a_1}{3\cdot 2} \\
a_5 &= -\frac{a_3}{5\cdot 4} = -\frac{1}{5\cdot 4}\left(-\frac{a_1}{3\cdot 2}\right) = \frac{a_1}{5\cdot 4\cdot 3\cdot 2}, \\
a_7 &= -\frac{a_5}{7\cdot 6} = -\frac{1}{7\cdot 6}\left(\frac{a_1}{5\cdot 4\cdot 3\cdot 2}\right) = -\frac{a_1}{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2},
\end{aligned}$$

and, in general,

$$a_{2m+1} = \frac{(-1)^m a_1}{(2m+1)!} \quad m \geqslant 0. \tag{4.2.14}$$

Thus, the general solution of (4.2.9) can be written as

$$y = \sum_{m=0}^{\infty} a_{2m}(x-x_0)^{2m} + \sum_{m=0}^{\infty} a_{2m+1}(x-x_0)^{2m+1},$$

or, from (4.2.13) and (4.2.14), as

$$y = a_0 \sum_{m=0}^{\infty} (-1)^m \frac{(x-x_0)^{2m}}{(2m)!} + a_1 \sum_{m=0}^{\infty} (-1)^m \frac{(x-x_0)^{2m+1}}{(2m+1)!}. \tag{4.2.15}$$

If we recall from calculus that

$$\sum_{m=0}^{\infty} (-1)^m \frac{(x-x_0)^{2m}}{(2m)!} = \cos(x-x_0) \quad \text{and} \quad \sum_{m=0}^{\infty} (-1)^m \frac{(x-x_0)^{2m+1}}{(2m+1)!} = \sin(x-x_0),$$

then (4.2.15) becomes

$$y = a_0 \cos(x-x_0) + a_1 \sin(x-x_0),$$

which should look familiar. ■

Equations like (4.2.10), (4.2.11), and (4.2.12), which define a given coefficient in the sequence $\{a_n\}$ in terms of one or more coefficients with lesser indices are called *recurrence relations*.

In the remainder of this section, we consider the problem of finding power series solutions in $x - x_0$ for equations of the form

$$\left(1 + \alpha(x-x_0)^2\right) y'' + \beta(x-x_0)y' + \gamma y = 0. \tag{4.2.16}$$

Many important equations that arise in applications are of this form with $x_0 = 0$, including Legendre's equation (4.2.2) and Airy's equation (4.2.3).

Since

$$P_2(x) = 1 + \alpha(x-x_0)^2$$

in (4.2.16), the point x_0 is an ordinary point of (4.2.16), and Theorem 4.2.1 implies that the solutions of (4.2.16) can be written as power series in $x - x_0$ that converge on the interval $(x_0 - 1/\sqrt{|\alpha|}, x_0 + 1/\sqrt{|\alpha|})$ if $\alpha \neq 0$, or on $(-\infty, \infty)$ if $\alpha = 0$. We will see that the coefficients in these power series can be obtained by methods similar to the one used in Example 4.2.1.

To simplify finding the coefficients, we introduce some notation for products:

$$\prod_{j=r}^{s} b_j = b_r b_{r+1} \cdots b_s \quad \text{if} \quad s \geqslant r.$$

Thus,

$$\prod_{j=2}^{7} b_j = b_2 b_3 b_4 b_5 b_6 b_7,$$

$$\prod_{j=0}^{4} (2j+1) = (1)(3)(5)(7)(9) = 945,$$

and

$$\prod_{j=2}^{2} j^2 = 2^2 = 4.$$

We define

$$\prod_{j=r}^{s} b_j = 1 \quad \text{if} \quad s < r,$$

no matter what the form of b_j.

Example 4.2.2 Find the power series in x for the general solution of

$$(1+2x^2)y'' + 6xy' + 2y = 0. \tag{4.2.17}$$

Solution Here

$$Ly = (1+2x^2)y'' + 6xy' + 2y.$$

If

$$y = \sum_{n=0}^{\infty} a_n x^n$$

then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2},$$

so

$$\begin{aligned} Ly &= (1+2x^2) \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + 6x \sum_{n=1}^{\infty} n a_n x^{n-1} + 2 \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} \left[2n(n-1) + 6n + 2\right] a_n x^n \\ &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + 2 \sum_{n=0}^{\infty} (n+1)^2 a_n x^n. \end{aligned}$$

To collect coefficients of x^n, we shift the summation index in the first sum. Ly is now

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + 2 \sum_{n=0}^{\infty} (n+1)^2 a_n x^n = \sum_{n=0}^{\infty} b_n x^n,$$

with

$$b_n = (n+2)(n+1)a_{n+2} + 2(n+1)^2 a_n, \quad n \geqslant 0.$$

To obtain solutions of (4.2.17), we set $b_n = 0$ for $n \geqslant 0$. This is equivalent to the recurrence relation

$$a_{n+2} = -2\frac{n+1}{n+2}a_n, \quad n \geqslant 0. \tag{4.2.18}$$

Since the indices on the left and right differ by two, we write (4.2.18) separately for $n = 2m$ and $n = 2m+1$, as in Example 4.2.1. This yields

$$a_{2m+2} = -2\frac{2m+1}{2m+2}a_{2m} = -\frac{2m+1}{m+1}a_{2m}, \quad m \geqslant 0, \tag{4.2.19}$$

and

$$a_{2m+3} = -2\frac{2m+2}{2m+3}a_{2m+1} = -4\frac{m+1}{2m+3}a_{2m+1}, \quad m \geqslant 0. \tag{4.2.20}$$

Computing the coefficients of even powers of x from (4.2.19) yields

$$\begin{aligned}
a_2 &= -\frac{1}{1}a_0, \\
a_4 &= -\frac{3}{2}a_2 = \left(-\frac{3}{2}\right)\left(-\frac{1}{1}\right)a_0 = \frac{1\cdot 3}{1\cdot 2}a_0, \\
a_6 &= -\frac{5}{3}a_4 = -\frac{5}{3}\left(\frac{1\cdot 3}{1\cdot 2}\right)a_0 = -\frac{1\cdot 3\cdot 5}{1\cdot 2\cdot 3}a_0, \\
a_8 &= -\frac{7}{4}a_6 = -\frac{7}{4}\left(-\frac{1\cdot 3\cdot 5}{1\cdot 2\cdot 3}\right)a_0 = \frac{1\cdot 3\cdot 5\cdot 7}{1\cdot 2\cdot 3\cdot 4}a_0.
\end{aligned}$$

In general,

$$a_{2m} = (-1)^m \frac{\prod_{j=1}^{m}(2j-1)}{m!}a_0, \quad m \geqslant 0. \tag{4.2.21}$$

(Note that (4.2.21) is correct for $m = 0$ because we defined $\prod_{j=1}^{0} b_j = 1$ for any b_j.)

Computing the coefficients of odd powers of x from (4.2.20) yields

$$\begin{aligned}
a_3 &= -4\,\frac{1}{3}a_1, \\
a_5 &= -4\,\frac{2}{5}a_3 = -4\,\frac{2}{5}\left(-4\frac{1}{3}\right)a_1 = 4^2\frac{1\cdot 2}{3\cdot 5}a_1, \\
a_7 &= -4\,\frac{3}{7}a_5 = -4\,\frac{3}{7}\left(4^2\frac{1\cdot 2}{3\cdot 5}\right)a_1 = -4^3\frac{1\cdot 2\cdot 3}{3\cdot 5\cdot 7}a_1, \\
a_9 &= -4\,\frac{4}{9}a_7 = -4\,\frac{4}{9}\left(4^3\frac{1\cdot 2\cdot 3}{3\cdot 5\cdot 7}\right)a_1 = 4^4\frac{1\cdot 2\cdot 3\cdot 4}{3\cdot 5\cdot 7\cdot 9}a_1.
\end{aligned}$$

In general,

$$a_{2m+1} = \frac{(-1)^m 4^m m!}{\prod_{j=1}^{m}(2j+1)} a_1, \quad m \geq 0. \tag{4.2.22}$$

From (4.2.21) and (4.2.22),

$$y = a_0 \sum_{m=0}^{\infty} (-1)^m \frac{\prod_{j=1}^{m}(2j-1)}{m!} x^{2m} + a_1 \sum_{m=0}^{\infty} (-1)^m \frac{4^m m!}{\prod_{j=1}^{m}(2j+1)} x^{2m+1}.$$

is the power series in x for the general solution of (4.2.17). Since $P_2(x) = 1 + 2x^2$ has no real zeros, Theorem 3.1.1 implies that every solution of (4.2.17) is defined on $(-\infty, \infty)$. However, since $P_2(\pm i/\sqrt{2}) = 0$, Theorem 4.2.1 implies only that the power series converges in $(-1/\sqrt{2}, 1/\sqrt{2})$ for any choice of a_0 and a_1.

The results in Examples 4.2.1 and 4.2.2 are consequences of the following general theorem.

Theorem 4.2.2 *The coefficients $\{a_n\}$ in any solution $y = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ of*

$$\left(1 + \alpha(x - x_0)^2\right) y'' + \beta(x - x_0)y' + \gamma y = 0 \tag{4.2.23}$$

satisfy the recurrence relation

$$a_{n+2} = -\frac{p(n)}{(n+2)(n+1)} a_n, \quad n \geq 0, \tag{4.2.24}$$

where

$$p(n) = \alpha n(n-1) + \beta n + \gamma. \tag{4.2.25}$$

Moreover, the coefficients of the even and odd powers of $x - x_0$ can be computed separately as

$$a_{2m+2} = -\frac{p(2m)}{(2m+2)(2m+1)} a_{2m}, \quad m \geq 0 \tag{4.2.26}$$

and

$$a_{2m+3} = -\frac{p(2m+1)}{(2m+3)(2m+2)} a_{2m+1}, \quad m \geq 0, \tag{4.2.27}$$

where a_0 and a_1 are arbitrary.

Proof Here

$$Ly = (1 + \alpha(x - x_0)^2)y'' + \beta(x - x_0)y' + \gamma y.$$

If

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n,$$

then

$$y' = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1} \quad \text{and} \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n (x - x_0)^{n-2}.$$

Hence,

$$\begin{aligned} Ly &= \sum_{n=2}^{\infty} n(n-1)a_n(x-x_0)^{n-2} + \sum_{n=0}^{\infty} \left[\alpha n(n-1)+\beta n+\gamma\right] a_n(x-x_0)^n \\ &= \sum_{n=2}^{\infty} n(n-1)a_n(x-x_0)^{n-2} + \sum_{n=0}^{\infty} p(n)a_n(x-x_0)^n, \end{aligned}$$

from (4.2.25). To collect coefficients of powers of $x-x_0$, we shift the summation index in the first sum. This yields

$$Ly = \sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} + p(n)a_n\right](x-x_0)^n.$$

Thus, $Ly = 0$ if and only if

$$(n+2)(n+1)a_{n+2} + p(n)a_n = 0, \quad n \geqslant 0,$$

which is equivalent to (4.2.24). Writing (4.2.24) separately for the cases where $n = 2m$ and $n = 2m+1$ yields (4.2.26) and (4.2.27). ■

Example 4.2.3 Find the power series in $x-1$ for the general solution of

$$(2+4x-2x^2)y'' - 12(x-1)y' - 12y = 0. \tag{4.2.28}$$

Solution We must first write the coefficient $P_2(x) = 2+4x-x^2$ in powers of $x-1$. To do this, we write $x = (x-1)+1$ in $P_2(x)$ and then expand the terms, collecting powers of $x-1$; thus,

$$\begin{aligned} 2+4x-2x^2 &= 2+4[(x-1)+1]-2[(x-1)+1]^2 \\ &= 4-2(x-1)^2. \end{aligned}$$

Therefore we can rewrite (4.2.28) as

$$\left(4-2(x-1)^2\right)y'' - 12(x-1)y' - 12y = 0,$$

or, equivalently,

$$\left(1-\frac{1}{2}(x-1)^2\right)y'' - 3(x-1)y' - 3y = 0.$$

This is of the form (4.2.23) with $\alpha = -1/2$, $\beta = -3$, and $\gamma = -3$. Therefore, from (4.2.25)

$$p(n) = -\frac{n(n-1)}{2} - 3n - 3 = -\frac{(n+2)(n+3)}{2}.$$

Hence, Theorem 4.2.2 implies that

$$\begin{aligned} a_{2m+2} &= -\frac{p(2m)}{(2m+2)(2m+1)}a_{2m} \\ &= \frac{(2m+2)(2m+3)}{2(2m+2)(2m+1)}a_{2m} = \frac{2m+3}{2(2m+1)}a_{2m}, \quad m \geqslant 0 \end{aligned}$$

and

$$\begin{aligned} a_{2m+3} &= -\frac{p(2m+1)}{(2m+3)(2m+2)}a_{2m+1} \\ &= \frac{(2m+3)(2m+4)}{2(2m+3)(2m+2)}a_{2m+1} = \frac{m+2}{2(m+1)}a_{2m+1}, \quad m \geqslant 0. \end{aligned}$$

We leave it to you to show that

$$a_{2m} = \frac{2m+1}{2^m}a_0 \quad \text{and} \quad a_{2m+1} = \frac{m+1}{2^m}a_1, \quad m \geqslant 0,$$

which implies that the power series in $x-1$ for the general solution of (4.2.28) is

$$y = a_0\sum_{m=0}^{\infty}\frac{2m+1}{2^m}(x-1)^{2m} + a_1\sum_{m=0}^{\infty}\frac{m+1}{2^m}(x-1)^{2m+1}. \blacksquare$$

In the examples considered so far we were able to express the coefficients in the power series solutions by using summation notation. In some cases this is impossible, and we must settle for computing a finite number of terms in the series. The next example illustrates this with an initial value problem.

Example 4.2.4 Compute a_0, a_1, ..., a_7 in the series solution $y = \sum_{n=0}^{\infty} a_n x^n$ of the initial value problem

$$(1+2x^2)y'' + 10xy' + 8y = 0, \quad y(0) = 2, \quad y'(0) = -3. \tag{4.2.29}$$

Solution Since $\alpha = 2$, $\beta = 10$, and $\gamma = 8$ in (4.2.29),

$$p(n) = 2n(n-1) + 10n + 8 = 2(n+2)^2.$$

Therefore

$$a_{n+2} = -2\frac{(n+2)^2}{(n+2)(n+1)}a_n = -2\frac{n+2}{n+1}a_n, \quad n \geqslant 0.$$

Writing this equation separately for $n = 2m$ and $n = 2m+1$ yields

$$a_{2m+2} = -2\frac{(2m+2)}{2m+1}a_{2m} = -4\frac{m+1}{2m+1}a_{2m}, \quad m \geqslant 0 \tag{4.2.30}$$

and

$$a_{2m+3} = -2\frac{2m+3}{2m+2}a_{2m+1} = -\frac{2m+3}{m+1}a_{2m+1}, \quad m \geqslant 0. \tag{4.2.31}$$

From the initial condition for the function, we start with $a_0 = 2$ and then we compute a_2, a_4, and a_6 from (4.2.30):

$$\begin{aligned} a_2 &= -4\,\frac{1}{1}2 = -8, \\ a_4 &= -4\,\frac{2}{3}(-8) = \frac{64}{3}, \\ a_6 &= -4\,\frac{3}{5}\left(\frac{64}{3}\right) = -\frac{256}{5}. \end{aligned}$$

Based on the initial condition for the derivative of the function, we start with $a_1 = -3$ and compute a_3, a_5 and a_7 from (4.2.31):

$$\begin{aligned} a_3 &= -\frac{3}{1}(-3) = 9, \\ a_5 &= -\frac{5}{2}9 = -\frac{45}{2}, \\ a_7 &= -\frac{7}{3}\left(-\frac{45}{2}\right) = \frac{105}{2}. \end{aligned}$$

Therefore the solution of (4.2.29) is

$$y = 2 - 3x - 8x^2 + 9x^3 + \frac{64}{3}x^4 - \frac{45}{2}x^5 - \frac{256}{5}x^6 + \frac{105}{2}x^7 + \cdots .$$

■

4.2 Exercises

In Exercises 1 –8 find the power series in x for the general solution.

1. $(1+x^2)y'' + 6xy' + 6y = 0$

2. $(1+x^2)y'' + 2xy' - 2y = 0$

3. $(1+x^2)y'' - 8xy' + 20y = 0$

4. $(1-x^2)y'' - 8xy' - 12y = 0$

5. $(1+2x^2)y'' + 7xy' + 2y = 0$

6. $(1+x^2)y'' + 2xy' + \frac{1}{4}y = 0$

7. $(1-x^2)y'' - 5xy' - 4y = 0$

8. $(1+x^2)y'' - 10xy' + 28y = 0$

In Exercises 9 –13 *find the power series in* $x - x_0$ *for the general solution.*

9. $y'' - y = 0; \quad x_0 = 3$

10. $y'' - (x-3)y' - y = 0; \quad x_0 = 3$

11. $(1 - 4x + 2x^2)y'' + 10(x-1)y' + 6y = 0; \quad x_0 = 1$

12. $(11 - 8x + 2x^2)y'' - 16(x-2)y' + 36y = 0; \quad x_0 = 2$

13. $(5 + 6x + 3x^2)y'' + 9(x+1)y' + 3y = 0; \quad x_0 = -1$

In Exercises 14 –19 *find* $a_0, \dots, a_N$ *for* N *at least* 5 *in the power series* $y = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ *for the solution of the initial value problem. Take* x_0 *to be the point where the initial conditions are imposed.*

14. $(x^2 - 4)y'' - xy' - 3y = 0, \quad y(0) = -1, \quad y'(0) = 2$

15. $y'' + (x-3)y' + 3y = 0, \quad y(3) = -2, \quad y'(3) = 3$

16. $(5 - 6x + 3x^2)y'' + (x-1)y' + 12y = 0, \quad y(1) = -1, \quad y'(1) = 1$

17. $(4x^2 - 24x + 37)y'' + y = 0, \quad y(3) = 4, \quad y'(3) = -6$

18. $(x^2 - 8x + 14)y'' - 8(x-4)y' + 20y = 0, \quad y(4) = 3, \quad y'(4) = -4$

19. $(2x^2 + 4x + 5)y'' - 20(x+1)y' + 60y = 0, \quad y(-1) = 3, \quad y'(-1) = -3$

4.3 SERIES SOLUTIONS NEAR AN ORDINARY POINT II

In this section we continue to find series solutions

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

of initial value problems

$$P_2(x)y'' + P_1(x)y' + P_0(x)y = 0, \quad y(x_0) = a_0, \quad y'(x_0) = a_1, \tag{4.3.1}$$

where P_0, P_1, and P_2 are polynomials and $P_2(x_0) \neq 0$, so x_0 is an ordinary point of (4.3.1). However, here we consider cases where the differential equation in (4.3.1) is not of the form

$$\left(1 + \alpha(x - x_0)^2\right) y'' + \beta(x - x_0)y' + \gamma y = 0,$$

so Theorem 4.2.2 does not apply and the computation of the coefficients $\{a_n\}$ is more complicated. For the equations considered here it is difficult or impossible to obtain an explicit formula for a_n in terms of n. Nevertheless, we can calculate as many coefficients as we wish. We provide three examples to illustrate this.

Example 4.3.1 Find the coefficients $a_0, \ldots, a_5$ in the series solution $y = \sum_{n=0}^{\infty} a_n x^n$ of the initial value problem

$$(1 + x + 2x^2)y'' + (1 + 7x)y' + 2y = 0, \quad y(0) = -1, \quad y'(0) = -2. \tag{4.3.2}$$

Solution Here

$$Ly = (1 + x + 2x^2)y'' + (1 + 7x)y' + 2y.$$

The zeros $(-1 \pm i\sqrt{7})/4$ of $P_2(x) = 1+x+2x^2$ have absolute value $1/\sqrt{2}$, so Theorem 4.2.2 implies that the series solution converges to the solution of (4.3.2) on $(-1/\sqrt{2}, 1/\sqrt{2})$. Since

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2},$$

$$\begin{aligned} Ly &= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} + 2\sum_{n=2}^{\infty} n(n-1)a_n x^n \\ &\quad + \sum_{n=1}^{\infty} n a_n x^{n-1} + 7\sum_{n=1}^{\infty} n a_n x^n + 2\sum_{n=0}^{\infty} a_n x^n. \end{aligned}$$

Shifting indices so the general term in each series is a constant multiple of x^n yields

$$\begin{aligned} Ly &= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n + \sum_{n=0}^{\infty} (n+1)n a_{n+1} x^n + 2\sum_{n=0}^{\infty} n(n-1)a_n x^n \\ &\quad + \sum_{n=0}^{\infty} (n+1)a_{n+1} x^n + 7\sum_{n=0}^{\infty} n a_n x^n + 2\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n, \end{aligned}$$

where

$$b_n = (n+2)(n+1)a_{n+2} + (n+1)^2 a_{n+1} + (n+2)(2n+1)a_n.$$

Therefore $y = \sum_{n=0}^{\infty} a_n x^n$ is a solution of $Ly = 0$ if and only if

$$a_{n+2} = -\frac{n+1}{n+2}\, a_{n+1} - \frac{2n+1}{n+1}\, a_n, \; n \geqslant 0. \tag{4.3.3}$$

From the initial conditions in (4.3.2), $a_0 = -1$ and $a_1 = -2$. Setting $n = 0$ in (4.3.3) yields

$$a_2 = -\frac{1}{2}a_1 - a_0 = -\frac{1}{2}(-2) - (-1) = 2.$$

Setting $n = 1$ in (4.3.3) yields

$$a_3 = -\frac{2}{3}a_2 - \frac{3}{2}a_1 = -\frac{2}{3}(2) - \frac{3}{2}(-2) = \frac{5}{3}.$$

We leave it to you to compute a_4 and a_5 from (4.3.3) and show that

$$y = -1 - 2x + 2x^2 + \frac{5}{3}x^3 - \frac{55}{12}x^4 + \frac{3}{4}x^5 + \cdots.$$

■

Example 4.3.2 Find the coefficients $a_0, \dots, a_5$ in the series solution

$$y = \sum_{n=0}^{\infty} a_n (x+1)^n$$

of the initial value problem

$$(3+x)y'' + (1+2x)y' - (2-x)y = 0, \quad y(-1) = 2, \quad y'(-1) = -3. \tag{4.3.4}$$

Solution Since the desired series is in powers of $x+1$ we rewrite the differential equation in (4.3.4) as $Ly = 0$, with

$$Ly = (2 + (x+1))\, y'' - (1 - 2(x+1))\, y' - (3 - (x+1))\, y.$$

Since

$$y = \sum_{n=0}^{\infty} a_n (x+1)^n, \quad y' = \sum_{n=1}^{\infty} n a_n (x+1)^{n-1} \quad \text{and} \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n (x+1)^{n-2},$$

$$\begin{aligned} Ly \;=\; & 2\sum_{n=2}^{\infty} n(n-1)a_n(x+1)^{n-2} + \sum_{n=2}^{\infty} n(n-1)a_n(x+1)^{n-1} \\ & - \sum_{n=1}^{\infty} n a_n (x+1)^{n-1} + 2\sum_{n=1}^{\infty} n a_n (x+1)^n \\ & -3\sum_{n=0}^{\infty} a_n(x+1)^n + \sum_{n=0}^{\infty} a_n (x+1)^{n+1}. \end{aligned}$$

Shifting indices so that the general term in each series is a constant multiple of $(x+1)^n$ yields

$$\begin{aligned} Ly \;=\; & 2\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x+1)^n + \sum_{n=0}^{\infty}(n+1)n a_{n+1}(x+1)^n \\ & - \sum_{n=0}^{\infty}(n+1)a_{n+1}(x+1)^n + \sum_{n=0}^{\infty}(2n-3)a_n(x+1)^n + \sum_{n=1}^{\infty} a_{n-1}(x+1)^n \\ =\; & \sum_{n=0}^{\infty} b_n (x+1)^n, \end{aligned}$$

where

$$b_0 = 4a_2 - a_1 - 3a_0$$

and

$$b_n = 2(n+2)(n+1)a_{n+2} + (n^2-1)a_{n+1} + (2n-3)a_n + a_{n-1}, \quad n \geqslant 1.$$

Therefore $y = \sum_{n=0}^{\infty} a_n(x+1)^n$ is a solution of $Ly = 0$ if and only if

$$a_2 = \frac{1}{4}(a_1 + 3a_0) \tag{4.3.5}$$

and

$$a_{n+2} = -\frac{1}{2(n+2)(n+1)}\left[(n^2-1)a_{n+1} + (2n-3)a_n + a_{n-1}\right], \quad n \geqslant 1. \tag{4.3.6}$$

From the initial conditions in (4.3.4), $a_0 = 2$ and $a_1 = -3$. We leave it to you to compute $a_2, \dots, a_5$ with (4.3.5) and (4.3.6) and show that the solution of (4.3.4) is

$$y = -2 - 3(x+1) + \frac{3}{4}(x+1)^2 - \frac{5}{12}(x+1)^3 + \frac{7}{48}(x+1)^4 - \frac{1}{60}(x+1)^5 + \cdots.$$

■

Example 4.3.3 Find the coefficients $a_0, \dots, a_5$ in the series solution $y = \sum_{n=0}^{\infty} a_n x^n$ of the initial value problem

$$y'' + 3xy' + (4+2x^2)y = 0, \quad y(0) = 2, \quad y'(0) = -3. \tag{4.3.7}$$

Solution Here

$$Ly = y'' + 3xy' + (4+2x^2)y.$$

Since

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad \text{and} \quad y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2},$$

$$Ly = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + 3\sum_{n=1}^{\infty} n a_n x^n + 4\sum_{n=0}^{\infty} a_n x^n + 2\sum_{n=0}^{\infty} a_n x^{n+2}.$$

Shifting indices so that the general term in each series is a constant multiple of x^n yields Ly as

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty} (3n+4)a_n x^n + 2\sum_{n=2}^{\infty} a_{n-2}x^n = \sum_{n=0}^{\infty} b_n x^n$$

where

$$b_0 = 2a_2 + 4a_0, \quad b_1 = 6a_3 + 7a_1,$$

and

$$b_n = (n+2)(n+1)a_{n+2} + (3n+4)a_n + 2a_{n-2}, \quad n \geqslant 2.$$

Therefore $y = \sum_{n=0}^{\infty} a_n x^n$ is a solution of $Ly = 0$ if and only if

$$a_2 = -2a_0, \quad a_3 = -\frac{7}{6}a_1, \tag{4.3.8}$$

and

$$a_{n+2} = -\frac{1}{(n+2)(n+1)}\left[(3n+4)a_n + 2a_{n-2}\right], \quad n \geqslant 2. \tag{4.3.9}$$

From the initial conditions in (4.3.7), $a_0 = 2$ and $a_1 = -3$. We leave it to you to compute $a_2, \dots, a_5$ with (4.3.8) and (4.3.9) and show that the solution of (4.3.7) is

$$y = 2 - 3x - 4x^2 + \frac{7}{2}x^3 + 3x^4 - \frac{79}{40}x^5 + \cdots.$$

■

4.3 Exercises

In Exercises 1–12 find the coefficients $a_0, \dots, a_N$ for N at least 5 in the series solution $y = \sum_{n=0}^{\infty} a_n x^n$ of the initial value problem.

1. $(1+3x)y'' + xy' + 2y = 0, \quad y(0) = 2, \quad y'(0) = -3$

2. $(1+x+2x^2)y'' + (2+8x)y' + 4y = 0, \quad y(0) = -1, \quad y'(0) = 2$

3. $(1-2x^2)y'' + (2-6x)y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = 0$

4. $(1+x+3x^2)y'' + (2+15x)y' + 12y = 0, \quad y(0) = 0, \quad y'(0) = 1$

5. $(2+x)y'' + (1+x)y' + 3y = 0, \quad y(0) = 4, \quad y'(0) = 3$

6. $(3+3x+x^2)y'' + (6+4x)y' + 2y = 0, \quad y(0) = 7, \quad y'(0) = 3$

7. $(4+x)y'' + (2+x)y' + 2y = 0, \quad y(0) = 2, \quad y'(0) = 5$

8. $(2-3x+2x^2)y'' - (4-6x)y' + 2y = 0, \quad y(1) = 1, \quad y'(1) = -1$

9. $(3x+2x^2)y'' + 10(1+x)y' + 8y = 0, \quad y(-1) = 1, \quad y'(-1) = -1$

10. $(1-x+x^2)y'' - (1-4x)y' + 2y = 0, \quad y(1) = 2, \quad y'(1) = -1$

11. $(2+x)y'' + (2+x)y' + y = 0, \quad y(-1) = -2, \quad y'(-1) = 3$

12. $x^2y'' - (6-7x)y' + 8y = 0, \quad y(1) = 1, \quad y'(1) = -2$

In Exercises 13–22 find the coefficients $a_0, \dots, a_N$ for N at least 5 in the series solution

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

of the initial value problem. Take x_0 to be the point where the initial conditions are imposed.

13. $(2+4x)y''-4y'-(6+4x)y=0, \quad y(0)=2, \quad y'(0)=-7$

14. $(1+2x)y''-(1-2x)y'-(3-2x)y=0, \quad y(1)=1, \quad y'(1)=-2$

15. $(5+2x)y''-y'+(5+x)y=0, \quad y(-2)=2, \quad y'(-2)=-1$

16. $(4+x)y''-(4+2x)y'+(6+x)y=0, \quad y(-3)=2, \quad y'(-3)=-2$

17. $(2+3x)y''-xy'+2xy=0, \quad y(0)=-1, \quad y'(0)=2$

18. $(3+2x)y''+3y'-xy=0, \quad y(-1)=2, \quad y'(-1)=-3$

19. $(3+2x)y''-3y'-(2+x)y=0, \quad y(-2)=-2, \quad y'(-2)=3$

20. $(10-2x)y''+(1+x)y=0, \quad y(2)=2, \quad y'(2)=-4$

21. $(7+x)y''+(8+2x)y'+(5+x)y=0, \quad y(-4)=1, \quad y'(-4)=2$

22. $(6+4x)y''+(1+2x)y=0, \quad y(-1)=-1, \quad y'(-1)=2$

In Exercises 23–29 find the coefficients $a_0, \dots, a_N$ *for* N *at least* 5 *in the series solution* $y=\sum_{n=0}^{\infty} a_n x^n$ *of the initial value problem.*

23. $y''+2xy'+(3+2x^2)y=0, \quad y(0)=1, \quad y'(0)=-2$

24. $y''-3xy'+(5+2x^2)y=0, \quad y(0)=1, \quad y'(0)=-2$

25. $y''+5xy'-(3-x^2)y=0, \quad y(0)=6, \quad y'(0)=-2$

26. $y''-2xy'-(2+3x^2)y=0, \quad y(0)=2, \quad y'(0)=-5$

27. $y''-3xy'+(2+4x^2)y=0, \quad y(0)=3, \quad y'(0)=6$

28. $2y''+5xy'+(4+2x^2)y=0, \quad y(0)=3, \quad y'(0)=-2$

29. $3y''+2xy'+(4-x^2)y=0, \quad y(0)=-2, \quad y'(0)=3$

4.4 SERIES SOLUTIONS NEAR A SINGULAR POINT

We continue to study equations of the form

$$P_2(x)y'' + P_1(x)y' + P_0(x)y = 0 \tag{4.4.1}$$

where P_0, P_1, and P_2 are polynomials, but the emphasis will be different from that of Sections 4.2 and 4.3, where we obtained solutions of (4.4.1) near an ordinary point x_0 in the form of power series in $x-x_0$. In this section, we consider cases where x_0 is a singular point of (4.4.1) (that is, where $P(x_0) = 0$). The solutions of such equations cannot in general be represented by power series in $x - x_0$. Nevertheless, it is often necessary in physical applications to study the behavior of solutions of (4.4.1) near a singular point. Although this can be difficult in the absence of some sort of assumption on the nature of the singular point, equations that satisfy the requirements of the next definition can be solved by series methods discussed in the next three sections. Fortunately, many equations arising in applications satisfy these requirements.

Definition 4.4.1 Let P_0, P_1, and P_2 be polynomials with no common factor and suppose $P_2(x_0) = 0$. Then x_0 is a *regular singular point* of the equation

$$P_2(x)y'' + P_1(x)y' + P_0(x)y = 0 \tag{4.4.2}$$

if (4.4.2) can be written as

$$(x - x_0)^2A(x)y'' + (x - x_0)B(x)y' + C(x)y = 0 \tag{4.4.3}$$

where A, B, and C are polynomials and $A(x_0) \neq 0$; otherwise, x_0 is an *irregular* singular point of (4.4.2).

Example 4.4.1 Bessel's equation,

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0, \tag{4.4.4}$$

has the singular point $x_0 = 0$. Since this equation is of the form (4.4.3) with $x_0 = 0$, $A(x) = 1$, $B(x) = 1$, and $C(x) = x^2 - \nu^2$, it follows that $x_0 = 0$ is a regular singular point of (4.4.4).

■

Example 4.4.2 Legendre's equation,

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0, \tag{4.4.5}$$

has the singular points $x_0 = \pm 1$. Mutiplying through by $1 - x$ yields

$$(x - 1)^2(x + 1)y'' + 2x(x - 1)y' - \alpha(\alpha + 1)(x - 1)y = 0,$$

which is of the form (4.4.3) with $x_0 = 1$, $A(x) = x + 1$, $B(x) = 2x$, and $C(x) = -\alpha(\alpha + 1)(x - 1)$. Therefore $x_0 = 1$ is a regular singular point of (4.4.5). We leave it to you to show that $x_0 = -1$ is also a regular singular point of (4.4.5).

■

Example 4.4.3 The equation

$$x^3y'' + xy' + y = 0$$

has an irregular singular point at $x_0 = 0$. (Verify.)

■

For convenience we restrict our attention to the case where $x_0 = 0$ is a regular singular point of (4.4.2). This is not really a restriction, since if $x_0 \neq 0$ is a regular singular point of (4.4.2) then introducing the new independent variable $t = x - x_0$ and the new unknown $Y(t) = y(t + x_0)$ leads to a differential equation with polynomial coefficients that has a regular singular point at $t_0 = 0$.

Euler Equations

The simplest kind of equation with a regular singular point at $x_0 = 0$ is the Euler equation, defined as follows.

Definition 4.4.2 An *Euler equation* is an equation that can be written in the form

$$ax^2y'' + bxy' + cy = 0, \tag{4.4.6}$$

where a, b, and c are real constants and $a \neq 0$.

Theorem 3.1.1 implies that (4.4.6) has solutions defined on $(0, \infty)$ and $(-\infty, 0)$, since (4.4.6) can be rewritten as

$$ay'' + \frac{b}{x}y' + \frac{c}{x^2}y = 0.$$

For convenience we restrict our attention to the interval $(0, \infty)$. The key to finding solutions on $(0, \infty)$ is that if $x > 0$ then x^r is defined as a real-valued function on $(0, \infty)$ for all values of r, and substituting $y = x^r$ into (4.4.6) produces

$$\begin{aligned} ax^2(x^r)'' + bx(x^r)' + cx^r &= ax^2r(r-1)x^{r-2} + bxrx^{r-1} + cx^r \\ &= [ar(r-1) + br + c]x^r. \end{aligned} \tag{4.4.7}$$

The polynomial

$$p(r) = ar(r-1) + br + c$$

is called the *indicial polynomial* of (4.4.6), and $p(r) = 0$ is its *indicial equation*. From (4.4.7) we can see that $y = x^r$ is a solution of (4.4.6) on $(0, \infty)$ if and only if $p(r) = 0$. Therefore, if the indicial equation has distinct real roots r_1 and r_2 then $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$ form a fundamental set of solutions of (4.4.6) on $(0, \infty)$, since $y_2/y_1 = x^{r_2 - r_1}$ is nonconstant. In this case

$$y = c_1x^{r_1} + c_2x^{r_2}$$

is the general solution of (4.4.6) on $(0, \infty)$.

Example 4.4.4 Find the general solution of

$$x^2y'' - xy' - 8y = 0 \tag{4.4.8}$$

on $(0, \infty)$.

Solution The indicial polynomial $p(r)$ of (4.4.8) is

$$r(r-1) - r - 8 = (r-4)(r+2).$$

Therefore $y_1 = x^4$ and $y_2 = x^{-2}$ are solutions of (4.4.8) on $(0, \infty)$, and its general solution on $(0, \infty)$ is

$$y = c_1x^4 + \frac{c_2}{x^2}.$$

■

Example 4.4.5 Find the general solution of

$$6x^2y'' + 5xy' - y = 0 \tag{4.4.9}$$

on $(0, \infty)$.

Solution The indicial polynomial $p(r)$ of (4.4.9) is

$$6r(r-1) + 5r - 1 = (2r-1)(3r+1).$$

Therefore the general solution of (4.4.9) on $(0, \infty)$ is

$$y = c_1x^{1/2} + c_2x^{-1/3}.$$

■

If the indicial equation has a repeated root r_1, then $y_1 = x^{r_1}$ is a solution of

$$ax^2y'' + bxy' + cy = 0, \tag{4.4.10}$$

on $(0, \infty)$, but (4.4.10) has no other solution of the form $y = x^r$. If the indicial equation has complex conjugate zeros then (4.4.10) has no real–valued solutions of the form $y = x^r$. Fortunately we can use the results of Section 3.2 for constant coefficient equations to solve (4.4.10) in any case.

Theorem 4.4.3 *Suppose the roots of the indicial equation*

$$ar(r-1) + br + c = 0 \tag{4.4.11}$$

are r_1 and r_2. Then the general solution of the Euler equation

$$ax^2y'' + bxy' + cy = 0 \tag{4.4.12}$$

on $(0, \infty)$ *is*

$$\begin{aligned} y &= c_1x^{r_1} + c_2x^{r_2} \textit{ if } r_1 \textit{ and } r_2 \textit{ are distinct real numbers };\\ y &= x^{r_1}(c_1 + c_2\ln x) \textit{ if } r_1 = r_2 ;\\ y &= x^{\lambda}\left[c_1\cos(\omega\ln x) + c_2\sin(\omega\ln x)\right] \textit{ if } r_1, r_2 = \lambda \pm i\omega \textit{ with } \omega > 0. \end{aligned}$$

Proof We first show that $y = y(x)$ satisfies (4.4.12) on $(0, \infty)$ if and only if $Y(t) = y(e^t)$ satisfies the constant coefficient equation

$$a\frac{d^2Y}{dt^2} + (b - a)\frac{dY}{dt} + cY = 0 \tag{4.4.13}$$

on $(-\infty, \infty)$. To do this, it is convenient to write $x = e^t$, or, equivalently, $t = \ln x$; thus, $Y(t) = y(x)$, where $x = e^t$. From the chain rule,

$$\frac{dY}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

and, since

$$\frac{dx}{dt} = e^t = x,$$

it follows that

$$\frac{dY}{dt} = x\frac{dy}{dx}. \tag{4.4.14}$$

Differentiating this with respect to t and using the chain rule again yields the second derivative as

$$\begin{aligned} \frac{d}{dt}\left(\frac{dY}{dt}\right) &= \frac{d}{dt}\left(x\frac{dy}{dx}\right)\\ &= \frac{dx}{dt}\frac{dy}{dx} + x\frac{d^2y}{dx^2}\frac{dx}{dt}\\ &= x\frac{dy}{dx} + x^2\frac{d^2y}{dx^2} \quad \left(\text{since } \frac{dx}{dt} = x\right). \end{aligned}$$

From this and (4.4.14),

$$x^2\frac{d^2y}{dx^2} = \frac{d^2Y}{dt^2} - \frac{dY}{dt}.$$

Substituting this and (4.4.14) into (4.4.12) yields (4.4.13). Since (4.4.11) is the characteristic equation of (4.4.13), Theorem 3.2.1 implies that the general solution of (4.4.13) on $(-\infty, \infty)$ is

$$\begin{aligned} Y(t) &= c_1e^{r_1t} + c_2e^{r_2t} \text{ if } r_1 \text{ and } r_2 \text{ are distinct real numbers;}\\ Y(t) &= e^{r_1t}(c_1 + c_2t) \text{ if } r_1 = r_2;\\ Y(t) &= e^{\lambda t}\left(c_1\cos\omega t + c_2\sin\omega t\right) \text{ if } r_1, r_2 = \lambda \pm i\omega \text{ with } \omega \neq 0. \end{aligned}$$

Since $Y(t) = y(e^t)$, substituting $t = \ln x$ in the last three equations shows that the general solution of (4.4.12) on $(0, \infty)$ has the form stated in the theorem. ■

Example 4.4.6 Find the general solution of

$$x^2y'' - 5xy' + 9y = 0 \tag{4.4.15}$$

on $(0, \infty)$.

Solution The indicial polynomial $p(r)$ of (4.4.15) is

$$r(r-1) - 5r + 9 = (r-3)^2.$$

Therefore the general solution of (4.4.15) on $(0, \infty)$ is

$$y = x^3(c_1 + c_2 \ln x).$$

■

Example 4.4.7 Find the general solution of

$$x^2y'' + 3xy' + 2y = 0 \tag{4.4.16}$$

on $(0, \infty)$.

Solution The indicial polynomial $p(r)$ of (4.4.16) is

$$r(r-1) + 3r + 2 = (r+1)^2 + 1.$$

The roots of the indicial equation are $r = -1 \pm i$ and the general solution of (4.4.16) on $(0, \infty)$ is

$$y = \frac{1}{x}\left[c_1 \cos(\ln x) + c_2 \sin(\ln x)\right].$$

■

4.4 Exercises

In Exercises 1–18 find the general solution of the given Euler equation on $(0, \infty)$.

1. $x^2y'' + 7xy' + 8y = 0$

2. $x^2y'' - 7xy' + 7y = 0$

3. $x^2y'' - xy' + y = 0$

4. $x^2y'' + 5xy' + 4y = 0$

5. $x^2y'' + xy' + y = 0$

6. $x^2y'' - 3xy' + 13y = 0$

7. $x^2y'' + 3xy' - 3y = 0$

8. $12x^2y'' - 5xy'' + 6y = 0$

9. $4x^2y'' + 8xy' + y = 0$

10. $3x^2y'' - xy' + y = 0$

11. $2x^2y'' - 3xy' + 2y = 0$

12. $x^2y'' + 3xy' + 5y = 0$

13. $9x^2y'' + 15xy' + y = 0$

14. $x^2y'' - xy' + 10y = 0$

15. $x^2y'' - 6y = 0$

16. $2x^2y'' + 3xy' - y = 0$

17. $x^2y'' - 3xy' + 4y = 0$

18. $2x^2y'' + 10xy' + 9y = 0$

CHAPTER 5

LAPLACE TRANSFORMS

IN THIS CHAPTER we study the method of *Laplace transforms*, which illustrates one of the basic problem solving techniques in mathematics: transform a difficult problem into an easier one, solve the latter, and then use its solution to obtain a solution of the original problem. The method discussed here transforms an initial value problem for a constant coefficient equation into an algebraic equation whose solution can then be used to solve the initial value problem. In some cases this method is merely an alternative procedure for solving problems that can be solved equally well by methods that we considered previously; however, in other cases the method of Laplace transforms is more efficient than the methods previously discussed. This is especially true in physical problems dealing with discontinuous forcing functions.

SECTION 8.1 defines the Laplace transform and developes its properties.

SECTION 8.2 deals with the problem of finding a function that has a given Laplace transform.

SECTION 8.3 applies the Laplace transform to solve initial value problems for constant coefficient second order differential equations on $(0, \infty)$.

SECTION 8.4 introduces the unit step function.

SECTION 8.5 uses the unit step function to solve constant coefficient equations with piecewise continuous forcing functions.

SECTION 8.6 deals with the convolution theorem, an important theoretical property of the Laplace transform.

SECTION 8.7 introduces the idea of impulsive force, and treats constant coefficient equations with impulsive forcing functions.

SECTION 8.8 is a brief table of Laplace transforms.

5.1 INTRODUCTION TO THE LAPLACE TRANSFORM

Definition of the Laplace Transform

To define the Laplace transform, we first recall the definition of an improper integral. If g is integrable over the interval $[a, T]$ for every $T > a$, then the *improper integral of g over* $[a, \infty)$ is defined as

$$\int_a^\infty g(t)\,dt = \lim_{T\to\infty}\int_a^T g(t)\,dt. \tag{5.1.1}$$

We say that the improper integral *converges* if the limit in (5.1.1) exists; otherwise, we say that the improper integral *diverges* or *does not exist*. Here's the definition of the Laplace transform of a function f.

Definition 5.1.1 Let f be defined for $t \geqslant 0$ and let s be a real number. Then the *Laplace transform* of f is the function F defined by

$$F(s) = \int_0^\infty e^{-st}f(t)\,dt, \tag{5.1.2}$$

for those values of s for which the improper integral converges.

It is important to keep in mind that the variable of integration in (5.1.2) is t, while s is a parameter independent of t. We use t as the independent variable for f because in applications the Laplace transform is usually applied to functions of time.

The Laplace transform can be viewed as an operator $\mathcal{L}$ that transforms the function $f = f(t)$ into the function $F = F(s)$. Thus, (5.1.2) can be expressed as

$$F = \mathcal{L}(f).$$

The functions f and F form a *transform pair*, which we'll sometimes denote by

$$f(t) \leftrightarrow F(s).$$

It can be shown that if $F(s)$ is defined for $s = s_0$ then it's defined for all $s > s_0$ (Exercise 14**(b)**).

Computation of Some Simple Laplace Transforms

Example 5.1.1 Find the Laplace transform of $f(t) = 1$.

Solution From (5.1.2) with $f(t) = 1$,

$$F(s) = \int_0^\infty e^{-st}\,dt = \lim_{T\to\infty}\int_0^T e^{-st}\,dt.$$

If $s \neq 0$ then

$$\int_0^T e^{-st}dt = -\frac{1}{s}e^{-st}\Big|_0^T = \frac{1-e^{-sT}}{s}. \tag{5.1.3}$$

Therefore

$$\lim_{T\to\infty}\int_0^T e^{-st}\,dt = \begin{cases} \dfrac{1}{s}, & s>0,\\ \infty, & s<0. \end{cases} \tag{5.1.4}$$

If $s=0$ the integrand reduces to the constant 1, and

$$\lim_{T\to\infty}\int_0^T 1\,dt = \lim_{T\to\infty}\int_0^T 1\,dt = \lim_{T\to\infty} T = \infty.$$

Therefore $F(0)$ is undefined, and

$$F(s)=\int_0^\infty e^{-st}\,dt = \frac{1}{s}, \quad s>0.$$

This result can be written in operator notation as

$$\mathcal{L}(1)=\frac{1}{s}, \quad s>0,$$

or as the transform pair

$$1 \leftrightarrow \frac{1}{s}, \quad s>0.$$

REMARK: It is convenient to combine the steps of integrating from 0 to T and letting $T\to\infty$. Therefore, instead of writing (5.1.3) and (5.1.4) as separate steps we write

$$\int_0^\infty e^{-st}\,dt = -\frac{1}{s}e^{-st}\Big|_0^\infty = \begin{cases} \dfrac{1}{s}, & s>0,\\ \infty, & s<0. \end{cases}$$

We'll follow this practice throughout this chapter.

Example 5.1.2 Find the Laplace transform of $f(t)=t$.

Solution From (5.1.2) with $f(t)=t$,

$$F(s)=\int_0^\infty e^{-st}t\,dt. \tag{5.1.5}$$

If $s\neq 0$, integrating by parts yields

$$\begin{aligned}\int_0^\infty e^{-st}t\,dt &= -\frac{te^{-st}}{s}\Big|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}\,dt = -\left[\frac{t}{s}+\frac{1}{s^2}\right]e^{-st}\Big|_0^\infty \\ &= \begin{cases} \dfrac{1}{s^2}, & s>0,\\ \infty, & s<0. \end{cases}\end{aligned}$$

If $s=0$, the integral in (5.1.5) becomes

$$\int_0^\infty t\,dt = \frac{t^2}{2}\Big|_0^\infty = \infty.$$

Therefore $F(0)$ is undefined and

$$F(s) = \frac{1}{s^2}, \quad s > 0.$$

This result can also be written as

$$\mathcal{L}(t) = \frac{1}{s^2}, \quad s > 0,$$

or as the transform pair

$$t \leftrightarrow \frac{1}{s^2}, \quad s > 0.$$

Example 5.1.3 Find the Laplace transform of $f(t) = e^{at}$, where a is a constant.

Solution From (5.1.2) with $f(t) = e^{at}$,

$$F(s) = \int_0^\infty e^{-st} e^{at}\, dt.$$

Combining the exponentials yields

$$F(s) = \int_0^\infty e^{-(s-a)t}\, dt.$$

However, we know from Example 5.1.1 that

$$\int_0^\infty e^{-st}\, dt = \frac{1}{s}, \quad s > 0.$$

Replacing s by $s - a$ here shows that

$$F(s) = \frac{1}{s-a}, \quad s > a.$$

This can also be written as

$$\mathcal{L}(e^{at}) = \frac{1}{s-a}, \quad s > a, \quad \text{or} \quad e^{at} \leftrightarrow \frac{1}{s-a}, \quad s > a.$$

Example 5.1.4 Find the Laplace transforms of $f(t) = \sin \omega t$ and $g(t) = \cos \omega t$, where ω is a constant.

Solution Define

$$F(s) = \int_0^\infty e^{-st} \sin \omega t\, dt \tag{5.1.6}$$

and

$$G(s) = \int_0^\infty e^{-st} \cos \omega t\, dt. \tag{5.1.7}$$

If $s > 0$, integrating (5.1.6) by parts yields

$$F(s) = -\frac{e^{-st}}{s}\sin\omega t\Big|_0^\infty + \frac{\omega}{s}\int_0^\infty e^{-st}\cos\omega t\,dt,$$

so

$$F(s) = \frac{\omega}{s}G(s). \tag{5.1.8}$$

If $s > 0$, integrating (5.1.7) by parts yields

$$G(s) = -\frac{e^{-st}\cos\omega t}{s}\Big|_0^\infty - \frac{\omega}{s}\int_0^\infty e^{-st}\sin\omega t\,dt,$$

so

$$G(s) = \frac{1}{s} - \frac{\omega}{s}F(s).$$

Now substitute from (5.1.8) into this to obtain

$$G(s) = \frac{1}{s} - \frac{\omega^2}{s^2}G(s).$$

Solving this for $G(s)$ yields

$$G(s) = \frac{s}{s^2+\omega^2}, \quad s > 0.$$

This and (5.1.8) imply that

$$F(s) = \frac{\omega}{s^2+\omega^2}, \quad s > 0.$$

Tables of Laplace transforms

Extensive tables of Laplace transforms have been compiled and are commonly used in applications. The brief table of Laplace transforms in the Appendix will be adequate for our purposes.

Example 5.1.5 Use the table of Laplace transforms to find $\mathcal{L}(t^3e^{4t})$.

Solution The table includes the transform pair

$$t^ne^{at} \leftrightarrow \frac{n!}{(s-a)^{n+1}}.$$

Setting $n = 3$ and $a = 4$ here yields

$$\mathcal{L}(t^3e^{4t}) = \frac{3!}{(s-4)^4} = \frac{6}{(s-4)^4}. \quad \blacksquare$$

We'll sometimes write Laplace transforms of specific functions without explicitly stating how they are obtained. In such cases you should refer to the table of Laplace transforms.

Linearity of the Laplace Transform

The next theorem presents an important property of the Laplace transform.

Theorem 5.1.2 *[Linearity Property] Suppose* $\mathcal{L}(f_i)$ *is defined for* $s > s_i$, $1 \leqslant i \leqslant n$). *Let* s_0 *be the largest of the numbers* $s_1, s_2, \dots, s_n$, *and let* $c_1, c_2, \dots, c_n$ *be constants. Then*

$$\mathcal{L}(c_1f_1 + c_2f_2 + \cdots + c_nf_n) = c_1\mathcal{L}(f_1) + c_2\mathcal{L}(f_2) + \cdots + c_n\mathcal{L}(f_n) \text{ for } s > s_0.$$

Proof We give the proof for the case where $n = 2$. If $s > s_0$ then

$$\begin{aligned}\mathcal{L}(c_1f_1 + c_2f_2) &= \int_0^\infty e^{-st}\left(c_1f_1(t) + c_2f_2(t)\right)\,dt \\ &= c_1\int_0^\infty e^{-st}f_1(t)\,dt + c_2\int_0^\infty e^{-st}f_2(t)\,dt \\ &= c_1\mathcal{L}(f_1) + c_2\mathcal{L}(f_2).\end{aligned}$$

Example 5.1.6 Use Theorem 5.1.2 and the known Laplace transform

$$\mathcal{L}(e^{at}) = \frac{1}{s-a}$$

to find $\mathcal{L}(\cosh bt)$ $(b \neq 0)$.

Solution By definition,

$$\cosh bt = \frac{e^{bt} + e^{-bt}}{2}.$$

Therefore

$$\begin{aligned}\mathcal{L}(\cosh bt) &= \mathcal{L}\left(\frac{1}{2}e^{bt} + \frac{1}{2}e^{-bt}\right) \\ &= \frac{1}{2}\mathcal{L}(e^{bt}) + \frac{1}{2}\mathcal{L}(e^{-bt}) \qquad \text{(linearity property)} \\ &= \frac{1}{2}\frac{1}{s-b} + \frac{1}{2}\frac{1}{s+b},\end{aligned} \tag{5.1.9}$$

where the first transform on the right is defined for $s > b$ and the second for $s > -b$; hence, both are defined for $s > |b|$. Simplifying the last expression in (5.1.9) yields

$$\mathcal{L}(\cosh bt) = \frac{s}{s^2 - b^2}, \quad s > |b|.$$

The First Shifting Theorem

The next theorem enables us to start with known transform pairs and derive others. (For other results of this kind, see Exercises 6 and 13.)

Theorem 5.1.3 *[First Shifting Theorem] If*

$$F(s) = \int_0^\infty e^{-st}f(t)\,dt \tag{5.1.10}$$

is the Laplace transform of $f(t)$ *for* $s > s_0$, *then* $F(s-a)$ *is the Laplace transform of* $e^{at}f(t)$ *for* $s > s_0 + a$.

PROOF. Replacing s by $s - a$ in (5.1.10) yields

$$F(s-a) = \int_0^\infty e^{-(s-a)t} f(t)\, dt \tag{5.1.11}$$

if $s - a > s_0$; that is, if $s > s_0 + a$. However, (5.1.11) can be rewritten as

$$F(s-a) = \int_0^\infty e^{-st}\left(e^{at} f(t)\right)\, dt,$$

which implies the conclusion.

Example 5.1.7 Use Theorem 5.1.3 and the known Laplace transforms of 1, t, $\cos \omega t$, and $\sin \omega t$ to find

$$\mathcal{L}(e^{at}), \quad \mathcal{L}(te^{at}), \quad \mathcal{L}(e^{\lambda t}\sin \omega t), \text{and } \mathcal{L}(e^{\lambda t}\cos \omega t).$$

Solution In the following table the known transform pairs are listed on the left and the required transform pairs listed on the right are obtained by applying Theorem 5.1.3.

$f(t) \leftrightarrow F(s)$	$e^{at}f(t) \leftrightarrow F(s-a)$
$1 \leftrightarrow \dfrac{1}{s}, \quad s > 0$	$e^{at} \leftrightarrow \dfrac{1}{(s-a)}, \quad s > a$
$t \leftrightarrow \dfrac{1}{s^2}, \quad s > 0$	$te^{at} \leftrightarrow \dfrac{1}{(s-a)^2}, \quad s > a$
$\sin \omega t \leftrightarrow \dfrac{\omega}{s^2+\omega^2}, \quad s > 0$	$e^{\lambda t}\sin \omega t \leftrightarrow \dfrac{\omega}{(s-\lambda)^2+\omega^2}, \quad s > \lambda$
$\cos \omega t \leftrightarrow \dfrac{s}{s^2+\omega^2}, \quad s > 0$	$e^{\lambda t}\sin \omega t \leftrightarrow \dfrac{s-\lambda}{(s-\lambda)^2+\omega^2}, \quad s > \lambda$

Existence of Laplace Transforms

Not every function has a Laplace transform. For example, it can be shown (Exercise 3) that

$$\int_0^\infty e^{-st}e^{t^2} dt = \infty$$

for every real number s. Hence, the function $f(t) = e^{t^2}$ does not have a Laplace transform.

Our next objective is to establish conditions that ensure the existence of the Laplace transform of a function. We first review some relevant definitions from calculus.

Recall that a limit

$$\lim_{t \to t_0} f(t)$$

Figure 5.1 A jump discontinuity

exists if and only if the one-sided limits

$$\lim_{t\to t_0-} f(t) \quad \text{and} \quad \lim_{t\to t_0+} f(t)$$

both exist and are equal; in this case,

$$\lim_{t\to t_0} f(t) = \lim_{t\to t_0-} f(t) = \lim_{t\to t_0+} f(t).$$

Recall also that f is continuous at a point t_0 in an open interval (a, b) if and only if

$$\lim_{t\to t_0} f(t) = f(t_0),$$

which is equivalent to

$$\lim_{t\to t_0+} f(t) = \lim_{t\to t_0-} f(t) = f(t_0). \tag{5.1.12}$$

For simplicity, we define

$$f(t_0+) = \lim_{t\to t_0+} f(t) \quad \text{and} \quad f(t_0-) = \lim_{t\to t_0-} f(t),$$

so (5.1.12) can be expressed as

$$f(t_0+) = f(t_0-) = f(t_0).$$

If $f(t_0+)$ and $f(t_0-)$ have finite but distinct values, we say that f has a *jump discontinuity* at t_0, and

$$f(t_0+) - f(t_0-)$$

is called the *jump* in f at t_0 (Figure 5.1).

If $f(t_0+)$ and $f(t_0-)$ are finite and equal, but either f isn't defined at t_0 or it's defined but

$$f(t_0) \neq f(t_0+) = f(t_0-),$$

we say that f has a *removable discontinuity* at t_0 (Figure 5.2). This terminolgy is appropriate since a function f with a removable discontinuity at t_0 can be made continuous at t_0 by defining (or redefining)

$$f(t_0) = f(t_0+) = f(t_0-).$$

Figure 5.2

Figure 5.3 A piecewise continuous function on $[a, b]$

REMARK: We know from calculus that a definite integral isn't affected by changing the values of its integrand at isolated points. Therefore, redefining a function f to make it continuous at removable discontinuities does not change $\mathcal{L}(f)$.

Definition 5.1.4

(i) A function f is said to be *piecewise continuous* on a finite closed interval $[0,T]$ if $f(0+)$ and $f(T-)$ are finite and f is continuous on the open interval $(0,T)$ except possibly at finitely many points, where f may have jump discontinuities or removable discontinuities.

(ii) A function f is said to be *piecewise continuous* on the infinite interval $[0,\infty)$ if it's piecewise continuous on $[0,T]$ for every $T>0$.

Figure 5.3 shows the graph of a typical piecewise continuous function.

It is shown in calculus that if a function is piecewise continuous on a finite closed interval then it's integrable on that interval. But if f is piecewise continuous on $[0,\infty)$, then so is $e^{-st}f(t)$, and therefore

$$\int_0^T e^{-st}f(t)\,dt$$

exists for every $T>0$. However, piecewise continuity alone does not guarantee that the improper integral

$$\int_0^\infty e^{-st}f(t)\,dt=\lim_{T\to\infty}\int_0^T e^{-st}f(t)\,dt \tag{5.1.13}$$

converges for s in some interval (s_0,∞). For example, we noted earlier that (5.1.13) diverges for all s if $f(t)=e^{t^2}$. Stated informally, this occurs because e^{t^2} increases too rapidly as $t\to\infty$. The next definition provides a constraint on the growth of a function that guarantees convergence of its Laplace transform for s in some interval (s_0,∞) .

Definition 5.1.5 A function f is said to be *of exponential order* s_0 if there are constants M and t_0 such that

$$|f(t)|\leqslant Me^{s_0t},\quad t\geqslant t_0. \tag{5.1.14}$$

In situations where the specific value of s_0 is irrelevant we say simply that f is *of exponential order.*

The next theorem gives useful sufficient conditions for a function f to have a Laplace transform. The proof is sketched in Exercise 10.

Theorem 5.1.6 *If f is piecewise continuous on $[0,\infty)$ and of exponential order s_0, then $\mathcal{L}(f)$ is defined for $s>s_0$.*

REMARK: We emphasize that the conditions of Theorem 5.1.6 are sufficient, but *not necessary*, for f to have a Laplace transform. For example, Exercise 14**(c)** shows that f may have a Laplace transform even though f isn't of exponential order.

Example 5.1.8 If f is bounded on some interval $[t_0,\infty)$, say

$$|f(t)|\leqslant M,\quad t\geqslant t_0,$$

then (5.1.14) holds with $s_0 = 0$, so f is of exponential order zero. Thus, for example, $\sin\omega t$ and $\cos\omega t$ are of exponential order zero, and Theorem 5.1.6 implies that $\mathcal{L}(\sin\omega t)$ and $\mathcal{L}(\cos\omega t)$ exist for $s > 0$. This is consistent with the conclusion of Example 5.1.4.

Example 5.1.9 It can be shown that if $\lim_{t\to\infty} e^{-s_0t}f(t)$ exists and is finite then f is of exponential order s_0 (Exercise 9). If α is any real number and $s_0 > 0$ then $f(t) = t^\alpha$ is of exponential order s_0, since

$$\lim_{t\to\infty} e^{-s_0t}t^\alpha = 0,$$

by L'Hôpital's rule. If $\alpha \geqslant 0$, f is also continuous on $[0,\infty)$. Therefore Exercise 9 and Theorem 5.1.6 imply that $\mathcal{L}(t^\alpha)$ exists for $s \geqslant s_0$. However, since s_0 is an arbitrary positive number, this really implies that $\mathcal{L}(t^\alpha)$ exists for all $s > 0$. This is consistent with the results of Example 5.1.2 and Exercises 6 and 8.

Example 5.1.10 Find the Laplace transform of the piecewise continuous function

$$f(t) = \begin{cases} 1, & 0 \leqslant t < 1, \\ -3e^{-t}, & t \geqslant 1. \end{cases}$$

Solution Since f is defined by different formulas on $[0,1)$ and $[1,\infty)$, we write

$$F(s) = \int_0^\infty e^{-st}f(t)\,dt = \int_0^1 e^{-st}(1)\,dt + \int_1^\infty e^{-st}(-3e^{-t})\,dt.$$

Since

$$\int_0^1 e^{-st}\,dt = \begin{cases} \dfrac{1-e^{-s}}{s}, & s \neq 0, \\ 1, & s = 0, \end{cases}$$

and

$$\int_1^\infty e^{-st}(-3e^{-t})\,dt = -3\int_1^\infty e^{-(s+1)t}\,dt = -\frac{3e^{-(s+1)}}{s+1}, \quad s > -1,$$

it follows that

$$F(s) = \begin{cases} \dfrac{1-e^{-s}}{s} - 3\dfrac{e^{-(s+1)}}{s+1}, & s > -1,\ s \neq 0, \\ 1 - \dfrac{3}{e}, & s = 0. \end{cases}$$

This is consistent with Theorem 5.1.6, since

$$|f(t)| \leqslant 3e^{-t}, \quad t \geqslant 1,$$

and therefore f is of exponential order $s_0 = -1$.

REMARK: In Section 8.4 we'll develop a more efficient method for finding Laplace transforms of piecewise continuous functions.

Example 5.1.11 We stated earlier that

$$\int_0^\infty e^{-st}e^{t^2}\,dt = \infty$$

for all s, so Theorem 5.1.6 implies that $f(t) = e^{t^2}$ is not of exponential order, since

$$\lim_{t\to\infty} \frac{e^{t^2}}{Me^{s_0 t}} = \lim_{t\to\infty} \frac{1}{M} e^{t^2 - s_0 t} = \infty,$$

so

$$e^{t^2} > Me^{s_0 t}$$

for sufficiently large values of t, for any choice of M and s_0 (Exercise 3).

5.1 Exercises

1. Find the Laplace transforms of the following functions by evaluating the integral $F(s) = \int_0^\infty e^{-st} f(t)\,dt$.

(a) t **(b)** te^{-t} **(c)** $\sinh bt$
(d) $e^{2t} - 3e^t$ **(e)** t^2

2. Use the table of Laplace transforms to find the Laplace transforms of the following functions.

(a) $\cosh t \sin t$ **(b)** $\sin^2 t$ **(c)** $\cos^2 2t$
(d) $\cosh^2 t$ **(e)** $t \sinh 2t$ **(f)** $\sin t \cos t$
(g) $\sin\left(t + \frac{\pi}{4}\right)$ **(h)** $\cos 2t - \cos 3t$ **(i)** $\sin 2t + \cos 4t$

3. Show that

$$\int_0^\infty e^{-st}e^{t^2}\,dt = \infty$$

for every real number s.

4. Graph the following piecewise continuous functions and evaluate $f(t+)$, $f(t-)$, and $f(t)$ at each point of discontinuity.

(a) $f(t) = \begin{cases} -t, & 0 \leqslant t < 2, \\ t-4, & 2 \leqslant t < 3, \\ 1, & t \geqslant 3. \end{cases}$ **(b)** $f(t) = \begin{cases} t^2+2, & 0 \leqslant t < 1, \\ 4, & t = 1, \\ t, & t > 1. \end{cases}$

(c) $f(t) = \begin{cases} \sin t, & 0 \leqslant t < \pi/2, \\ 2\sin t, & \pi/2 \leqslant t < \pi, \\ \cos t, & t \geqslant \pi. \end{cases}$ **(d)** $f(t) = \begin{cases} t, & 0 \leqslant t < 1, \\ 2, & t = 1, \\ 2-t, & 1 \leqslant t < 2, \\ 3, & t = 2, \\ 6, & t > 2. \end{cases}$

5. Find the Laplace transform:

(a) $f(t) = \begin{cases} e^{-t}, & 0 \leqslant t < 1, \\ e^{-2t}, & t \geqslant 1. \end{cases}$ **(b)** $f(t) = \begin{cases} 1, & 0 \leqslant t < 4, \\ t, & t \geqslant 4. \end{cases}$

(c) $f(t) = \begin{cases} t, & 0 \leqslant t < 1, \\ 1, & t \geqslant 1. \end{cases}$ **(d)** $f(t) = \begin{cases} te^{t}, & 0 \leqslant t < 1, \\ e^{t}, & t \geqslant 1. \end{cases}$

6. Prove that if $f(t) \leftrightarrow F(s)$ then $t^k f(t) \leftrightarrow (-1)^k F^{(k)}(s)$. HINT: *Assume that it's permissible to differentiate the integral* $\int_0^\infty e^{-st} f(t)\,dt$ *with respect to* s *under the integral sign.*

7. Use the known Laplace transforms

$$\mathcal{L}(e^{\lambda t}\sin\omega t) = \frac{\omega}{(s-\lambda)^2+\omega^2} \quad \text{and} \quad \mathcal{L}(e^{\lambda t}\cos\omega t) = \frac{s-\lambda}{(s-\lambda)^2+\omega^2}$$

and the result of Exercise 6 to find $\mathcal{L}(te^{\lambda t}\cos\omega t)$ and $\mathcal{L}(te^{\lambda t}\sin\omega t)$.

8. Use the known Laplace transform $\mathcal{L}(1) = 1/s$ and the result of Exercise 6 to show that

$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}, \quad n = \text{integer}.$$

9. **(a)** Show that if $\lim_{t\to\infty} e^{-s_0 t} f(t)$ exists and is finite then f is of exponential order s_0.

(b) Show that if f is of exponential order s_0 then $\lim_{t\to\infty} e^{-st} f(t) = 0$ for all $s > s_0$.

(c) Show that if f is of exponential order s_0 and $g(t) = f(t+\tau)$ where $\tau > 0$, then g is also of exponential order s_0.

10. Recall the next theorem from calculus.

THEOREM A. *Let* g *be integrable on* $[0, T]$ *for every* $T > 0$. *Suppose there's a function* w *defined on some interval* $[\tau, \infty)$ *(with* $\tau \geqslant 0$*) such that* $|g(t)| \leqslant w(t)$ *for* $t \geqslant \tau$ *and* $\int_\tau^\infty w(t)\,dt$ *converges. Then* $\int_0^\infty g(t)\,dt$ *converges.*

Use Theorem A to show that if f is piecewise continuous on $[0, \infty)$ and of exponential order s_0, then f has a Laplace transform $F(s)$ defined for $s > s_0$.

11. Prove: If f is piecewise continuous and of exponential order then $\lim_{s\to\infty} F(s) = 0$.

12. Prove: If f is continuous on $[0, \infty)$ and of exponential order $s_0 > 0$, then

$$\mathcal{L}\left(\int_0^t f(\tau)\,d\tau\right) = \frac{1}{s}\mathcal{L}(f), \quad s > s_0.$$

HINT: *Use integration by parts to evaluate the transform on the left.*

13. Suppose f is piecewise continuous and of exponential order, and that $\lim_{t\to 0+} f(t)/t$ exists. Show that

$$\mathcal{L}\left(\frac{f(t)}{t}\right) = \int_s^\infty F(r)\,dr.$$

HINT: *Use the results of Exercises* 6 *and* 11.

14. Suppose f is piecewise continuous on $[0,\infty)$.

(a) Prove: If the integral $g(t)=\int_0^t e^{-s_0\tau}f(\tau)\,d\tau$ satisfies the inequality $|g(t)|\leqslant M\ (t\geqslant 0)$, then f has a Laplace transform $F(s)$ defined for $s>s_0$. HINT: *Use integration by parts to show that*

$$\int_0^T e^{-st}f(t)\,dt=e^{-(s-s_0)T}g(T)+(s-s_0)\int_0^T e^{-(s-s_0)t}g(t)\,dt.$$

(b) Show that if $\mathcal{L}(f)$ exists for $s=s_0$ then it exists for $s>s_0$. Show that the function

$$f(t)=te^{t^2}\cos(e^{t^2})$$

has a Laplace transform defined for $s>0$, even though f isn't of exponential order.

(c) Show that the function

$$f(t)=te^{t^2}\cos(e^{t^2})$$

has a Laplace transform defined for $s>0$, even though f isn't of exponential order.

15. Use the table of Laplace transforms and the result of Exercise 13 to find the Laplace transforms of the following functions.

(a) $\dfrac{\sin\omega t}{t}\quad(\omega>0)$ **(b)** $\dfrac{\cos\omega t-1}{t}\quad(\omega>0)$ **(c)** $\dfrac{e^{at}-e^{bt}}{t}$

(d) $\dfrac{\cosh t-1}{t}$ **(e)** $\dfrac{\sinh^2 t}{t}$

16. The *gamma function* is defined by

$$\Gamma(\alpha)=\int_0^\infty x^{\alpha-1}e^{-x}\,dx,$$

which can be shown to converge if $\alpha>0$.

(a) Use integration by parts to show that

$$\Gamma(\alpha+1)=\alpha\Gamma(\alpha),\quad \alpha>0.$$

(b) Show that $\Gamma(n+1)=n!$ if $n=1,2,3,\ldots$.

(c) From **(b)** and the table of Laplace transforms,

$$\mathcal{L}(t^\alpha)=\frac{\Gamma(\alpha+1)}{s^{\alpha+1}},\quad s>0,$$

if α is a nonnegative integer. Show that this formula is valid for any $\alpha>-1$. HINT: *Change the variable of integration in the integral for* $\Gamma(\alpha+1)$.

17. Suppose f is continuous on $[0, T]$ and $f(t + T) = f(t)$ for all $t \geqslant 0$. (We say in this case that f is *periodic with period* T.)

(a) Conclude from Theorem 5.1.6 that the Laplace transform of f is defined for $s > 0$. HINT: *Since f is continuous on $[0, T]$ and periodic with period T, it's bounded on $[0, \infty)$.*

(b) **(b)** Show that

$$F(s) = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t)\, dt, \quad s > 0.$$

HINT: *Write*

$$F(s) = \sum_{n=0}^{\infty} \int_{nT}^{(n+1)T} e^{-st} f(t)\, dt.$$

Then show that

$$\int_{nT}^{(n+1)T} e^{-st} f(t)\, dt = e^{-nsT} \int_0^T e^{-st} f(t)\, dt,$$

and recall the formula for the sum of a geometric series.

18. Use the formula given in Exercise 17**(b)** to find the Laplace transforms of the given periodic functions:

(a) $f(t) = \begin{cases} t, & 0 \leqslant t < 1, \\ 2 - t, & 1 \leqslant t < 2, \end{cases} \qquad f(t+2) = f(t), \quad t \geqslant 0$

(b) $f(t) = \begin{cases} 1, & 0 \leqslant t < \frac{1}{2}, \\ -1, & \frac{1}{2} \leqslant t < 1, \end{cases} \qquad f(t+1) = f(t), \quad t \geqslant 0$

(c) $f(t) = |\sin t|$

(d) $f(t) = \begin{cases} \sin t, & 0 \leqslant t < \pi, \\ 0, & \pi \leqslant t < 2\pi, \end{cases} \qquad f(t + 2\pi) = f(t)$

5.2 THE INVERSE LAPLACE TRANSFORM

Definition of the Inverse Laplace Transform

In Section 8.1 we defined the Laplace transform of f by

$$F(s) = \mathcal{L}(f) = \int_0^{\infty} e^{-st} f(t)\, dt.$$

We'll also say that f is an *inverse Laplace Transform* of F, and write

$$f = \mathcal{L}^{-1}(F).$$

To solve differential equations with the Laplace transform, we must be able to obtain f from its transform F. There's a formula for doing this, but we can't use it because it requires the theory of functions of a complex variable. Fortunately, we can use the table of Laplace transforms to find inverse transforms that we'll need.

Example 5.2.1 Use the table of Laplace transforms to find

$$\textbf{(a)}\ \mathcal{L}^{-1}\left(\frac{1}{s^2-1}\right) \quad \text{and} \quad \textbf{(b)}\ \mathcal{L}^{-1}\left(\frac{s}{s^2+9}\right).$$

SOLUTION(**a**) Setting $b = 1$ in the transform pair

$$\sinh bt \leftrightarrow \frac{b}{s^2-b^2}$$

shows that

$$\mathcal{L}^{-1}\left(\frac{1}{s^2-1}\right) = \sinh t.$$

SOLUTION(**b**) Setting $\omega = 3$ in the transform pair

$$\cos \omega t \leftrightarrow \frac{s}{s^2+\omega^2}$$

shows that

$$\mathcal{L}^{-1}\left(\frac{s}{s^2+9}\right) = \cos 3t. \ \blacksquare$$

The next theorem enables us to find inverse transforms of linear combinations of transforms in the table. We omit the proof.

Theorem 5.2.1 *[Linearity Property] If $F_1, F_2, \dots, F_n$ are Laplace transforms and $c_1, c_2, \dots, c_n$ are constants, then*

$$\mathcal{L}^{-1}(c_1F_1 + c_2F_2 + \cdots + c_nF_n) = c_1\mathcal{L}^{-1}(F_1) + c_2\mathcal{L}^{-1}(F_2) + \cdots + c_n\mathcal{L}^{-1}F_n.$$

Example 5.2.2 Find

$$\mathcal{L}^{-1}\left(\frac{8}{s+5} + \frac{7}{s^2+3}\right).$$

Solution From the table of Laplace transforms in Section 8.8,,

$$e^{at} \leftrightarrow \frac{1}{s-a} \quad \text{and} \quad \sin \omega t \leftrightarrow \frac{\omega}{s^2+\omega^2}.$$

Theorem 5.2.1 with $a = -5$ and $\omega = \sqrt{3}$ yields

$$\begin{aligned}
\mathcal{L}^{-1}\left(\frac{8}{s+5} + \frac{7}{s^2+3}\right) &= 8\mathcal{L}^{-1}\left(\frac{1}{s+5}\right) + 7\mathcal{L}^{-1}\left(\frac{1}{s^2+3}\right) \\
&= 8\mathcal{L}^{-1}\left(\frac{1}{s+5}\right) + \frac{7}{\sqrt{3}}\mathcal{L}^{-1}\left(\frac{\sqrt{3}}{s^2+3}\right) \\
&= 8e^{-5t} + \frac{7}{\sqrt{3}}\sin\sqrt{3}t.
\end{aligned}$$

Example 5.2.3 Find

$$\mathcal{L}^{-1}\left(\frac{3s+8}{s^2+2s+5}\right).$$

Solution Completing the square in the denominator yields

$$\frac{3s+8}{s^2+2s+5}=\frac{3s+8}{(s+1)^2+4}.$$

Because of the form of the denominator, we consider the transform pairs

$$e^{-t}\cos 2t \leftrightarrow \frac{s+1}{(s+1)^2+4} \quad \text{and} \quad e^{-t}\sin 2t \leftrightarrow \frac{2}{(s+1)^2+4},$$

and write

$$\begin{aligned}
\mathcal{L}^{-1}\left(\frac{3s+8}{(s+1)^2+4}\right) &= \mathcal{L}^{-1}\left(\frac{3s+3}{(s+1)^2+4}\right)+\mathcal{L}^{-1}\left(\frac{5}{(s+1)^2+4}\right)\\
&= 3\mathcal{L}^{-1}\left(\frac{s+1}{(s+1)^2+4}\right)+\frac{5}{2}\mathcal{L}^{-1}\left(\frac{2}{(s+1)^2+4}\right)\\
&= e^{-t}(3\cos 2t+\frac{5}{2}\sin 2t).
\end{aligned}$$

REMARK: We'll often write inverse Laplace transforms of specific functions without explicitly stating how they are obtained. In such cases you should refer to the table of Laplace transforms in Section 8.8.

Inverse Laplace Transforms of Rational Functions

Using the Laplace transform to solve differential equations often requires finding the inverse transform of a rational function

$$F(s)=\frac{P(s)}{Q(s)},$$

where P and Q are polynomials in s with no common factors. Since it can be shown that $\lim_{s\to\infty} F(s)=0$ if F is a Laplace transform, we need only consider the case where degree$(P)<$ degree(Q). To obtain $\mathcal{L}^{-1}(F)$, we find the partial fraction expansion of F, obtain inverse transforms of the individual terms in the expansion from the table of Laplace transforms, and use the linearity property of the inverse transform. The next two examples illustrate this.

Example 5.2.4 Find the inverse Laplace transform of

$$F(s)=\frac{3s+2}{s^2-3s+2}. \tag{5.2.1}$$

Solution (METHOD 1) Factoring the denominator in (5.2.1) yields

$$F(s) = \frac{3s + 2}{(s - 1)(s - 2)}. \tag{5.2.2}$$

The form for the partial fraction expansion is

$$\frac{3s + 2}{(s - 1)(s - 2)} = \frac{A}{s - 1} + \frac{B}{s - 2}. \tag{5.2.3}$$

Multiplying this by $(s - 1)(s - 2)$ yields

$$3s + 2 = (s - 2)A + (s - 1)B.$$

Setting $s = 2$ yields $B = 8$ and setting $s = 1$ yields $A = -5$. Therefore

$$F(s) = -\frac{5}{s - 1} + \frac{8}{s - 2}$$

and

$$\mathcal{L}^{-1}(F) = -5\mathcal{L}^{-1}\left(\frac{1}{s - 1}\right) + 8\mathcal{L}^{-1}\left(\frac{1}{s - 2}\right) = -5e^{t} + 8e^{2t}.$$

Solution (METHOD 2) We don't really have to multiply (5.2.3) by $(s-1)(s-2)$ to compute A and B. We can obtain A by simply ignoring the factor $s - 1$ in the denominator of (5.2.2) and setting $s = 1$ elsewhere; thus,

$$A = \left.\frac{3s + 2}{s - 2}\right|_{s=1} = \frac{3 \cdot 1 + 2}{1 - 2} = -5. \tag{5.2.4}$$

Similarly, we can obtain B by ignoring the factor $s - 2$ in the denominator of (5.2.2) and setting $s = 2$ elsewhere; thus,

$$B = \left.\frac{3s + 2}{s - 1}\right|_{s=2} = \frac{3 \cdot 2 + 2}{2 - 1} = 8. \tag{5.2.5}$$

To justify this, we observe that multiplying (5.2.3) by $s - 1$ yields

$$\frac{3s + 2}{s - 2} = A + (s - 1)\frac{B}{s - 2},$$

and setting $s = 1$ leads to (5.2.4). Similarly, multiplying (5.2.3) by $s - 2$ yields

$$\frac{3s + 2}{s - 1} = (s - 2)\frac{A}{s - 2} + B$$

and setting $s = 2$ leads to (5.2.5). (It isn't necesary to write the last two equations. We wrote them only to justify the shortcut procedure indicated in (5.2.4) and (5.2.5).) ■

The shortcut employed in the second solution of Example 5.2.4 is *Heaviside's method*. The next theorem states this method formally. For a proof and an extension of this theorem, see Exercise 10.

Theorem 5.2.2 *Suppose*

$$F(s) = \frac{P(s)}{(s - s_1)(s - s_2) \cdots (s - s_n)}, \tag{5.2.6}$$

where $s_1, s_2, \dots, s_n$ *are distinct and* P *is a polynomial of degree less than* n*. Then*

$$F(s) = \frac{A_1}{s - s_1} + \frac{A_2}{s - s_2} + \cdots + \frac{A_n}{s - s_n},$$

where A_i *can be computed from* (5.2.6) *by ignoring the factor* $s - s_i$ *and setting* $s = s_i$ *elsewhere.*

Example 5.2.5 Find the inverse Laplace transform of

$$F(s) = \frac{6 + (s + 1)(s^2 - 5s + 11)}{s(s - 1)(s - 2)(s + 1)}. \tag{5.2.7}$$

Solution The partial fraction expansion of (5.2.7) is of the form

$$F(s) = \frac{A}{s} + \frac{B}{s - 1} + \frac{C}{s - 2} + \frac{D}{s + 1}. \tag{5.2.8}$$

To find A, we ignore the factor s in the denominator of (5.2.7) and set $s = 0$ elsewhere. This yields

$$A = \frac{6 + (1)(11)}{(-1)(-2)(1)} = \frac{17}{2}.$$

Similarly, the other coefficients are given by

$$B = \frac{6 + (2)(7)}{(1)(-1)(2)} = -10,$$

$$C = \frac{6 + 3(5)}{2(1)(3)} = \frac{7}{2},$$

and

$$D = \frac{6}{(-1)(-2)(-3)} = -1.$$

Therefore

$$F(s) = \frac{17}{2}\frac{1}{s} - \frac{10}{s - 1} + \frac{7}{2}\frac{1}{s - 2} - \frac{1}{s + 1}$$

and

$$\begin{aligned} \mathcal{L}^{-1}(F) &= \frac{17}{2}\mathcal{L}^{-1}\left(\frac{1}{s}\right) - 10\mathcal{L}^{-1}\left(\frac{1}{s - 1}\right) + \frac{7}{2}\mathcal{L}^{-1}\left(\frac{1}{s - 2}\right) - \mathcal{L}^{-1}\left(\frac{1}{s + 1}\right) \\ &= \frac{17}{2} - 10e^t + \frac{7}{2}e^{2t} - e^{-t}. \end{aligned}$$

REMARK: We didn't "multiply out" the numerator in (5.2.7) before computing the coefficients in (5.2.8), since it wouldn't simplify the computations.

Example 5.2.6 Find the inverse Laplace transform of

$$F(s) = \frac{8-(s+2)(4s+10)}{(s+1)(s+2)^2}. \tag{5.2.9}$$

Solution The form for the partial fraction expansion is

$$F(s) = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{(s+2)^2}. \tag{5.2.10}$$

Because of the repeated factor $(s+2)^2$ in (5.2.9), Heaviside's method doesn't work. Instead, we find a common denominator in (5.2.10). This yields

$$F(s) = \frac{A(s+2)^2 + B(s+1)(s+2) + C(s+1)}{(s+1)(s+2)^2}. \tag{5.2.11}$$

If (5.2.9) and (5.2.11) are to be equivalent, then

$$A(s+2)^2 + B(s+1)(s+2) + C(s+1) = 8-(s+2)(4s+10). \tag{5.2.12}$$

The two sides of this equation are polynomials of degree two. From a theorem of algebra, they will be equal for all s if they are equal for any three distinct values of s. We may determine A, B and C by choosing convenient values of s.

The left side of (5.2.12) suggests that we take $s=-2$ to obtain $C=-8$, and $s=-1$ to obtain $A=2$. We can now choose any third value of s to determine B. Taking $s=0$ yields $4A+2B+C=-12$. Since $A=2$ and $C=-8$ this implies that $B=-6$. Therefore

$$F(s) = \frac{2}{s+1} - \frac{6}{s+2} - \frac{8}{(s+2)^2}$$

and

$$\begin{aligned} \mathcal{L}^{-1}(F) &= 2\mathcal{L}^{-1}\left(\frac{1}{s+1}\right) - 6\mathcal{L}^{-1}\left(\frac{1}{s+2}\right) - 8\mathcal{L}^{-1}\left(\frac{1}{(s+2)^2}\right) \\ &= 2e^{-t} - 6e^{-2t} - 8te^{-2t}. \end{aligned}$$

Example 5.2.7 Find the inverse Laplace transform of

$$F(s) = \frac{s^2-5s+7}{(s+2)^3}.$$

Solution The form for the partial fraction expansion is

$$F(s) = \frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{C}{(s+2)^3}.$$

The easiest way to obtain A, B, and C is to expand the numerator in powers of $s+2$. This yields

$$s^2-5s+7=[(s+2)-2]^2-5[(s+2)-2]+7=(s+2)^2-9(s+2)+21.$$

Therefore

$$\begin{aligned} F(s) &= \frac{(s+2)^2-9(s+2)+21}{(s+2)^3} \\ &= \frac{1}{s+2}-\frac{9}{(s+2)^2}+\frac{21}{(s+2)^3} \end{aligned}$$

and

$$\begin{aligned} \mathcal{L}^{-1}(F) &= \mathcal{L}^{-1}\left(\frac{1}{s+2}\right)-9\mathcal{L}^{-1}\left(\frac{1}{(s+2)^2}\right)+\frac{21}{2}\mathcal{L}^{-1}\left(\frac{2}{(s+2)^3}\right) \\ &= e^{-2t}\left(1-9t+\frac{21}{2}t^2\right). \end{aligned}$$

Example 5.2.8 Find the inverse Laplace transform of

$$F(s)=\frac{1-s(5+3s)}{s\left[(s+1)^2+1\right]}. \tag{5.2.13}$$

Solution One form for the partial fraction expansion of F is

$$F(s)=\frac{A}{s}+\frac{Bs+C}{(s+1)^2+1}. \tag{5.2.14}$$

However, we see from the table of Laplace transforms that the inverse transform of the second fraction on the right of (5.2.14) will be a linear combination of the inverse transforms

$$e^{-t}\cos t \quad \text{and} \quad e^{-t}\sin t$$

of

$$\frac{s+1}{(s+1)^2+1} \quad \text{and} \quad \frac{1}{(s+1)^2+1}$$

respectively. Therefore, instead of (5.2.14) we write

$$F(s)=\frac{A}{s}+\frac{B(s+1)+C}{(s+1)^2+1}. \tag{5.2.15}$$

Finding a common denominator yields

$$F(s)=\frac{A\left[(s+1)^2+1\right]+B(s+1)s+Cs}{s\left[(s+1)^2+1\right]}. \tag{5.2.16}$$

If (5.2.13) and (5.2.16) are to be equivalent, then

$$A\left[(s+1)^2+1\right]+B(s+1)s+Cs=1-s(5+3s).$$

This is true for all s if it's true for three distinct values of s. Choosing $s=0,-1$, and 1 yields the system

$$\begin{aligned} 2A &= 1 \\ A-C &= 3 \\ 5A+2B+C &= -7. \end{aligned}$$

Solving this system yields

$$A=\frac{1}{2}, \quad B=-\frac{7}{2}, \quad C=-\frac{5}{2}.$$

Hence, from (5.2.15),

$$F(s)=\frac{1}{2s}-\frac{7}{2}\frac{s+1}{(s+1)^2+1}-\frac{5}{2}\frac{1}{(s+1)^2+1}.$$

Therefore

$$\begin{aligned} \mathcal{L}^{-1}(F) &= \frac{1}{2}\mathcal{L}^{-1}\left(\frac{1}{s}\right)-\frac{7}{2}\mathcal{L}^{-1}\left(\frac{s+1}{(s+1)^2+1}\right)-\frac{5}{2}\mathcal{L}^{-1}\left(\frac{1}{(s+1)^2+1}\right) \\ &= \frac{1}{2}-\frac{7}{2}e^{-t}\cos t-\frac{5}{2}e^{-t}\sin t. \end{aligned}$$

Example 5.2.9 Find the inverse Laplace transform of

$$F(s)=\frac{8+3s}{(s^2+1)(s^2+4)}. \tag{5.2.17}$$

Solution The form for the partial fraction expansion is

$$F(s)=\frac{A+Bs}{s^2+1}+\frac{C+Ds}{s^2+4}.$$

The coefficients A, B, C and D can be obtained by finding a common denominator and equating the resulting numerator to the numerator in (5.2.17). However, since there's no first power of s in the denominator of (5.2.17), there's an easier way: the expansion of

$$F_1(s)=\frac{1}{(s^2+1)(s^2+4)}$$

can be obtained quickly by using Heaviside's method to expand

$$\frac{1}{(x+1)(x+4)}=\frac{1}{3}\left(\frac{1}{x+1}-\frac{1}{x+4}\right)$$

and then setting $x = s^2$ to obtain

$$\frac{1}{(s^2+1)(s^2+4)} = \frac{1}{3}\left(\frac{1}{s^2+1} - \frac{1}{s^2+4}\right).$$

Multiplying this by $8+3s$ yields

$$F(s) = \frac{8+3s}{(s^2+1)(s^2+4)} = \frac{1}{3}\left(\frac{8+3s}{s^2+1} - \frac{8+3s}{s^2+4}\right).$$

Therefore

$$\mathcal{L}^{-1}(F) = \frac{8}{3}\sin t + \cos t - \frac{4}{3}\sin 2t - \cos 2t.$$

USING TECHNOLOGY

Some software packages that do symbolic algebra can find partial fraction expansions very easily. We recommend that you use such a package if one is available to you, but only after you've done enough partial fraction expansions on your own to master the technique.

5.2 Exercises

1. Use the table of Laplace transforms to find the inverse Laplace transform.

(a) $\dfrac{3}{(s-7)^4}$ **(b)** $\dfrac{2s-4}{s^2-4s+13}$ **(c)** $\dfrac{1}{s^2+4s+20}$

(d) $\dfrac{2}{s^2+9}$ **(e)** $\dfrac{s^2-1}{(s^2+1)^2}$ **(f)** $\dfrac{1}{(s-2)^2-4}$

(g) $\dfrac{12s-24}{(s^2-4s+85)^2}$ **(h)** $\dfrac{2}{(s-3)^2-9}$ **(i)** $\dfrac{s^2-4s+3}{(s^2-4s+5)^2}$

2. Use Theorem 5.2.1 and the table of Laplace transforms to find the inverse Laplace transform.

(a) $\dfrac{2s+3}{(s-7)^4}$ **(b)** $\dfrac{s^2-1}{(s-2)^6}$ **(c)** $\dfrac{s+5}{s^2+6s+18}$

(d) $\dfrac{2s+1}{s^2+9}$ **(e)** $\dfrac{s}{s^2+2s+1}$ **(f)** $\dfrac{s+1}{s^2-9}$

(g) $\dfrac{s^3+2s^2-s-3}{(s+1)^4}$ **(h)** $\dfrac{2s+3}{(s-1)^2+4}$ **(i)** $\dfrac{1}{s} - \dfrac{s}{s^2+1}$

(j) $\dfrac{3s+4}{s^2-1}$ **(k)** $\dfrac{3}{s-1} + \dfrac{4s+1}{s^2+9}$ **(l)** $\dfrac{3}{(s+2)^2} - \dfrac{2s+6}{s^2+4}$

3. Use Heaviside's method to find the inverse Laplace transform.

(a) $\dfrac{3-(s+1)(s-2)}{(s+1)(s+2)(s-2)}$ **(b)** $\dfrac{7+(s+4)(18-3s)}{(s-3)(s-1)(s+4)}$

(c) $\dfrac{2+(s-2)(3-2s)}{(s-2)(s+2)(s-3)}$ **(d)** $\dfrac{3-(s-1)(s+1)}{(s+4)(s-2)(s-1)}$

(e) $\dfrac{3+(s-2)(10-2s-s^2)}{(s-2)(s+2)(s-1)(s+3)}$ **(f)** $\dfrac{3+(s-3)(2s^2+s-21)}{(s-3)(s-1)(s+4)(s-2)}$

4. Find the inverse Laplace transform.

(a) $\dfrac{2+3s}{(s^2+1)(s+2)(s+1)}$ **(b)** $\dfrac{3s^2+2s+1}{(s^2+1)(s^2+2s+2)}$

(c) $\dfrac{3s+2}{(s-2)(s^2+2s+5)}$ **(d)** $\dfrac{3s^2+2s+1}{(s-1)^2(s+2)(s+3)}$

(e) $\dfrac{2s^2+s+3}{(s-1)^2(s+2)^2}$ **(f)** $\dfrac{3s+2}{(s^2+1)(s-1)^2}$

5. Use the method of Example 5.2.9 to find the inverse Laplace transform.

(a) $\dfrac{3s+2}{(s^2+4)(s^2+9)}$ **(b)** $\dfrac{-4s+1}{(s^2+1)(s^2+16)}$ **(c)** $\dfrac{5s+3}{(s^2+1)(s^2+4)}$

(d) $\dfrac{-s+1}{(4s^2+1)(s^2+1)}$ **(e)** $\dfrac{17s-34}{(s^2+16)(16s^2+1)}$ **(f)** $\dfrac{2s-1}{(4s^2+1)(9s^2+1)}$

6. Find the inverse Laplace transform.

(a) $\dfrac{17s-15}{(s^2-2s+5)(s^2+2s+10)}$ **(b)** $\dfrac{8s+56}{(s^2-6s+13)(s^2+2s+5)}$

(c) $\dfrac{s+9}{(s^2+4s+5)(s^2-4s+13)}$ **(d)** $\dfrac{3s-2}{(s^2-4s+5)(s^2-6s+13)}$

(e) $\dfrac{3s-1}{(s^2-2s+2)(s^2+2s+5)}$ **(f)** $\dfrac{20s+40}{(4s^2-4s+5)(4s^2+4s+5)}$

7. Find the inverse Laplace transform.

(a) $\dfrac{1}{s(s^2+1)}$ **(b)** $\dfrac{1}{(s-1)(s^2-2s+17)}$

(c) $\dfrac{3s+2}{(s-2)(s^2+2s+10)}$ **(d)** $\dfrac{34-17s}{(2s-1)(s^2-2s+5)}$

(e) $\dfrac{s+2}{(s-3)(s^2+2s+5)}$ **(f)** $\dfrac{2s-2}{(s-2)(s^2+2s+10)}$

8. Find the inverse Laplace transform.

(a) $\dfrac{2s+1}{(s^2+1)(s-1)(s-3)}$ **(b)** $\dfrac{s+2}{(s^2+2s+2)(s^2-1)}$

(c) $\dfrac{2s-1}{(s^2-2s+2)(s+1)(s-2)}$ **(d)** $\dfrac{s-6}{(s^2-1)(s^2+4)}$

(e) $\dfrac{2s-3}{s(s-2)(s^2-2s+5)}$ **(f)** $\dfrac{5s-15}{(s^2-4s+13)(s-2)(s-1)}$

9. Given that $f(t) \leftrightarrow F(s)$, find the inverse Laplace transform of $F(as-b)$, where $a>0$.

10. **(a)** If $s_1, s_2, \dots, s_n$ are distinct and P is a polynomial of degree less than n, then

$$\frac{P(s)}{(s-s_1)(s-s_2)\cdots(s-s_n)} = \frac{A_1}{s-s_1} + \frac{A_2}{s-s_2} + \cdots + \frac{A_n}{s-s_n}.$$

Multiply through by $s-s_i$ to show that A_i can be obtained by ignoring the factor $s-s_i$ on the left and setting $s=s_i$ elsewhere.

(b) Suppose P and Q_1 are polynomials such that $\text{degree}(P) \leqslant \text{degree}(Q_1)$ and $Q_1(s_1) \neq 0$. Show that the coefficient of $1/(s-s_1)$ in the partial fraction expansion of

$$F(s) = \frac{P(s)}{(s-s_1)Q_1(s)}$$

is $P(s_1)/Q_1(s_1)$.

(c) Explain how the results of **(a)** and **(b)** are related.

5.3 SOLUTION OF INITIAL VALUE PROBLEMS

Laplace Transforms of Derivatives

In the rest of this chapter we'll use the Laplace transform to solve initial value problems for constant coefficient second order equations. To do this, we must know how the Laplace transform of f' is related to the Laplace transform of f. The next theorem answers this question.

Theorem 5.3.1 *Suppose f is continuous on $[0,\infty)$ and of exponential order s_0, and f' is piecewise continuous on $[0,\infty)$. Then f and f' have Laplace transforms for $s>s_0$, and*

$$\mathcal{L}(f') = s\mathcal{L}(f) - f(0). \tag{5.3.1}$$

Proof

We know from Theorem 8.1.6 that $\mathcal{L}(f)$ is defined for $s>s_0$. We first consider the case where f' is continuous on $[0,\infty)$. Integration by parts yields

$$\begin{aligned} \int_0^T e^{-st} f'(t)\,dt &= e^{-st} f(t)\Big|_0^T + s\int_0^T e^{-st} f(t)\,dt \\ &= e^{-sT} f(T) - f(0) + s\int_0^T e^{-st} f(t)\,dt \end{aligned} \tag{5.3.2}$$

for any $T>0$. Since f is of exponential order s_0, $\lim_{T\to\infty} e^{-sT} f(T) = 0$ and the last integral in (5.3.2) converges as $T\to\infty$ if $s>s_0$. Therefore

$$\begin{aligned} \int_0^\infty e^{-st} f'(t)\,dt &= -f(0) + s\int_0^\infty e^{-st} f(t)\,dt \\ &= -f(0) + s\mathcal{L}(f), \end{aligned}$$

which proves (5.3.1). Now suppose $T > 0$ and f' is only piecewise continuous on $[0, T]$, with discontinuities at $t_1 < t_2 < \cdots < t_{n-1}$. For convenience, let $t_0 = 0$ and $t_n = T$. Integrating by parts yields

$$\begin{aligned}\int_{t_{i-1}}^{t_i} e^{-st} f'(t)\, dt &= e^{-st} f(t)\Big|_{t_{i-1}}^{t_i} + s\int_{t_{i-1}}^{t_i} e^{-st} f(t)\, dt \\ &= e^{-st_i} f(t_i) - e^{-st_{i-1}} f(t_{i-1}) + s\int_{t_{i-1}}^{t_i} e^{-st} f(t)\, dt.\end{aligned}$$

Summing both sides of this equation from $i = 1$ to n and noting that

$$\left(e^{-st_1} f(t_1) - e^{-st_0} f(t_0)\right) + \left(e^{-st_2} f(t_2) - e^{-st_1} f(t_1)\right) + \cdots + \left(e^{-st_N} f(t_N) - e^{-st_{N-1}} f(t_{N-1})\right)$$

$$= e^{-st_N} f(t_N) - e^{-st_0} f(t_0) = e^{-sT} f(T) - f(0)$$

yields (5.3.2), so (5.3.1) follows as before.

Example 5.3.1 In Example 5.1.4 we saw that

$$\mathcal{L}(\cos\omega t) = \frac{s}{s^2+\omega^2}.$$

Applying (5.3.1) with $f(t) = \cos\omega t$ shows that

$$\mathcal{L}(-\omega\sin\omega t) = s\frac{s}{s^2+\omega^2} - 1 = -\frac{\omega^2}{s^2+\omega^2}.$$

Therefore

$$\mathcal{L}(\sin\omega t) = \frac{\omega}{s^2+\omega^2},$$

which agrees with the corresponding result obtained in 5.1.4. ■

In Section 2.1 we showed that the solution of the initial value problem

$$y' = ay, \quad y(0) = y_0, \tag{5.3.3}$$

is $y = y_0 e^{at}$. We'll now obtain this result by using the Laplace transform.

Let $Y(s) = \mathcal{L}(y)$ be the Laplace transform of the unknown solution of (5.3.3). Taking Laplace transforms of both sides of (5.3.3) yields

$$\mathcal{L}(y') = \mathcal{L}(ay),$$

which, by Theorem 5.3.1, can be rewritten as

$$s\mathcal{L}(y) - y(0) = a\mathcal{L}(y),$$

or

$$sY(s) - y_0 = aY(s).$$

Solving for $Y(s)$ yields

$$Y(s)=\frac{y_0}{s-a},$$

so

$$y=\mathcal{L}^{-1}(Y(s))=\mathcal{L}^{-1}\left(\frac{y_0}{s-a}\right)=y_0\mathcal{L}^{-1}\left(\frac{1}{s-a}\right)=y_0e^{at},$$

which agrees with the known result.

We need the next theorem to solve second order differential equations using the Laplace transform.

Theorem 5.3.2 *Suppose f and f' are continuous on $[0,\infty)$ and of exponential order s_0, and that f'' is piecewise continuous on $[0,\infty)$. Then f, f', and f'' have Laplace transforms for $s>s_0$,*

$$\mathcal{L}(f')=s\mathcal{L}(f)-f(0), \tag{5.3.4}$$

and

$$\mathcal{L}(f'')=s^2\mathcal{L}(f)-f'(0)-sf(0). \tag{5.3.5}$$

Proof Theorem 5.3.1 implies that $\mathcal{L}(f')$ exists and satisfies (5.3.4) for $s>s_0$. To prove that $\mathcal{L}(f'')$ exists and satisfies (5.3.5) for $s>s_0$, we first apply Theorem 5.3.1 to $g=f'$. Since g satisfies the hypotheses of Theorem 5.3.1, we conclude that $\mathcal{L}(g')$ is defined and satisfies

$$\mathcal{L}(g')=s\mathcal{L}(g)-g(0)$$

for $s>s_0$. However, since $g'=f''$, this can be rewritten as

$$\mathcal{L}(f'')=s\mathcal{L}(f')-f'(0).$$

Substituting (5.3.4) into this yields (5.3.5).

Solving Second Order Equations with the Laplace Transform

We'll now use the Laplace transform to solve initial value problems for second order equations.

Example 5.3.2 Use the Laplace transform to solve the initial value problem

$$y''-6y'+5y=3e^{2t},\quad y(0)=2,\quad y'(0)=3. \tag{5.3.6}$$

Solution Taking Laplace transforms of both sides of the differential equation in (5.3.6) yields

$$\mathcal{L}(y''-6y'+5y)=\mathcal{L}\left(3e^{2t}\right)=\frac{3}{s-2},$$

which we rewrite as

$$\mathcal{L}(y'')-6\mathcal{L}(y')+5\mathcal{L}(y)=\frac{3}{s-2}. \tag{5.3.7}$$

Now denote $\mathcal{L}(y) = Y(s)$. Theorem 5.3.2 and the initial conditions in (5.3.6) imply that

$$\mathcal{L}(y') = sY(s) - y(0) = sY(s) - 2$$

and

$$\mathcal{L}(y'') = s^2Y(s) - y'(0) - sy(0) = s^2Y(s) - 3 - 2s.$$

Substituting from the last two equations into (5.3.7) yields

$$\left(s^2Y(s) - 3 - 2s\right) - 6\left(sY(s) - 2\right) + 5Y(s) = \frac{3}{s-2}.$$

Therefore

$$(s^2 - 6s + 5)Y(s) = \frac{3}{s-2} + (3 + 2s) + 6(-2), \tag{5.3.8}$$

so

$$(s-5)(s-1)Y(s) = \frac{3 + (s-2)(2s-9)}{s-2},$$

and

$$Y(s) = \frac{3 + (s-2)(2s-9)}{(s-2)(s-5)(s-1)}.$$

Heaviside's method yields the partial fraction expansion

$$Y(s) = -\frac{1}{s-2} + \frac{1}{2}\frac{1}{s-5} + \frac{5}{2}\frac{1}{s-1},$$

and taking the inverse transform of this yields

$$y = -e^{2t} + \frac{1}{2}e^{5t} + \frac{5}{2}e^{t}$$

as the solution of (5.3.6). ■

It isn't necessary to write all the steps that we used to obtain (5.3.8). To see how to avoid this, let's apply the method of Example 5.3.2 to the general initial value problem

$$ay'' + by' + cy = f(t), \quad y(0) = k_0, \quad y'(0) = k_1. \tag{5.3.9}$$

Taking Laplace transforms of both sides of the differential equation in (5.3.9) yields

$$a\mathcal{L}(y'') + b\mathcal{L}(y') + c\mathcal{L}(y) = F(s). \tag{5.3.10}$$

Now let $Y(s) = \mathcal{L}(y)$. Theorem 5.3.2 and the initial conditions in (5.3.9) imply that

$$\mathcal{L}(y') = sY(s) - k_0 \quad \text{and} \quad \mathcal{L}(y'') = s^2Y(s) - k_1 - k_0s.$$

Substituting these into (5.3.10) yields

$$a\left(s^2Y(s) - k_1 - k_0s\right) + b\left(sY(s) - k_0\right) + cY(s) = F(s). \tag{5.3.11}$$

The coefficient of $Y(s)$ on the left is the characteristic polynomial

$$p(s) = as^2 + bs + c$$

of the complementary equation for (5.3.9). Using this and moving the terms involving k_0 and k_1 to the right side of (5.3.11) yields

$$p(s)Y(s) = F(s) + a(k_1 + k_0 s) + bk_0. \tag{5.3.12}$$

This equation corresponds to (5.3.8) of Example 5.3.2. Having established the form of this equation in the general case, it is preferable to go directly from the initial value problem to this equation. You may find it easier to remember (5.3.12) rewritten as

$$p(s)Y(s) = F(s) + a\left(y'(0) + sy(0)\right) + by(0). \tag{5.3.13}$$

Example 5.3.3 Use the Laplace transform to solve the initial value problem

$$2y'' + 3y' + y = 8e^{-2t}, \quad y(0) = -4, \; y'(0) = 2. \tag{5.3.14}$$

Solution The characteristic polynomial is

$$p(s) = 2s^2 + 3s + 1 = (2s + 1)(s + 1)$$

and

$$F(s) = \mathcal{L}(8e^{-2t}) = \frac{8}{s+2},$$

so (5.3.13) becomes

$$(2s + 1)(s + 1)Y(s) = \frac{8}{s+2} + 2(2 - 4s) + 3(-4).$$

Solving for $Y(s)$ yields

$$Y(s) = \frac{4\left(1 - (s + 2)(s + 1)\right)}{(s + 1/2)(s + 1)(s + 2)}.$$

Heaviside's method yields the partial fraction expansion

$$Y(s) = \frac{4}{3}\frac{1}{s + 1/2} - \frac{8}{s+1} + \frac{8}{3}\frac{1}{s+2},$$

so the solution of (5.3.14) is

$$y = \mathcal{L}^{-1}(Y(s)) = \frac{4}{3}e^{-t/2} - 8e^{-t} + \frac{8}{3}e^{-2t}$$

(Figure 5.1).

Example 5.3.4 Solve the initial value problem

$$y'' + 2y' + 2y = 1, \quad y(0) = -3, \; y'(0) = 1. \tag{5.3.15}$$

Figure 5.1 $y = \frac{4}{3}e^{-t/2} - 8e^{-t} + \frac{8}{3}e^{-2t}$ Figure 5.2 $y = \frac{1}{2} - \frac{7}{2}e^{-t}\cos t - \frac{5}{2}e^{-t}\sin t$

Solution The characteristic polynomial is

$$p(s) = s^2 + 2s + 2 = (s+1)^2 + 1$$

and

$$F(s) = \mathcal{L}(1) = \frac{1}{s},$$

so (5.3.13) becomes

$$\left[(s+1)^2 + 1\right] Y(s) = \frac{1}{s} + 1 \cdot (1 - 3s) + 2(-3).$$

Solving for $Y(s)$ yields

$$Y(s) = \frac{1 - s(5 + 3s)}{s\left[(s+1)^2 + 1\right]}.$$

In Example 5.2.8 we found the inverse transform of this function to be

$$y = \frac{1}{2} - \frac{7}{2}e^{-t}\cos t - \frac{5}{2}e^{-t}\sin t$$

(Figure 5.2), which is therefore the solution of (5.3.15).

REMARK: In our examples we applied Theorems 5.3.1 and 5.3.2 without verifying that the unknown function y satisfies their hypotheses. This is characteristic of the formal manipulative way in which the Laplace transform is used to solve differential equations. Any doubts about the validity of the method for solving a given equation can be resolved by verifying that the resulting function y is the solution of the given problem.

5.3 Exercises

In Exercises 1–31 use the Laplace transform to solve the initial value problem.

1. $y'' + 3y' + 2y = e^t, \quad y(0) = 1, \quad y'(0) = -6$

2. $y'' - y' - 6y = 2, \quad y(0) = 1, \quad y'(0) = 0$

3. $y'' + y' - 2y = 2e^{3t}, \quad y(0) = -1, \quad y'(0) = 4$

4. $y'' - 4y = 2e^{3t}, \quad y(0) = 1, \quad y'(0) = -1$

5. $y'' + y' - 2y = e^{3t}, \quad y(0) = 1, \quad y'(0) = -1$

6. $y'' + 3y' + 2y = 6e^t, \quad y(0) = 1, \quad y'(0) = -1$

7. $y'' + y = \sin 2t, \quad y(0) = 0, \quad y'(0) = 1$

8. $y'' - 3y' + 2y = 2e^{3t}, \quad y(0) = 1, \quad y'(0) = -1$

9. $y'' - 3y' + 2y = e^{4t}, \quad y(0) = 1, \quad y'(0) = -2$

10. $y'' - 3y' + 2y = e^{3t}, \quad y(0) = -1, \quad y'(0) = -4$

11. $y'' + 3y' + 2y = 2e^{t}, \quad y(0) = 0, \quad y'(0) = -1$

12. $y'' + y' - 2y = -4, \quad y(0) = 2, \quad y'(0) = 3$

13. $y'' + 4y = 4, \quad y(0) = 0, \quad y'(0) = 1$

14. $y'' - y' - 6y = 2, \quad y(0) = 1, \quad y'(0) = 0$

15. $y'' + 3y' + 2y = e^{t}, \quad y(0) = 0, \quad y'(0) = 1$

16. $y'' - y = 1, \quad y(0) = 1, \quad y'(0) = 0$

17. $y'' + 4y = 3\sin t, \quad y(0) = 1, \quad y'(0) = -1$

18. $y'' + y' = 2e^{3t}, \quad y(0) = -1, \quad y'(0) = 4$

19. $y'' + y = 1, \quad y(0) = 2, \quad y'(0) = 0$

20. $y'' + y = t, \quad y(0) = 0, \quad y'(0) = 2$

21. $y'' + y = t - 3\sin 2t, \quad y(0) = 1, \quad y'(0) = -3$

22. $y'' + 5y' + 6y = 2e^{-t}, \quad y(0) = 1, \quad y'(0) = 3$

23. $y'' + 2y' + y = 6\sin t - 4\cos t, \quad y(0) = -1,\ y'(0) = 1$

24. $y'' - 2y' - 3y = 10\cos t, \quad y(0) = 2, \quad y'(0) = 7$

25. $y'' + y = 4\sin t + 6\cos t, \quad y(0) = -6,\ y'(0) = 2$

26. $y'' + 4y = 8\sin 2t + 9\cos t, \quad y(0) = 1, \quad y'(0) = 0$

27. $y'' - 5y' + 6y = 10e^{t}\cos t, \quad y(0) = 2, \quad y'(0) = 1$

28. $y'' + 2y' + 2y = 2t, \quad y(0) = 2, \quad y'(0) = -7$

29. $y'' - 2y' + 2y = 5\sin t + 10\cos t, \quad y(0) = 1,\ y'(0) = 2$

30. $y'' + 4y' + 13y = 10e^{-t} - 36e^{t}, \quad y(0) = 0,\ y'(0) = -16$

31. $y'' + 4y' + 5y = e^{-t}(\cos t + 3\sin t), \quad y(0) = 0, \quad y'(0) = 4$

32. $2y'' - 3y' - 2y = 4e^{t}, \quad y(0) = 1,\ y'(0) = -2$

33. $6y'' - y' - y = 3e^{2t}, \quad y(0) = 0,\ y'(0) = 0$

34. $2y'' + 2y' + y = 2t, \quad y(0) = 1,\ y'(0) = -1$

35. $4y'' - 4y' + 5y = 4\sin t - 4\cos t, \quad y(0) = 0,\ y'(0) = 11/17$

36. $4y'' + 4y' + y = 3\sin t + \cos t, \quad y(0) = 2,\ y'(0) = -1$

37. $9y'' + 6y' + y = 3e^{3t}, \quad y(0) = 0,\ y'(0) = -3$

38. Suppose a, b, and c are constants and $a \neq 0$. Let

$$y_1 = \mathcal{L}^{-1}\left(\frac{as + b}{as^2 + bs + c}\right) \quad \text{and} \quad y_2 = \mathcal{L}^{-1}\left(\frac{a}{as^2 + bs + c}\right).$$

Show that

$$y_1(0) = 1, \quad y_1'(0) = 0 \quad \text{and} \quad y_2(0) = 0, \quad y_2'(0) = 1.$$

HINT: *Use the Laplace transform to solve the initial value problems*

$$\begin{aligned} ay'' + by' + cy &= 0, \quad y(0) = 1, \quad y'(0) = 0 \\ ay'' + by' + cy &= 0, \quad y(0) = 0, \quad y'(0) = 1. \end{aligned}$$

5.4 THE UNIT STEP FUNCTION

In the next section we'll consider initial value problems

$$ay'' + by' + cy = f(t), \quad y(0) = k_0, \quad y'(0) = k_1,$$

where a, b, and c are constants and f is piecewise continuous. In this section we'll develop procedures for using the table of Laplace transforms to find Laplace transforms of piecewise continuous functions, and to find the piecewise continuous inverses of Laplace transforms.

Example 5.4.1 Use the table of Laplace transforms to find the Laplace transform of

$$f(t) = \begin{cases} 2t+1, & 0 \leqslant t < 2, \\ 3t, & t \geqslant 2 \end{cases} \tag{5.4.1}$$

(Figure 5.1).

Solution Since the formula for f changes at $t = 2$, we write

$$\begin{aligned} \mathcal{L}(f) &= \int_0^\infty e^{-st} f(t)\, dt \\ &= \int_0^2 e^{-st}(2t+1)\, dt + \int_2^\infty e^{-st}(3t)\, dt. \end{aligned} \tag{5.4.2}$$

To relate the first term to a Laplace transform, we add and subtract

$$\int_2^\infty e^{-st}(2t+1)\, dt$$

in (5.4.2) to obtain

$$\begin{aligned} \mathcal{L}(f) &= \int_0^\infty e^{-st}(2t+1)\, dt + \int_2^\infty e^{-st}(3t-2t-1)\, dt \\ &= \int_0^\infty e^{-st}(2t+1)\, dt + \int_2^\infty e^{-st}(t-1)\, dt \\ &= \mathcal{L}(2t+1) + \int_2^\infty e^{-st}(t-1)\, dt. \end{aligned} \tag{5.4.3}$$

To relate the last integral to a Laplace transform, we make the change of variable $x = t-2$ and rewrite the integral as

$$\begin{aligned}\int_2^\infty e^{-st}(t-1)\,dt &= \int_0^\infty e^{-s(x+2)}(x+1)\,dx \\ &= e^{-2s}\int_0^\infty e^{-sx}(x+1)\,dx.\end{aligned}$$

Since the symbol used for the variable of integration has no effect on the value of a definite integral, we can now replace x by the more standard t and write

$$\int_2^\infty e^{-st}(t-1)\,dt = e^{-2s}\int_0^\infty e^{-st}(t+1)\,dt = e^{-2s}\mathcal{L}(t+1).$$

This and (5.4.3) imply that

$$\mathcal{L}(f) = \mathcal{L}(2t+1) + e^{-2s}\mathcal{L}(t+1).$$

Now we can use the table of Laplace transforms to find that

$$\mathcal{L}(f) = \frac{2}{s^2} + \frac{1}{s} + e^{-2s}\left(\frac{1}{s^2}+\frac{1}{s}\right). \blacksquare$$

Figure 5.1 The piecewise continuous function (5.4.1)

Figure 5.2 $y = u(t-\tau)$

Laplace Transforms of Piecewise Continuous Functions

We'll now develop the method of Example 5.4.1 into a systematic way to find the Laplace transform of a piecewise continuous function. It is convenient to introduce the *unit step function*, defined as

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geqslant 0. \end{cases} \tag{5.4.4}$$

Thus, $u(t)$ "steps" from the constant value 0 to the constant value 1 at $t = 0$. If we replace t by $t-\tau$ in (5.4.4), then

$$u(t-\tau) = \begin{cases} 0, & t < \tau, \\ 1, & t \geqslant \tau \end{cases};$$

that is, the step now occurs at $t = \tau$ (Figure 5.2).

The step function enables us to represent piecewise continuous functions conveniently. For example, consider the function

$$f(t) = \begin{cases} f_0(t), & 0 \leqslant t < t_1, \\ f_1(t), & t \geqslant t_1, \end{cases} \tag{5.4.5}$$

where we assume that f_0 and f_1 are defined on $[0,\infty)$, even though they equal f only on the indicated intervals. This assumption enables us to rewrite (5.4.5) as

$$f(t) = f_0(t) + u(t-t_1)\left(f_1(t) - f_0(t)\right). \qquad (5.4.6)$$

To verify this, note that if $t < t_1$ then $u(t-t_1) = 0$ and (5.4.6) becomes

$$f(t) = f_0(t) + (0)\left(f_1(t) - f_0(t)\right) = f_0(t).$$

If $t \geqslant t_1$ then $u(t-t_1) = 1$ and (5.4.6) becomes

$$f(t) = f_0(t) + (1)\left(f_1(t) - f_0(t)\right) = f_1(t).$$

We need the next theorem to show how (5.4.6) can be used to find $\mathcal{L}(f)$.

Theorem 5.4.1 *Let g be defined on $[0,\infty)$. Suppose $\tau \geqslant 0$ and $\mathcal{L}\left(g(t+\tau)\right)$ exists for $s > s_0$. Then $\mathcal{L}\left(u(t-\tau)g(t)\right)$ exists for $s > s_0$, and*

$$\mathcal{L}(u(t-\tau)g(t)) = e^{-s\tau}\mathcal{L}\left(g(t+\tau)\right).$$

Proof By definition,

$$\mathcal{L}\left(u(t-\tau)g(t)\right) = \int_0^\infty e^{-st}u(t-\tau)g(t)\,dt.$$

From this and the definition of $u(t-\tau)$,

$$\mathcal{L}\left(u(t-\tau)g(t)\right) = \int_0^\tau e^{-st}(0)\,dt + \int_\tau^\infty e^{-st}g(t)\,dt.$$

The first integral on the right equals zero. Introducing the new variable of integration $x = t - \tau$ in the second integral yields

$$\mathcal{L}\left(u(t-\tau)g(t)\right) = \int_0^\infty e^{-s(x+\tau)}g(x+\tau)\,dx = e^{-s\tau}\int_0^\infty e^{-sx}g(x+\tau)\,dx.$$

Changing the name of the variable of integration in the last integral from x to t yields

$$\mathcal{L}\left(u(t-\tau)g(t)\right) = e^{-s\tau}\int_0^\infty e^{-st}g(t+\tau)\,dt = e^{-s\tau}\mathcal{L}(g(t+\tau)). \blacksquare$$

Example 5.4.2 Find

$$\mathcal{L}\left(u(t-1)(t^2+1)\right).$$

Solution Here $\tau = 1$ and $g(t) = t^2 + 1$, so

$$g(t+1) = (t+1)^2 + 1 = t^2 + 2t + 2.$$

Since

$$\mathcal{L}\left(g(t+1)\right) = \frac{2}{s^3} + \frac{2}{s^2} + \frac{2}{s},$$

Theorem 5.4.1 implies that

$$\mathcal{L}\left(u(t-1)(t^2+1)\right) = e^{-s}\left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{2}{s}\right).$$

Figure 5.3 The piecewise contnuous function (5.4.7)

Example 5.4.3 Use Theorem 5.4.1 to find the Laplace transform of the function

$$f(t) = \begin{cases} 2t+1, & 0 \leqslant t < 2, \\ 3t, & t \geqslant 2, \end{cases}$$

from Example 5.4.1.

Solution We first write f in the form (5.4.6) as

$$f(t) = 2t + 1 + u(t-2)(t-1).$$

Therefore

$$\begin{aligned} \mathcal{L}(f) &= \mathcal{L}(2t+1) + \mathcal{L}\left(u(t-2)(t-1)\right) \\ &= \mathcal{L}(2t+1) + e^{-2s}\mathcal{L}(t+1) \quad \text{(from Theorem 5.4.1)} \\ &= \frac{2}{s^2} + \frac{1}{s} + e^{-2s}\left(\frac{1}{s^2} + \frac{1}{s}\right), \end{aligned}$$

which is the result obtained in Example 5.4.1. ■

Formula (5.4.6) can be extended to more general piecewise continuous functions. For example, we can write

$$f(t) = \begin{cases} f_0(t), & 0 \leqslant t < t_1, \\ f_1(t), & t_1 \leqslant t < t_2, \\ f_2(t), & t \geqslant t_2, \end{cases}$$

as

$$f(t) = f_0(t) + u(t-t_1)\left(f_1(t) - f_0(t)\right) + u(t-t_2)\left(f_2(t) - f_1(t)\right)$$

if f_0, f_1, and f_2 are all defined on $[0, \infty)$.

Example 5.4.4 Find the Laplace transform of

$$f(t) = \begin{cases} 1, & 0 \leqslant t < 2, \\ -2t+1, & 2 \leqslant t < 3, \\ 3t, & 3 \leqslant t < 5, \\ t-1, & t \geqslant 5 \end{cases} \tag{5.4.7}$$

(Figure 5.3).

Solution In terms of step functions,

$$\begin{aligned} f(t) &= 1 + u(t-2)(-2t+1-1) + u(t-3)(3t+2t-1) \\ &\quad + u(t-5)(t-1-3t), \end{aligned}$$

Figure 5.4 The piecewise continuous function (5.4.10)

or

$$f(t) = 1 - 2u(t-2)t + u(t-3)(5t-1) - u(t-5)(2t+1).$$

Now Theorem 5.4.1 implies that

$$\begin{aligned} \mathcal{L}(f) &= \mathcal{L}(1) - 2e^{-2s}\mathcal{L}(t+2) + e^{-3s}\mathcal{L}\left(5(t+3)-1\right) - e^{-5s}\mathcal{L}\left(2(t+5)+1\right) \\ &= \mathcal{L}(1) - 2e^{-2s}\mathcal{L}(t+2) + e^{-3s}\mathcal{L}(5t+14) - e^{-5s}\mathcal{L}(2t+11) \\ &= \frac{1}{s} - 2e^{-2s}\left(\frac{1}{s^2}+\frac{2}{s}\right) + e^{-3s}\left(\frac{5}{s^2}+\frac{14}{s}\right) - e^{-5s}\left(\frac{2}{s^2}+\frac{11}{s}\right). \ \blacksquare \end{aligned}$$

The trigonometric identities

$$\begin{aligned} \sin(A+B) &= \sin A\cos B + \cos A\sin B & (5.4.8) \\ \cos(A+B) &= \cos A\cos B - \sin A\sin B & (5.4.9) \end{aligned}$$

are useful in problems that involve shifting the arguments of trigonometric functions. We'll use these identities in the next example.

Example 5.4.5 Find the Laplace transform of

$$f(t) = \begin{cases} \sin t, & 0 \leqslant t < \dfrac{\pi}{2}, \\ \cos t - 3\sin t, & \dfrac{\pi}{2} \leqslant t < \pi, \\ 3\cos t, & t \geqslant \pi \end{cases} \tag{5.4.10}$$

(Figure 5.4).

Solution In terms of step functions,

$$f(t) = \sin t + u(t-\pi/2)(\cos t - 4\sin t) + u(t-\pi)(2\cos t + 3\sin t).$$

Now Theorem 5.4.1 implies that

$$\begin{aligned} \mathcal{L}(f) = \ & \mathcal{L}(\sin t) + e^{-\frac{\pi}{2}s}\mathcal{L}\left(\cos\left(t+\tfrac{\pi}{2}\right) - 4\sin\left(t+\tfrac{\pi}{2}\right)\right) \\ & + e^{-\pi s}\mathcal{L}\left(2\cos(t+\pi) + 3\sin(t+\pi)\right). \end{aligned} \tag{5.4.11}$$

Since

$$\cos\left(t+\frac{\pi}{2}\right) - 4\sin\left(t+\frac{\pi}{2}\right) = -\sin t - 4\cos t$$

and

$$2\cos(t+\pi) + 3\sin(t+\pi) = -2\cos t - 3\sin t,$$

we see from (5.4.11) that

$$\begin{aligned}\mathcal{L}(f) &= \mathcal{L}(\sin t) - e^{-\pi s/2}\mathcal{L}(\sin t + 4\cos t) - e^{-\pi s}\mathcal{L}(2\cos t + 3\sin t) \\ &= \frac{1}{s^2+1} - e^{-\frac{\pi}{2}s}\left(\frac{1+4s}{s^2+1}\right) - e^{-\pi s}\left(\frac{3+2s}{s^2+1}\right). \blacksquare\end{aligned}$$

The Second Shifting Theorem

Replacing $g(t)$ by $g(t-\tau)$ in Theorem 5.4.1 yields the next theorem.

Theorem 5.4.2 *[Second Shifting Theorem] If $\tau \geqslant 0$ and $\mathcal{L}(g)$ exists for $s > s_0$ then $\mathcal{L}\left(u(t-\tau)g(t-\tau)\right)$ exists for $s > s_0$ and*

$$\mathcal{L}(u(t-\tau)g(t-\tau)) = e^{-s\tau}\mathcal{L}(g(t)),$$

or, equivalently,

$$\textit{if } g(t) \leftrightarrow G(s), \textit{ then } u(t-\tau)g(t-\tau) \leftrightarrow e^{-s\tau}G(s). \tag{5.4.12}$$

REMARK: Recall that the First Shifting Theorem (Theorem 5.1.3 states that multiplying a function by e^{at} corresponds to shifting the argument of its transform by a units. Theorem 5.4.2 states that multiplying a Laplace transform by the exponential $e^{-\tau s}$ corresponds to shifting the argument of the inverse transform by τ units.

Example 5.4.6 Use (5.4.12) to find

$$\mathcal{L}^{-1}\left(\frac{e^{-2s}}{s^2}\right).$$

Solution To apply (5.4.12) we let $\tau = 2$ and $G(s) = 1/s^2$. Then $g(t) = t$ and (5.4.12) implies that

$$\mathcal{L}^{-1}\left(\frac{e^{-2s}}{s^2}\right) = u(t-2)(t-2). \blacksquare$$

Example 5.4.7 Find the inverse Laplace transform h of

$$H(s) = \frac{1}{s^2} - e^{-s}\left(\frac{1}{s^2} + \frac{2}{s}\right) + e^{-4s}\left(\frac{4}{s^3} + \frac{1}{s}\right),$$

and find distinct formulas for h on appropriate intervals.

Solution Let

$$G_0(s) = \frac{1}{s^2}, \quad G_1(s) = \frac{1}{s^2} + \frac{2}{s}, \quad G_2(s) = \frac{4}{s^3} + \frac{1}{s}.$$

Then

$$g_0(t) = t, \ g_1(t) = t + 2, \ g_2(t) = 2t^2 + 1.$$

Hence, (5.4.12) and the linearity of $\mathcal{L}^{-1}$ imply that

$$\begin{aligned} h(t) &= \mathcal{L}^{-1}\left(G_0(s)\right) - \mathcal{L}^{-1}\left(e^{-s}G_1(s)\right) + \mathcal{L}^{-1}\left(e^{-4s}G_2(s)\right) \\ &= t - u(t-1)\left[(t-1)+2\right] + u(t-4)\left[2(t-4)^2+1\right] \\ &= t - u(t-1)(t+1) + u(t-4)(2t^2-16t+33), \end{aligned}$$

which can also be written as

$$h(t) = \begin{cases} t, & 0 \leqslant t < 1, \\ -1, & 1 \leqslant t < 4, \\ 2t^2 - 16t + 32, & t \geqslant 4. \end{cases} \quad \blacksquare$$

Example 5.4.8 Find the inverse transform of

$$H(s) = \frac{2s}{s^2+4} - e^{-\frac{\pi}{2}s}\frac{3s+1}{s^2+9} + e^{-\pi s}\frac{s+1}{s^2+6s+10}.$$

Solution Let

$$G_0(s) = \frac{2s}{s^2+4}, \quad G_1(s) = -\frac{(3s+1)}{s^2+9},$$

and

$$G_2(s) = \frac{s+1}{s^2+6s+10} = \frac{(s+3)-2}{(s+3)^2+1}.$$

Then

$$g_0(t) = 2\cos 2t, \quad g_1(t) = -3\cos 3t - \frac{1}{3}\sin 3t,$$

and

$$g_2(t) = e^{-3t}(\cos t - 2\sin t).$$

Therefore (5.4.12) and the linearity of $\mathcal{L}^{-1}$ imply that

$$\begin{aligned} h(t) &= 2\cos 2t - u(t-\pi/2)\left[3\cos 3(t-\pi/2) + \frac{1}{3}\sin 3\left(t-\frac{\pi}{2}\right)\right] \\ &\quad + u(t-\pi)e^{-3(t-\pi)}\left[\cos(t-\pi) - 2\sin(t-\pi)\right]. \end{aligned}$$

Using the trigonometric identities (5.4.8) and (5.4.9), we can rewrite this as

$$\begin{aligned} h(t) &= 2\cos 2t + u(t-\pi/2)\left(3\sin 3t - \tfrac{1}{3}\cos 3t\right) \\ &\quad - u(t-\pi)e^{-3(t-\pi)}(\cos t - 2\sin t) \end{aligned} \tag{5.4.13}$$

(Figure 5.5).

Figure 5.5 The piecewise continouous function (5.4.13)

5.4 Exercises

In Exercises 1–6 find the Laplace transform by the method of Example 5.4.1. Then express the given function f in terms of unit step functions as in Eqn. (5.4.6), and use Theorem 5.4.1 to find $\mathcal{L}(f)$. Where indicated by C/G, graph f.

1. $f(t) = \begin{cases} 1, & 0 \leqslant t < 4, \\ t, & t \geqslant 4. \end{cases}$

2. $f(t) = \begin{cases} t, & 0 \leqslant t < 1, \\ 1, & t \geqslant 1. \end{cases}$

3. C/G $f(t) = \begin{cases} 2t-1, & 0 \leqslant t < 2, \\ t, & t \geqslant 2. \end{cases}$

4. C/G $f(t) = \begin{cases} 1, & 0 \leqslant t < 1, \\ t+2, & t \geqslant 1. \end{cases}$

5. $f(t) = \begin{cases} t-1, & 0 \leqslant t < 2, \\ 4, & t \geqslant 2. \end{cases}$

6. $f(t) = \begin{cases} t^2, & 0 \leqslant t < 1, \\ 0, & t \geqslant 1. \end{cases}$

In Exercises 7–18 express the given function f in terms of unit step functions and use Theorem 5.4.1 to find $\mathcal{L}(f)$. Where indicated by C/G, graph f.

7. $f(t) = \begin{cases} 0, & 0 \leqslant t < 2, \\ t^2+3t, & t \geqslant 2. \end{cases}$

8. $f(t) = \begin{cases} t^2+2, & 0 \leqslant t < 1, \\ t, & t \geqslant 1. \end{cases}$

9. $f(t) = \begin{cases} te^t, & 0 \leqslant t < 1, \\ e^t, & t \geqslant 1. \end{cases}$

10. $f(t) = \begin{cases} e^{-t}, & 0 \leqslant t < 1, \\ e^{-2t}, & t \geqslant 1. \end{cases}$

11. $f(t) = \begin{cases} -t, & 0 \leqslant t < 2, \\ t-4, & 2 \leqslant t < 3, \\ 1, & t \geqslant 3. \end{cases}$

12. $f(t) = \begin{cases} 0, & 0 \leqslant t < 1, \\ t, & 1 \leqslant t < 2, \\ 0, & t \geqslant 2. \end{cases}$

13. $f(t) = \begin{cases} t, & 0 \leqslant t < 1, \\ t^2, & 1 \leqslant t < 2, \\ 0, & t \geqslant 2. \end{cases}$

14. $f(t) = \begin{cases} t, & 0 \leqslant t < 1, \\ 2-t, & 1 \leqslant t < 2, \\ 6, & t > 2. \end{cases}$

15. C/G $f(t) = \begin{cases} \sin t, & 0 \leqslant t < \frac{\pi}{2}, \\ 2\sin t, & \frac{\pi}{2} \leqslant t < \pi, \\ \cos t, & t \geqslant \pi. \end{cases}$

16. C/G $f(t) = \begin{cases} 2, & 0 \leqslant t < 1, \\ -2t+2, & 1 \leqslant t < 3, \\ 3t, & t \geqslant 3. \end{cases}$

17. C/G $f(t) = \begin{cases} 3, & 0 \leqslant t < 2, \\ 3t+2, & 2 \leqslant t < 4, \\ 4t, & t \geqslant 4. \end{cases}$

18. C/G $f(t) = \begin{cases} (t+1)^2, & 0 \leqslant t < 1, \\ (t+2)^2, & t \geqslant 1. \end{cases}$

In Exercises 19–28 *use Theorem* 5.4.2 *to express the inverse transforms in terms of step functions, and then find distinct formulas the for inverse transforms on the appropriate intervals, as in Example* 5.4.7. *Where indicated by* C/G , *graph the inverse transform.*

19. $H(s) = \frac{e^{-2s}}{s-2}$

20. $H(s) = \frac{e^{-s}}{s(s+1)}$

21. C/G $H(s) = \frac{e^{-s}}{s^3} + \frac{e^{-2s}}{s^2}$

22. C/G $H(s) = \left(\frac{2}{s} + \frac{1}{s^2}\right) + e^{-s}\left(\frac{3}{s} - \frac{1}{s^2}\right) + e^{-3s}\left(\frac{1}{s} + \frac{1}{s^2}\right)$

23. $H(s) = \left(\frac{5}{s} - \frac{1}{s^2}\right) + e^{-3s}\left(\frac{6}{s} + \frac{7}{s^2}\right) + \frac{3e^{-6s}}{s^3}$

24. $H(s) = \frac{e^{-\pi s}(1-2s)}{s^2+4s+5}$

25. C/G $H(s) = \left(\frac{1}{s} - \frac{s}{s^2+1}\right) + e^{-\frac{\pi}{2}s}\left(\frac{3s-1}{s^2+1}\right)$

26. $H(s) = e^{-2s}\left[\frac{3(s-3)}{(s+1)(s-2)} - \frac{s+1}{(s-1)(s-2)}\right]$

27. $H(s) = \frac{1}{s} + \frac{1}{s^2} + e^{-s}\left(\frac{3}{s} + \frac{2}{s^2}\right) + e^{-3s}\left(\frac{4}{s} + \frac{3}{s^2}\right)$

28. $H(s) = \frac{1}{s} - \frac{2}{s^3} + e^{-2s}\left(\frac{3}{s} - \frac{1}{s^3}\right) + \frac{e^{-4s}}{s^2}$

29. Find $\mathcal{L}\left(u(t-\tau)\right)$.

30. Let $\{t_m\}_{m=0}^{\infty}$ be a sequence of points such that $t_0 = 0$, $t_{m+1} > t_m$, and $\lim_{m\to\infty} t_m = \infty$. For each nonnegative integer m, let f_m be continuous on $[t_m, \infty)$, and let f be defined on $[0, \infty)$ by

$$f(t) = f_m(t), \; t_m \leqslant t < t_{m+1} \quad (m = 0, 1, \dots).$$

Show that f is piecewise continuous on $[0,\infty)$ and that it has the step function representation

$$f(t)=f_0(t)+\sum_{m=1}^{\infty}u(t-t_m)\left(f_m(t)-f_{m-1}(t)\right),\ 0\leqslant t<\infty.$$

How do we know that the series on the right converges for all t in $[0,\infty)$?

31. In addition to the assumptions of Exercise 30, assume that

$$|f_m(t)|\leqslant Me^{s_0 t},\ t\geqslant t_m,\ m=0,1,\dots, \tag{A}$$

and that the series

$$\sum_{m=0}^{\infty}e^{-\rho t_m} \tag{B}$$

converges for some $\rho>0$. Using the steps listed below, show that $\mathcal{L}(f)$ is defined for $s>s_0$ and

$$\mathcal{L}(f)=\mathcal{L}(f_0)+\sum_{m=1}^{\infty}e^{-st_m}\mathcal{L}(g_m) \tag{C}$$

for $s>s_0+\rho$, where

$$g_m(t)=f_m(t+t_m)-f_{m-1}(t+t_m).$$

(a) Use (A) and Theorem 8.1.6 to show that

$$\mathcal{L}(f)=\sum_{m=0}^{\infty}\int_{t_m}^{t_{m+1}}e^{-st}f_m(t)\,dt \tag{D}$$

is defined for $s>s_0$.

(b) Show that (D) can be rewritten as

$$\mathcal{L}(f)=\sum_{m=0}^{\infty}\left(\int_{t_m}^{\infty}e^{-st}f_m(t)\,dt-\int_{t_{m+1}}^{\infty}e^{-st}f_m(t)\,dt\right). \tag{E}$$

(c) Use (A), the assumed convergence of (B), and the comparison test to show that the series

$$\sum_{m=0}^{\infty}\int_{t_m}^{\infty}e^{-st}f_m(t)\,dt \quad\text{and}\quad \sum_{m=0}^{\infty}\int_{t_{m+1}}^{\infty}e^{-st}f_m(t)\,dt$$

both converge (absolutely) if $s>s_0+\rho$.

(d) Show that (E) can be rewritten as

$$\mathcal{L}(f)=\mathcal{L}(f_0)+\sum_{m=1}^{\infty}\int_{t_m}^{\infty}e^{-st}\left(f_m(t)-f_{m-1}(t)\right)\,dt$$

if $s>s_0+\rho$.

(e) Complete the proof of (C).

32. Suppose $\{t_m\}_{m=0}^{\infty}$ and $\{f_m\}_{m=0}^{\infty}$ satisfy the assumptions of Exercises 30 and 31, and there's a positive constant K such that $t_m \geqslant Km$ for m sufficiently large. Show that the series (B) of Exercise 31 converges for any $\rho > 0$, and conclude from this that (C) of Exercise 31 holds for $s > s_0$.

In Exercises 33–36 find the step function representation of f *and use the result of Exercise 32 to find* $\mathcal{L}(f)$. HINT: *You will need formulas related to the formula for the sum of a geometric series.*

33. $f(t) = m+1,\ m \leqslant t < m+1\ (m = 0, 1, 2, \dots)$

34. $f(t) = (-1)^m,\ m \leqslant t < m+1\ (m = 0, 1, 2, \dots)$

35. $f(t) = (m+1)^2,\ m \leqslant t < m+1\ (m = 0, 1, 2, \dots)$

36. $f(t) = (-1)^m m,\ m \leqslant t < m+1\ (m = 0, 1, 2, \dots)$

5.5 CONSTANT COEEFFICIENT EQUATIONS WITH PIECEWISE CONTINUOUS FORCING FUNCTIONS

We'll now consider initial value problems of the form

$$ay'' + by' + cy = f(t), \quad y(0) = k_0, \quad y'(0) = k_1, \qquad (5.5.1)$$

where a, b, and c are constants ($a \neq 0$) and f is piecewise continuous on $[0, \infty)$. Problems of this kind occur in situations where the input to a physical system undergoes instantaneous changes, as when a switch is turned on or off or the forces acting on the system change abruptly.

It can be shown (Exercises 23 and 24) that the differential equation in (5.5.1) has no solutions on an open interval that contains a jump discontinuity of f. Therefore we must define what we mean by a solution of (5.5.1) on $[0, \infty)$ in the case where f has jump discontinuities. The next theorem motivates our definition. We omit the proof.

Theorem 5.5.1 *Suppose* $a, b,$ *and* c *are constants* $(a \neq 0)$, *and* f *is piecewise continuous on* $[0, \infty)$. *with jump discontinuities at* $t_1, \dots, t_n$, *where*

$$0 < t_1 < \cdots < t_n.$$

Let k_0 *and* k_1 *be arbitrary real numbers. Then there is a unique function* y *defined on* $[0, \infty)$ *with these properties:*

(a) $y(0) = k_0$ *and* $y'(0) = k_1$.

(b) y *and* y' *are continuous on* $[0, \infty)$.

(c) y'' *is defined on every open subinterval of* $[0, \infty)$ *that does not contain any of the points* $t_1, \dots, t_n$, *and*

$$ay'' + by' + cy = f(t)$$

on every such subinterval.

(d) *y'' has limits from the right and left at $t_1, \dots, t_n$.*

We define the function y of Theorem 5.5.1 to be the solution of the initial value problem (5.5.1).

We begin by considering initial value problems of the form

$$ay'' + by' + cy = \begin{cases} f_0(t), & 0 \leqslant t < t_1, \\ f_1(t), & t \geqslant t_1, \end{cases} \quad y(0) = k_0, \quad y'(0) = k_1, \tag{5.5.2}$$

where the forcing function has a single jump discontinuity at t_1.

We can solve (5.5.2) by the these steps:

Step 1. Find the solution y_0 of the initial value problem

$$ay'' + by' + cy = f_0(t), \quad y(0) = k_0, \quad y'(0) = k_1.$$

Step 2. Compute $c_0 = y_0(t_1)$ and $c_1 = y_0'(t_1)$.

Step 3. Find the solution y_1 of the initial value problem

$$ay'' + by' + cy = f_1(t), \quad y(t_1) = c_0, \quad y'(t_1) = c_1.$$

Step 4. Obtain the solution y of (5.5.2) as

$$y = \begin{cases} y_0(t), & 0 \leqslant t < t_1 \\ y_1(t), & t \geqslant t_1. \end{cases}$$

It is shown in Exercise 23 that y' exists and is continuous at t_1. The next example illustrates this procedure.

Example 5.5.1 Solve the initial value problem

$$y'' + y = f(t), \quad y(0) = 2, \ y'(0) = -1, \tag{5.5.3}$$

where

$$f(t) = \begin{cases} 1, & 0 \leqslant t < \dfrac{\pi}{2}, \\ -1, & t \geqslant \dfrac{\pi}{2}. \end{cases}$$

Solution The initial value problem in Step 1 is

$$y'' + y = 1, \quad y(0) = 2, \quad y'(0) = -1.$$

We leave it to you to verify that its solution is

$$y_0 = 1 + \cos t - \sin t.$$

Figure 5.1 Graph of (5.5.4)

Doing Step 2 yields $y_0(\pi/2) = 0$ and $y_0'(\pi/2) = -1$, so the second initial value problem is

$$y'' + y = -1, \quad y\left(\frac{\pi}{2}\right) = 0,\ y'\left(\frac{\pi}{2}\right) = -1.$$

We leave it to you to verify that the solution of this problem is

$$y_1 = -1 + \cos t + \sin t.$$

Hence, the solution of (5.5.3) is

$$y = \begin{cases} 1 + \cos t - \sin t, & 0 \leqslant t < \dfrac{\pi}{2}, \\ -1 + \cos t + \sin t, & t \geqslant \dfrac{\pi}{2} \end{cases} \tag{5.5.4}$$

(Figure:8.5.1).

If f_0 and f_1 are defined on $[0, \infty)$, we can rewrite (5.5.2) as

$$ay'' + by' + cy = f_0(t) + u(t - t_1)\left(f_1(t) - f_0(t)\right), \quad y(0) = k_0, \quad y'(0) = k_1,$$

and apply the method of Laplace transforms. We'll now solve the problem considered in Example 5.5.1 by this method.

Example 5.5.2 Use the Laplace transform to solve the initial value problem

$$y'' + y = f(t), \quad y(0) = 2,\ y'(0) = -1, \tag{5.5.5}$$

where

$$f(t) = \begin{cases} 1, & 0 \leqslant t < \dfrac{\pi}{2}, \\ -1, & t \geqslant \dfrac{\pi}{2}. \end{cases}$$

Solution Here

$$f(t) = 1 - 2u\left(t - \frac{\pi}{2}\right),$$

so Theorem 5.4.1 (with $g(t) = 1$) implies that

$$\mathcal{L}(f) = \frac{1 - 2e^{-\pi s/2}}{s}.$$

Therefore, transforming (5.5.5) yields

$$(s^2 + 1)Y(s) = \frac{1 - 2e^{-\pi s/2}}{s} - 1 + 2s,$$

so

$$Y(s) = (1 - 2e^{-\pi s/2})G(s) + \frac{2s - 1}{s^2 + 1}, \tag{5.5.6}$$

with

$$G(s) = \frac{1}{s(s^2+1)}.$$

The form for the partial fraction expansion of G is

$$\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1}. \tag{5.5.7}$$

Multiplying through by $s(s^2+1)$ yields

$$A(s^2+1) + (Bs+C)s = 1,$$

or

$$(A+B)s^2 + Cs + A = 1.$$

Equating coefficients of like powers of s on the two sides of this equation shows that $A = 1$, $B = -A = -1$ and $C = 0$. Hence, from (5.5.7),

$$G(s) = \frac{1}{s} - \frac{s}{s^2+1}.$$

Therefore

$$g(t) = 1 - \cos t.$$

From this, (5.5.6), and Theorem 5.4.2,

$$y = 1 - \cos t - 2u\left(t - \frac{\pi}{2}\right)\left(1 - \cos\left(t - \frac{\pi}{2}\right)\right) + 2\cos t - \sin t.$$

Simplifying this (recalling that $\cos(t - \pi/2) = \sin t$) yields

$$y = 1 + \cos t - \sin t - 2u\left(t - \frac{\pi}{2}\right)(1 - \sin t),$$

or

$$y = \begin{cases} 1 + \cos t - \sin t, & 0 \leqslant t < \dfrac{\pi}{2}, \\ -1 + \cos t + \sin t, & t \geqslant \dfrac{\pi}{2}, \end{cases}$$

which is the result obtained in Example 5.5.1.

REMARK: It isn't obvious that using the Laplace transform to solve (5.5.2) as we did in Example 5.5.2 yields a function y with the properties stated in Theorem 5.5.1; that is, such that y and y' are continuous on $[0, \infty)$ and y'' has limits from the right and left at t_1. However, this is true if f_0 and f_1 are continuous and of exponential order on $[0, \infty)$. A proof is sketched in Exercises 8.6.11–8.613.

Example 5.5.3 Solve the initial value problem

$$y'' - y = f(t), \quad y(0) = -1,\ y'(0) = 2, \tag{5.5.8}$$

where

$$f(t) = \begin{cases} t, & 0 \leqslant t < 1, \\ 1, & t \geqslant 1. \end{cases}$$

Solution Here

$$f(t) = t - u(t-1)(t-1),$$

so

$$\begin{aligned}
\mathcal{L}(f) &= \mathcal{L}(t) - \mathcal{L}\left(u(t-1)(t-1)\right) \\
&= \mathcal{L}(t) - e^{-s}\mathcal{L}(t) \text{ (from Theorem 5.4.1)} \\
&= \frac{1}{s^2} - \frac{e^{-s}}{s^2}.
\end{aligned}$$

Since transforming (5.5.8) yields

$$(s^2-1)Y(s) = \mathcal{L}(f) + 2 - s,$$

we see that

$$Y(s) = (1-e^{-s})H(s) + \frac{2-s}{s^2-1}, \tag{5.5.9}$$

where

$$H(s) = \frac{1}{s^2(s^2-1)} = \frac{1}{s^2-1} - \frac{1}{s^2};$$

therefore

$$h(t) = \sinh t - t. \tag{5.5.10}$$

Since

$$\mathcal{L}^{-1}\left(\frac{2-s}{s^2-1}\right) = 2\sinh t - \cosh t,$$

we conclude from (5.5.9), (5.5.10), and Theorem 5.4.1 that

$$y = \sinh t - t - u(t-1)\left(\sinh(t-1) - t + 1\right) + 2\sinh t - \cosh t,$$

or

$$y = 3\sinh t - \cosh t - t - u(t-1)\left(\sinh(t-1) - t + 1\right) \tag{5.5.11}$$

We leave it to you to verify that y and y' are continuous and y'' has limits from the right and left at $t_1 = 1$.

Example 5.5.4 Solve the initial value problem

$$y'' + y = f(t), \quad y(0) = 0, \; y'(0) = 0, \tag{5.5.12}$$

where

$$f(t) = \begin{cases} 0, & 0 \leqslant t < \dfrac{\pi}{4}, \\ \cos 2t, & \dfrac{\pi}{4} \leqslant t < \pi, \\ 0, & t \geqslant \pi. \end{cases}$$

Solution Here

$$f(t) = u(t - \pi/4)\cos 2t - u(t - \pi)\cos 2t,$$

so

$$\begin{aligned} \mathcal{L}(f) &= \mathcal{L}\left(u(t - \pi/4)\cos 2t\right) - \mathcal{L}\left(u(t - \pi)\cos 2t\right) \\ &= e^{-\pi s/4}\mathcal{L}\left(\cos 2(t + \pi/4)\right) - e^{-\pi s}\mathcal{L}\left(\cos 2(t + \pi)\right) \\ &= -e^{-\pi s/4}\mathcal{L}(\sin 2t) - e^{-\pi s}\mathcal{L}(\cos 2t) \\ &= -\frac{2e^{-\pi s/4}}{s^2 + 4} - \frac{se^{-\pi s}}{s^2 + 4}. \end{aligned}$$

Since transforming (5.5.12) yields

$$(s^2 + 1)Y(s) = \mathcal{L}(f),$$

we see that

$$Y(s) = e^{-\pi s/4}H_1(s) + e^{-\pi s}H_2(s), \tag{5.5.13}$$

where

$$H_1(s) = -\frac{2}{(s^2 + 1)(s^2 + 4)} \quad \text{and} \quad H_2(s) = -\frac{s}{(s^2 + 1)(s^2 + 4)}. \tag{5.5.14}$$

To simplify the required partial fraction expansions, we first write

$$\frac{1}{(x + 1)(x + 4)} = \frac{1}{3}\left[\frac{1}{x + 1} - \frac{1}{x + 4}\right].$$

Setting $x = s^2$ and substituting the result in (5.5.14) yields

$$H_1(s) = -\frac{2}{3}\left[\frac{1}{s^2 + 1} - \frac{1}{s^2 + 4}\right] \quad \text{and} \quad H_2(s) = -\frac{1}{3}\left[\frac{s}{s^2 + 1} - \frac{s}{s^2 + 4}\right].$$

The inverse transforms are

$$h_1(t) = -\frac{2}{3}\sin t + \frac{1}{3}\sin 2t \quad \text{and} \quad h_2(t) = -\frac{1}{3}\cos t + \frac{1}{3}\cos 2t.$$

From (5.5.13) and Theorem 8.4.2,

$$y = u\left(t - \frac{\pi}{4}\right)h_1\left(t - \frac{\pi}{4}\right) + u(t - \pi)h_2(t - \pi). \tag{5.5.15}$$

Since

$$\begin{aligned} h_1\left(t - \frac{\pi}{4}\right) &= -\frac{2}{3}\sin\left(t - \frac{\pi}{4}\right) + \frac{1}{3}\sin 2\left(t - \frac{\pi}{4}\right) \\ &= -\frac{\sqrt{2}}{3}(\sin t - \cos t) - \frac{1}{3}\cos 2t \end{aligned}$$

Figure 5.2 Graph of (5.5.16)

and

$$\begin{aligned} h_2(t-\pi) &= -\frac{1}{3}\cos(t-\pi) + \frac{1}{3}\cos 2(t-\pi) \\ &= \frac{1}{3}\cos t + \frac{1}{3}\cos 2t, \end{aligned}$$

(5.5.15) can be rewritten as

$$y = -\frac{1}{3}u\left(t-\frac{\pi}{4}\right)\left(\sqrt{2}(\sin t - \cos t) + \cos 2t\right) + \frac{1}{3}u(t-\pi)(\cos t + \cos 2t)$$

or

$$y = \begin{cases} 0, & 0 \leqslant t < \frac{\pi}{4}, \\ -\frac{\sqrt{2}}{3}(\sin t - \cos t) - \frac{1}{3}\cos 2t, & \frac{\pi}{4} \leqslant t < \pi, \\ -\frac{\sqrt{2}}{3}\sin t + \frac{1+\sqrt{2}}{3}\cos t, & t \geqslant \pi. \end{cases} \tag{5.5.16}$$

We leave it to you to verify that y and y' are continuous and y'' has limits from the right and left at $t_1 = \pi/4$ and $t_2 = \pi$ (Figure 5.2).

5.5 Exercises

In Exercises 1–20 use the Laplace transform to solve the initial value problem. Where indicated by
C/G , graph the solution.

1. $y'' + y = \begin{cases} 3, & 0 \leqslant t < \pi, \\ 0, & t \geqslant \pi, \end{cases} \quad y(0) = 0, \quad y'(0) = 0$

2. $y'' + y = \begin{cases} 3, & 0 \leqslant t < 4, \\ ;2t-5, & t > 4, \end{cases} \quad y(0) = 1, \quad y'(0) = 0$

3. $y'' - 2y' = \begin{cases} 4, & 0 \leqslant t < 1, \\ 6, & t \geqslant 1, \end{cases} \quad y(0) = -6, \quad y'(0) = 1$

4. $y'' - y = \begin{cases} e^{2t}, & 0 \leqslant t < 2, \\ 1, & t \geqslant 2, \end{cases} \quad y(0) = 3, \quad y'(0) = -1$

5. $y'' - 3y' + 2y = \begin{cases} 0, & 0 \leqslant t < 1, \\ 1, & 1 \leqslant t < 2, \\ -1, & t \geqslant 2, \end{cases} \quad y(0) = -3, \quad y'(0) = 1$

6. C/G $y'' + 4y = \begin{cases} |\sin t|, & 0 \leqslant t < 2\pi, \\ 0, & t \geqslant 2\pi, \end{cases} \quad y(0) = -3, \quad y'(0) = 1$

7. $y'' - 5y' + 4y = \begin{cases} 1, & 0 \leqslant t < 1 \\ -1, & 1 \leqslant t < 2, \\ 0, & t \geqslant 2, \end{cases} \qquad y(0) = 3, \quad y'(0) = -5$

8. $y'' + 9y = \begin{cases} \cos t, & 0 \leqslant t < \frac{3\pi}{2}, \\ \sin t, & t \geqslant \frac{3\pi}{2}, \end{cases} \qquad y(0) = 0,\ y'(0) = 0$

9. C/G $y'' + 4y = \begin{cases} t, & 0 \leqslant t < \frac{\pi}{2}, \\ \pi, & t \geqslant \frac{\pi}{2}, \end{cases} \qquad y(0) = 0, \quad y'(0) = 0$

10. $y'' + y = \begin{cases} t, & 0 \leqslant t < \pi, \\ -t, & t \geqslant \pi, \end{cases} \qquad y(0) = 0,\ y'(0) = 0$

11. $y'' - 3y' + 2y = \begin{cases} 0, & 0 \leqslant t < 2, \\ 2t - 4, & t \geqslant 2, \end{cases} , \quad y(0) = 0, \quad y'(0) = 0$

12. $y'' + y = \begin{cases} t, & 0 \leqslant t < 2\pi, \\ -2t, & t \geqslant 2\pi, \end{cases} \qquad y(0) = 1, \quad y'(0) = 2$

13. C/G $y'' + 3y' + 2y = \begin{cases} 1, & 0 \leqslant t < 2, \\ -1, & t \geqslant 2, \end{cases} \qquad y(0) = 0,\ y'(0) = 0$

14. $y'' - 4y' + 3y = \begin{cases} -1, & 0 \leqslant t < 1, \\ 1, & t \geqslant 1, \end{cases} \qquad y(0) = 0,\ y'(0) = 0$

15. $y'' + 2y' + y = \begin{cases} e^t, & 0 \leqslant t < 1, \\ e^t - 1, & t \geqslant 1, \end{cases} \qquad y(0) = 3,\ y'(0) = -1$

16. $y'' + 2y' + y = \begin{cases} 4e^t, & 0 \leqslant t < 1, \\ 0, & t \geqslant 1, \end{cases} \qquad y(0) = 0,\ y'(0) = 0$

17. $y'' + 3y' + 2y = \begin{cases} e^{-t}, & 0 \leqslant t < 1, \\ 0, & t \geqslant 1, \end{cases} \qquad y(0) = 1,\ y'(0) = -1$

18. $y'' - 4y' + 4y = \begin{cases} e^{2t}, & 0 \leqslant t < 2, \\ -e^{2t}, & t \geqslant 2, \end{cases} \qquad y(0) = 0,\ y'(0) = -1$

19. C/G $y'' = \begin{cases} t^2, & 0 \leqslant t < 1, \\ -t, & 1 \leqslant t < 2, \\ t + 1, & t \geqslant 2, \end{cases} \quad y(0) = 1,\ y'(0) = 0$

20. $y'' + 2y' + 2y = \begin{cases} 1, & 0 \leqslant t < 2\pi, \\ t, & 2\pi \leqslant t < 3\pi, \\ -1, & t \geqslant 3\pi, \end{cases} \quad y(0) = 2, \quad y'(0) = -1$

21. Solve the initial value problem

$$y'' = f(t), \quad y(0) = 0, \quad y'(0) = 0,$$

where

$$f(t) = m + 1, \quad m \leqslant t < m + 1, \quad m = 0, 1, 2, \ldots.$$

22. Solve the given initial value problem and find a formula that does not involve step functions and represents y on each interval of continuity of f.

(a) $y'' + y = f(t), \quad y(0) = 0, \quad y'(0) = 0;$
$f(t) = m + 1, \quad m\pi \leqslant t < (m + 1)\pi, \quad m = 0, 1, 2, \ldots.$

(b) $y'' + y = f(t), \quad y(0) = 0, \quad y'(0) = 0;$
$f(t) = (m + 1)t, \quad 2m\pi \leqslant t < 2(m + 1)\pi, \quad m = 0, 1, 2, \ldots$ HINT: *You'll need the formula*

$$1 + 2 + \cdots + m = \frac{m(m + 1)}{2}.$$

(c) $y'' + y = f(t), \quad y(0) = 0, \quad y'(0) = 0;$
$f(t) = (-1)^m, \quad m\pi \leqslant t < (m + 1)\pi, \quad m = 0, 1, 2, \ldots.$

(d) $y'' - y = f(t), \quad y(0) = 0, \quad y'(0) = 0;$
$f(t) = m + 1, \quad m \leqslant t < (m + 1), \quad m = 0, 1, 2, \ldots.$
HINT: *You will need the formula*

$$1 + r + \cdots + r^m = \frac{1 - r^{m+1}}{1 - r} \; (r \neq 1).$$

(e) $y'' + 2y' + 2y = f(t), \quad y(0) = 0, \quad y'(0) = 0;$
$f(t) = (m + 1)(\sin t + 2\cos t), \quad 2m\pi \leqslant t < 2(m + 1)\pi, \quad m = 0, 1, 2, \ldots.$
(See the hint in **(d)**.)

(f) $y'' - 3y' + 2y = f(t), \quad y(0) = 0, \quad y'(0) = 0;$
$f(t) = m + 1, \quad m \leqslant t < m + 1, \quad m = 0, 1, 2, \ldots.$
(See the hints in **(b)** and **(d)**.)

23. **(a)** Let g be continuous on (α, β) and differentiable on the (α, t_0) and (t_0, β). Suppose $A = \lim_{t \to t_0-} g'(t)$ and $B = \lim_{t \to t_0+} g'(t)$ both exist. Use the mean value theorem to show that

$$\lim_{t \to t_0-} \frac{g(t) - g(t_0)}{t - t_0} = A \quad \text{and} \quad \lim_{t \to t_0+} \frac{g(t) - g(t_0)}{t - t_0} = B.$$

(b) Conclude from **(a)** that $g'(t_0)$ exists and g' is continuous at t_0 if $A = B$.

(c) Conclude from **(a)** that if g is differentiable on (α, β) then g' can't have a jump discontinuity on (α, β).

24. **(a)** Let a, b, and c be constants, with $a \neq 0$. Let f be piecewise continuous on an interval (α, β), with a single jump discontinuity at a point t_0 in (α, β). Suppose y and y' are continuous on (α, β) and y'' on (α, t_0) and (t_0, β). Suppose also that

$$ay'' + by' + cy = f(t) \tag{A}$$

on (α, t_0) and (t_0, β). Show that

$$y''(t_0+) - y''(t_0-) = \frac{f(t_0+) - f(t_0-)}{a} \neq 0.$$

(b) Use **(a)** and Exercise 23**(c)** to show that (A) does not have solutions on any interval (α, β) that contains a jump discontinuity of f.

25. Suppose P_0, P_1, and P_2 are continuous and P_0 has no zeros on an open interval (a, b), and that F has a jump discontinuity at a point t_0 in (a, b). Show that the differential equation

$$P_0(t)y'' + P_1(t)y' + P_2(t)y = F(t)$$

has no solutions on (a, b).HINT: *Generalize the result of Exercise* 24 *and use Exercise* 23(**c**).

26. Let $0 = t_0 < t_1 < \cdots < t_n$. Suppose f_m is continuous on $[t_m, \infty)$ for $m = 1, \dots, n$. Let

$$f(t) = \begin{cases} f_m(t), & t_m \leqslant t < t_{m+1}, \quad m = 1, \dots, n-1, \\ f_n(t), & t \geqslant t_n. \end{cases}$$

Show that the solution of

$$ay'' + by' + cy = f(t), \quad y(0) = k_0, \quad y'(0) = k_1,$$

as defined following Theorem 8.5.1, is given by

$$y = \begin{cases} z_0(t), & 0 \leqslant t < t_1, \\ z_0(t) + z_1(t), & t_1 \leqslant t < t_2, \\ & \vdots \\ z_0 + \cdots + z_{n-1}(t), & t_{n-1} \leqslant t < t_n, \\ z_0 + \cdots + z_n(t), & t \geqslant t_n, \end{cases}$$

where z_0 is the solution of

$$az'' + bz' + cz = f_0(t), \quad z(0) = k_0, \quad z'(0) = k_1$$

and z_m is the solution of

$$az'' + bz' + cz = f_m(t) - f_{m-1}(t), \quad z(t_m) = 0, \quad z'(t_m) = 0$$

for $m = 1, \dots, n$.

5.6 CONVOLUTION

In this section we consider the problem of finding the inverse Laplace transform of a product $H(s) = F(s)G(s)$, where F and G are the Laplace transforms of known functions f and g. To motivate our interest in this problem, consider the initial value problem

$$ay'' + by' + cy = f(t), \quad y(0) = 0, \quad y'(0) = 0.$$

Taking Laplace transforms yields

$$(as^2 + bs + c)Y(s) = F(s),$$

so

$$Y(s) = F(s)G(s), \tag{5.6.1}$$

where

$$G(s) = \frac{1}{as^2 + bs + c}.$$

Until now wen't been interested in the factorization indicated in (5.6.1), since we dealt only with differential equations with specific forcing functions. Hence, we could simply do the indicated multiplication in (5.6.1) and use the table of Laplace transforms to find $y = \mathcal{L}^{-1}(Y)$. However, this isn't possible if we want a *formula* for y in terms of f, which may be unspecified.

To motivate the formula for $\mathcal{L}^{-1}(FG)$, consider the initial value problem

$$y' - ay = f(t), \quad y(0) = 0, \tag{5.6.2}$$

which we first solve without using the Laplace transform. The solution of the differential equation in (5.6.2) is of the form $y = ue^{at}$ where

$$u' = e^{-at} f(t).$$

Integrating this from 0 to t and imposing the initial condition $u(0) = y(0) = 0$ yields

$$u = \int_0^t e^{-a\tau} f(\tau)\, d\tau.$$

Therefore

$$y(t) = e^{at} \int_0^t e^{-a\tau} f(\tau)\, d\tau = \int_0^t e^{a(t-\tau)} f(\tau)\, d\tau. \tag{5.6.3}$$

Now we'll use the Laplace transform to solve (5.6.2) and compare the result to (5.6.3). Taking Laplace transforms in (5.6.2) yields

$$(s - a)Y(s) = F(s),$$

so

$$Y(s) = F(s)\frac{1}{s - a},$$

which implies that

$$y(t) = \mathcal{L}^{-1}\left(F(s)\frac{1}{s-a}\right). \tag{5.6.4}$$

If we now let $g(t) = e^{at}$, so that

$$G(s) = \frac{1}{s-a},$$

then (5.6.3) and (5.6.4) can be written as

$$y(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau$$

and

$$y = \mathcal{L}^{-1}(FG),$$

respectively. Therefore

$$\mathcal{L}^{-1}(FG) = \int_0^t f(\tau)g(t-\tau)\,d\tau \tag{5.6.5}$$

in this case.

This motivates the next definition.

Definition 5.6.1 The *convolution* $f * g$ of two functions f and g is defined by

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau.$$

It can be shown (Exercise 6) that $f * g = g * f$; that is,

$$\int_0^t f(t-\tau)g(\tau)\,d\tau = \int_0^t f(\tau)g(t-\tau)\,d\tau.$$

Eqn. (5.6.5) shows that $\mathcal{L}^{-1}(FG) = f * g$ in the special case where $g(t) = e^{at}$. This next theorem states that this is true in general.

Theorem 5.6.2 *[The Convolution Theorem] If $\mathcal{L}(f) = F$ and $\mathcal{L}(g) = G$, then*

$$\mathcal{L}(f * g) = FG.$$

A complete proof of the convolution theorem is beyond the scope of this book. However, we'll assume that $f * g$ has a Laplace transform and verify the conclusion of the theorem in a purely computational way. By the definition of the Laplace transform,

$$\mathcal{L}(f * g) = \int_0^\infty e^{-st}(f * g)(t)\,dt = \int_0^\infty e^{-st}\int_0^t f(\tau)g(t-\tau)\,d\tau\,dt.$$

This iterated integral equals a double integral over the region shown in Figure 5.1. Reversing the order of integration yields

$$\mathcal{L}(f * g) = \int_0^\infty f(\tau)\int_\tau^\infty e^{-st}g(t-\tau)\,dt\,d\tau. \tag{5.6.6}$$

Figure 5.1

However, the substitution $x = t - \tau$ shows that

$$\begin{aligned}\int_\tau^\infty e^{-st} g(t-\tau)\,dt &= \int_0^\infty e^{-s(x+\tau)} g(x)\,dx \\ &= e^{-s\tau}\int_0^\infty e^{-sx} g(x)\,dx = e^{-s\tau}G(s).\end{aligned}$$

Substituting this into (5.6.6) and noting that $G(s)$ is independent of τ yields

$$\begin{aligned}\mathcal{L}(f*g) &= \int_0^\infty e^{-s\tau} f(\tau)G(s)\,d\tau \\ &= G(s)\int_0^\infty e^{-st} f(\tau)\,d\tau = F(s)G(s).\end{aligned}$$

Example 5.6.1 Let

$$f(t) = e^{at} \quad \text{and} \quad g(t) = e^{bt} \qquad (a \neq b).$$

Verify that $\mathcal{L}(f*g) = \mathcal{L}(f)\mathcal{L}(g)$, as implied by the convolution theorem.

Solution We first compute

$$\begin{aligned}(f*g)(t) &= \int_0^t e^{a\tau} e^{b(t-\tau)}\,d\tau &= e^{bt}\int_0^t e^{(a-b)\tau}\,d\tau \\ &= e^{bt}\frac{e^{(a-b)\tau}}{a-b}\Bigg|_0^t &= \frac{e^{bt}\left[e^{(a-b)t}-1\right]}{a-b} \\ &= \frac{e^{at}-e^{bt}}{a-b}.\end{aligned}$$

Since

$$e^{at} \leftrightarrow \frac{1}{s-a} \quad \text{and} \quad e^{bt} \leftrightarrow \frac{1}{s-b},$$

it follows that

$$\begin{aligned}\mathcal{L}(f*g) &= \frac{1}{a-b}\left[\frac{1}{s-a} - \frac{1}{s-b}\right] \\ &= \frac{1}{(s-a)(s-b)} \\ &= \mathcal{L}(e^{at})\mathcal{L}(e^{bt}) = \mathcal{L}(f)\mathcal{L}(g).\end{aligned}$$

A Formula for the Solution of an Initial Value Problem

The convolution theorem provides a formula for the solution of an initial value problem for a linear constant coefficient second order equation with an unspecified. The next three examples illustrate this.

Example 5.6.2 Find a formula for the solution of the initial value problem

$$y'' - 2y' + y = f(t), \quad y(0) = k_0, \quad y'(0) = k_1. \tag{5.6.7}$$

Solution Taking Laplace transforms in (5.6.7) yields

$$(s^2 - 2s + 1)Y(s) = F(s) + (k_1 + k_0 s) - 2k_0.$$

Therefore

$$\begin{aligned} Y(s) &= \frac{1}{(s-1)^2}F(s) + \frac{k_1 + k_0 s - 2k_0}{(s-1)^2} \\ &= \frac{1}{(s-1)^2}F(s) + \frac{k_0}{s-1} + \frac{k_1 - k_0}{(s-1)^2}. \end{aligned}$$

From the table of Laplace transforms,

$$\mathcal{L}^{-1}\left(\frac{k_0}{s-1} + \frac{k_1 - k_0}{(s-1)^2}\right) = e^t\left(k_0 + (k_1 - k_0)t\right).$$

Since

$$\frac{1}{(s-1)^2} \leftrightarrow te^t \quad \text{and} \quad F(s) \leftrightarrow f(t),$$

the convolution theorem implies that

$$\mathcal{L}^{-1}\left(\frac{1}{(s-1)^2}F(s)\right) = \int_0^t \tau e^\tau f(t-\tau)\,d\tau.$$

Therefore the solution of (5.6.7) is

$$y(t) = e^t\left(k_0 + (k_1 - k_0)t\right) + \int_0^t \tau e^\tau f(t-\tau)\,d\tau.$$

Example 5.6.3 Find a formula for the solution of the initial value problem

$$y'' + 4y = f(t), \quad y(0) = k_0, \quad y'(0) = k_1. \tag{5.6.8}$$

Solution Taking Laplace transforms in (5.6.8) yields

$$(s^2 + 4)Y(s) = F(s) + k_1 + k_0 s.$$

Therefore

$$Y(s) = \frac{1}{(s^2+4)}F(s) + \frac{k_1 + k_0 s}{s^2 + 4}.$$

From the table of Laplace transforms,

$$\mathcal{L}^{-1}\left(\frac{k_1 + k_0 s}{s^2 + 4}\right) = k_0 \cos 2t + \frac{k_1}{2}\sin 2t.$$

Since

$$\frac{1}{(s^2+4)} \leftrightarrow \frac{1}{2}\sin 2t \quad \text{and} \quad F(s) \leftrightarrow f(t),$$

the convolution theorem implies that

$$\mathcal{L}^{-1}\left(\frac{1}{(s^2+4)}F(s)\right) = \frac{1}{2}\int_0^t f(t-\tau)\sin 2\tau \, d\tau.$$

Therefore the solution of (5.6.8) is

$$y(t) = k_0\cos 2t + \frac{k_1}{2}\sin 2t + \frac{1}{2}\int_0^t f(t-\tau)\sin 2\tau \, d\tau.$$

Example 5.6.4 Find a formula for the solution of the initial value problem

$$y'' + 2y' + 2y = f(t), \quad y(0) = k_0, \quad y'(0) = k_1. \tag{5.6.9}$$

Solution Taking Laplace transforms in (5.6.9) yields

$$(s^2 + 2s + 2)Y(s) = F(s) + k_1 + k_0 s + 2k_0.$$

Therefore

$$\begin{aligned} Y(s) &= \frac{1}{(s+1)^2+1}F(s) + \frac{k_1 + k_0 s + 2k_0}{(s+1)^2+1} \\ &= \frac{1}{(s+1)^2+1}F(s) + \frac{(k_1+k_0) + k_0(s+1)}{(s+1)^2+1}. \end{aligned}$$

From the table of Laplace transforms,

$$\mathcal{L}^{-1}\left(\frac{(k_1+k_0)+k_0(s+1)}{(s+1)^2+1}\right) = e^{-t}\left((k_1+k_0)\sin t + k_0\cos t\right).$$

Since

$$\frac{1}{(s+1)^2+1} \leftrightarrow e^{-t}\sin t \quad \text{and} \quad F(s) \leftrightarrow f(t),$$

the convolution theorem implies that

$$\mathcal{L}^{-1}\left(\frac{1}{(s+1)^2+1}F(s)\right) = \int_0^t f(t-\tau)e^{-\tau}\sin\tau \, d\tau.$$

Therefore the solution of (5.6.9) is

$$y(t) = e^{-t}\left((k_1+k_0)\sin t + k_0\cos t\right) + \int_0^t f(t-\tau)e^{-\tau}\sin\tau \, d\tau. \tag{5.6.10}$$

Evaluating Convolution Integrals

We'll say that an integral of the form $\int_0^t u(\tau)v(t-\tau)\,d\tau$ is a *convolution integral*. The convolution theorem provides a convenient way to evaluate convolution integrals.

Example 5.6.5 Evaluate the convolution integral

$$h(t) = \int_0^t (t-\tau)^5 \tau^7 \, d\tau.$$

Solution We could evaluate this integral by expanding $(t-\tau)^5$ in powers of τ and then integrating. However, the convolution theorem provides an easier way. The integral is the convolution of $f(t) = t^5$ and $g(t) = t^7$. Since

$$t^5 \leftrightarrow \frac{5!}{s^6} \quad \text{and} \quad t^7 \leftrightarrow \frac{7!}{s^8},$$

the convolution theorem implies that

$$h(t) \leftrightarrow \frac{5!7!}{s^{14}} = \frac{5!7!}{13!} \frac{13!}{s^{14}},$$

where we have written the second equality because

$$\frac{13!}{s^{14}} \leftrightarrow t^{13}.$$

Hence,

$$h(t) = \frac{5!7!}{13!} t^{13}.$$

Example 5.6.6 Use the convolution theorem and a partial fraction expansion to evaluate the convolution integral

$$h(t) = \int_0^t \sin a(t-\tau) \cos b\tau \, d\tau \quad (|a| \neq |b|).$$

Solution Since

$$\sin at \leftrightarrow \frac{a}{s^2+a^2} \quad \text{and} \quad \cos bt \leftrightarrow \frac{s}{s^2+b^2},$$

the convolution theorem implies that

$$H(s) = \frac{a}{s^2+a^2} \frac{s}{s^2+b^2}.$$

Expanding this in a partial fraction expansion yields

$$H(s) = \frac{a}{b^2-a^2} \left[\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2} \right].$$

Therefore

$$h(t) = \frac{a}{b^2-a^2} \left(\cos at - \cos bt \right).$$

Volterra Integral Equations

An equation of the form

$$y(t) = f(t) + \int_0^t k(t-\tau)y(\tau)\,d\tau \tag{5.6.11}$$

is a *Volterra integral equation*. Here f and k are given functions and y is unknown. Since the integral on the right is a convolution integral, the convolution theorem provides a convenient formula for solving (5.6.11). Taking Laplace transforms in (5.6.11) yields

$$Y(s) = F(s) + K(s)Y(s),$$

and solving this for $Y(s)$ yields

$$Y(s) = \frac{F(s)}{1-K(s)}.$$

We then obtain the solution of (5.6.11) as $y = \mathcal{L}^{-1}(Y)$.

Example 5.6.7 Solve the integral equation

$$y(t) = 1 + 2\int_0^t e^{-2(t-\tau)}y(\tau)\,d\tau. \tag{5.6.12}$$

Solution Taking Laplace transforms in (5.6.12) yields

$$Y(s) = \frac{1}{s} + \frac{2}{s+2}Y(s),$$

and solving this for $Y(s)$ yields

$$Y(s) = \frac{1}{s} + \frac{2}{s^2}.$$

Hence,

$$y(t) = 1 + 2t.$$

Transfer Functions

The next theorem presents a formula for the solution of the general initial value problem

$$ay'' + by' + cy = f(t), \quad y(0) = k_0, \quad y'(0) = k_1,$$

where we assume for simplicity that f is continuous on $[0,\infty)$ and that $\mathcal{L}(f)$ exists. In Exercises 11–14 it's shown that the formula is valid under much weaker conditions on f.

Theorem 5.6.3 *Suppose* f *is continuous on* $[0, \infty)$ *and has a Laplace transform. Then the solution of the initial value problem*

$$ay'' + by' + cy = f(t), \quad y(0) = k_0, \quad y'(0) = k_1, \tag{5.6.13}$$

is

$$y(t) = k_0 y_1(t) + k_1 y_2(t) + \int_0^t w(\tau) f(t-\tau)\, d\tau, \tag{5.6.14}$$

where y_1 *and* y_2 *satisfy*

$$ay_1'' + by_1' + cy_1 = 0, \quad y_1(0) = 1, \quad y_1'(0) = 0, \tag{5.6.15}$$

and

$$ay_2'' + by_2' + cy_2 = 0, \quad y_2(0) = 0, \quad y_2'(0) = 1, \tag{5.6.16}$$

and

$$w(t) = \frac{1}{a} y_2(t). \tag{5.6.17}$$

Proof Taking Laplace transforms in (5.6.13) yields

$$p(s)Y(s) = F(s) + a(k_1 + k_0 s) + bk_0,$$

where

$$p(s) = as^2 + bs + c.$$

Hence,

$$Y(s) = W(s)F(s) + V(s) \tag{5.6.18}$$

with

$$W(s) = \frac{1}{p(s)} \tag{5.6.19}$$

and

$$V(s) = \frac{a(k_1 + k_0 s) + bk_0}{p(s)}. \tag{5.6.20}$$

Taking Laplace transforms in (5.6.15) and (5.6.16) shows that

$$p(s)Y_1(s) = as + b \quad \text{and} \quad p(s)Y_2(s) = a.$$

Therefore

$$Y_1(s) = \frac{as + b}{p(s)}$$

and

$$Y_2(s) = \frac{a}{p(s)}. \tag{5.6.21}$$

Hence, (5.6.20) can be rewritten as

$$V(s) = k_0 Y_1(s) + k_1 Y_2(s).$$

Substituting this into (5.6.18) yields

$$Y(s) = k_0 Y_1(s) + k_1 Y_2(s) + \frac{1}{a} Y_2(s) F(s).$$

Taking inverse transforms and invoking the convolution theorem yields (5.6.14). Finally, (5.6.19) and (5.6.21) imply (5.6.17). ■

It is useful to note from (5.6.14) that y is of the form

$$y = v + h,$$

where

$$v(t) = k_0 y_1(t) + k_1 y_2(t)$$

depends on the initial conditions and is independent of the forcing function, while

$$h(t) = \int_0^t w(\tau) f(t - \tau)\, d\tau$$

depends on the forcing function and is independent of the initial conditions. If the zeros of the characteristic polynomial

$$p(s) = as^2 + bs + c$$

of the complementary equation have negative real parts, then y_1 and y_2 both approach zero as $t \to \infty$, so $\lim_{t\to\infty} v(t) = 0$ for any choice of initial conditions. Moreover, the value of $h(t)$ is essentially independent of the values of $f(t - \tau)$ for large τ, since $\lim_{\tau\to\infty} w(\tau) = 0$. In this case we say that v and h are *transient* and *steady state components*, respectively, of the solution y of (5.6.13). These definitions apply to the initial value problem of Example 5.6.4, where the zeros of

$$p(s) = s^2 + 2s + 2 = (s + 1)^2 + 1$$

are $-1 \pm i$. From (5.6.10), we see that the solution of the general initial value problem of Example 5.6.4 is $y = v + h$, where

$$v(t) = e^{-t}\left((k_1 + k_0)\sin t + k_0 \cos t\right)$$

is the transient component of the solution and

$$h(t) = \int_0^t f(t - \tau) e^{-\tau} \sin \tau \, d\tau$$

is the steady state component. The definitions don't apply to the initial value problems considered in Examples 5.6.2 and 5.6.3, since the zeros of the characteristic polynomials in these two examples don't have negative real parts.

In physical applications where the input f and the output y of a device are related by (5.6.13), the zeros of the characteristic polynomial usually do have negative real parts. Then $W = \mathcal{L}(w)$ is called the *transfer function* of the device. Since

$$H(s) = W(s)F(s),$$

we see that

$$W(s) = \frac{H(s)}{F(s)}$$

is the ratio of the transform of the steady state output to the transform of the input.

Because of the form of

$$h(t) = \int_0^t w(\tau)f(t-\tau)\,d\tau,$$

w is sometimes called the *weighting function* of the device, since it assigns weights to past values of the input f. It is also called the *impulse response* of the device, for reasons discussed in the next section.

Formula (5.6.14) is given in more detail in Exercises 8–10 for the three possible cases where the zeros of $p(s)$ are real and distinct, real and repeated, or complex conjugates, respectively.

5.6 Exercises

1. Express the inverse transform as an integral.

(a) $\dfrac{1}{s^2(s^2+4)}$ **(b)** $\dfrac{s}{(s+2)(s^2+9)}$

(c) $\dfrac{s}{(s^2+4)(s^2+9)}$ **(d)** $\dfrac{s}{(s^2+1)^2}$

(e) $\dfrac{1}{s(s-a)}$ **(f)** $\dfrac{1}{(s+1)(s^2+2s+2)}$

(g) $\dfrac{1}{(s+1)^2(s^2+4s+5)}$ **(h)** $\dfrac{1}{(s-1)^3(s+2)^2}$

(i) $\dfrac{s-1}{s^2(s^2-2s+2)}$ **(j)** $\dfrac{s(s+3)}{(s^2+4)(s^2+6s+10)}$

(k) $\dfrac{1}{(s-3)^5s^6}$ **(l)** $\dfrac{1}{(s-1)^3(s^2+4)}$

(m) $\dfrac{1}{s^2(s-2)^3}$ **(n)** $\dfrac{1}{s^7(s-2)^6}$

2. Find the Laplace transform.

(a) $\displaystyle\int_0^t \sin a\tau \cos b(t-\tau)\,d\tau$ **(b)** $\displaystyle\int_0^t e^{\tau} \sin a(t-\tau)\,d\tau$

(c) $\int_0^t \sinh a\tau \cosh a(t-\tau)\,d\tau$ **(d)** $\int_0^t \tau(t-\tau)\sin\omega\tau\cos\omega(t-\tau)\,d\tau$

(e) $e^t\int_0^t \sin\omega\tau\cos\omega(t-\tau)\,d\tau$ **(f)** $e^t\int_0^t \tau^2(t-\tau)e^\tau\,d\tau$

(g) $e^{-t}\int_0^t e^{-\tau}\tau\cos\omega(t-\tau)\,d\tau$ **(h)** $e^t\int_0^t e^{2\tau}\sinh(t-\tau)\,d\tau$

(i) $\int_0^t \tau e^{2\tau}\sin 2(t-\tau)\,d\tau$ **(j)** $\int_0^t (t-\tau)^3e^\tau\,d\tau$

(k) $\int_0^t \tau^6e^{-(t-\tau)}\sin 3(t-\tau)\,d\tau$ **(l)** $\int_0^t \tau^2(t-\tau)^3\,d\tau$

(m) $\int_0^t (t-\tau)^7e^{-\tau}\sin 2\tau\,d\tau$ **(n)** $\int_0^t (t-\tau)^4\sin 2\tau\,d\tau$

3. Find a formula for the solution of the initial value problem.

(a) $y''+3y'+y=f(t),\quad y(0)=0,\quad y'(0)=0$
(b) $y''+4y=f(t),\quad y(0)=0,\quad y'(0)=0$
(c) $y''+2y'+y=f(t),\quad y(0)=0,\quad y'(0)=0$
(d) $y''+k^2y=f(t),\quad y(0)=1,\quad y'(0)=-1$
(e) $y''+6y'+9y=f(t),\quad y(0)=0,\quad y'(0)=-2$
(f) $y''-4y=f(t),\quad y(0)=0,\quad y'(0)=3$
(g) $y''-5y'+6y=f(t),\quad y(0)=1,\quad y'(0)=3$
(h) $y''+\omega^2y=f(t),\quad y(0)=k_0,\quad y'(0)=k_1$

4. Solve the integral equation.

(a) $y(t)=t-\int_0^t (t-\tau)y(\tau)\,d\tau$

(b) $y(t)=\sin t-2\int_0^t \cos(t-\tau)y(\tau)\,d\tau$

(c) $y(t)=1+2\int_0^t y(\tau)\cos(t-\tau)\,d\tau$ **(d)** $y(t)=t+\int_0^t y(\tau)e^{-(t-\tau)}\,d\tau$

(e) $y'(t)=t+\int_0^t y(\tau)\cos(t-\tau)\,d\tau,\ y(0)=4$

(f) $y(t)=\cos t-\sin t+\int_0^t y(\tau)\sin(t-\tau)\,d\tau$

5. Use the convolution theorem to evaluate the integral.

(a) $\int_0^t (t-\tau)^7\tau^8\,d\tau$ **(b)** $\int_0^t (t-\tau)^{13}\tau^7\,d\tau$

(c) $\int_0^t (t-\tau)^6\tau^7\,d\tau$ **(d)** $\int_0^t e^{-\tau}\sin(t-\tau)\,d\tau$

(e) $\int_0^t \sin\tau\cos 2(t-\tau)\,d\tau$

6. Show that

$$\int_0^t f(t-\tau)g(\tau)\,d\tau = \int_0^t f(\tau)g(t-\tau)\,d\tau$$

by introducing the new variable of integration $x = t-\tau$ in the first integral.

7. Use the convolution theorem to show that if $f(t) \leftrightarrow F(s)$ then

$$\int_0^t f(\tau)\,d\tau \leftrightarrow \frac{F(s)}{s}.$$

8. Show that if $p(s) = as^2 + bs + c$ has distinct real zeros r_1 and r_2 then the solution of

$$ay'' + by' + cy = f(t), \quad y(0) = k_0, \quad y'(0) = k_1$$

is

$$\begin{aligned} y(t) &= k_0\frac{r_2e^{r_1t} - r_1e^{r_2t}}{r_2 - r_1} + k_1\frac{e^{r_2t} - e^{r_1t}}{r_2 - r_1} \\ &\quad + \frac{1}{a(r_2 - r_1)}\int_0^t (e^{r_2\tau} - e^{r_1\tau})f(t-\tau)\,d\tau. \end{aligned}$$

9. Show that if $p(s) = as^2 + bs + c$ has a repeated real zero r_1 then the solution of

$$ay'' + by' + cy = f(t), \quad y(0) = k_0, \quad y'(0) = k_1$$

is

$$y(t) = k_0(1 - r_1t)e^{r_1t} + k_1te^{r_1t} + \frac{1}{a}\int_0^t \tau e^{r_1\tau}f(t-\tau)\,d\tau.$$

10. Show that if $p(s) = as^2 + bs + c$ has complex conjugate zeros $\lambda \pm i\omega$ then the solution of

$$ay'' + by' + cy = f(t), \quad y(0) = k_0, \quad y'(0) = k_1$$

is

$$\begin{aligned} y(t) &= e^{\lambda t}\left[k_0(\cos\omega t - \frac{\lambda}{\omega}\sin\omega t) + \frac{k_1}{\omega}\sin\omega t\right] \\ &\quad + \frac{1}{a\omega}\int_0^t e^{\lambda t}f(t-\tau)\sin\omega\tau\,d\tau. \end{aligned}$$

11. Let

$$w = \mathcal{L}^{-1}\left(\frac{1}{as^2+bs+c}\right),$$

where a, b, and c are constants and $a \neq 0$.

(a) Show that w is the solution of

$$aw'' + bw' + cw = 0, \quad w(0) = 0, \quad w'(0) = \frac{1}{a}.$$

(b) Let f be continuous on $[0, \infty)$ and define

$$h(t) = \int_0^t w(t-\tau)f(\tau)\,d\tau.$$

Use *Leibniz's rule* for differentiating an integral with respect to a parameter to show that h is the solution of

$$ah'' + bh' + ch = f, \quad h(0) = 0, \quad h'(0) = 0.$$

(c) Show that the function y in Eqn. (5.6.14) is the solution of Eqn. (5.6.13) provided that f is continuous on $[0, \infty)$; thus, it's not necessary to assume that f has a Laplace transform.

12. Consider the initial value problem

$$ay'' + by' + cy = f(t), \quad y(0) = 0, \quad y'(0) = 0, \tag{A}$$

where a, b, and c are constants, $a \neq 0$, and

$$f(t) = \begin{cases} f_0(t), & 0 \leqslant t < t_1, \\ f_1(t), & t \geqslant t_1. \end{cases}$$

Assume that f_0 is continuous and of exponential order on $[0, \infty)$ and f_1 is continuous and of exponential order on $[t_1, \infty)$. Let

$$p(s) = as^2 + bs + c.$$

(a) Show that the Laplace transform of the solution of (A) is

$$Y(s) = \frac{F_0(s) + e^{-st_1}G(s)}{p(s)}$$

where $g(t) = f_1(t+t_1) - f_0(t+t_1)$.

(b) Let w be as in Exercise 11. Use Theorem 5.4.2 and the convolution theorem to show that the solution of (A) is

$$y(t) = \int_0^t w(t-\tau)f_0(\tau)\,d\tau + u(t-t_1)\int_0^{t-t_1} w(t-t_1-\tau)g(\tau)\,d\tau$$

for $t > 0$.

(c) Henceforth, assume only that f_0 is continuous on $[0,\infty)$ and f_1 is continuous on $[t_1,\infty)$. Use Exercise 11 **(a)** and **(b)** to show that

$$y'(t)=\int_0^t w'(t-\tau)f_0(\tau)\,d\tau+u(t-t_1)\int_0^{t-t_1} w'(t-t_1-\tau)g(\tau)\,d\tau$$

for $t>0$, and

$$y''(t)=\frac{f(t)}{a}+\int_0^t w''(t-\tau)f_0(\tau)\,d\tau+u(t-t_1)\int_0^{t-t_1} w''(t-t_1-\tau)g(\tau)\,d\tau$$

for $0<t<t_1$ and $t>t_1$. Also, show y satisfies the differential equation in (A) on$(0,t_1)$ and (t_1,∞).

(d) Show that y and y' are continuous on $[0,\infty)$.

13. Suppose

$$f(t)=\begin{cases} f_0(t), & 0\leqslant t<t_1,\\ f_1(t), & t_1\leqslant t<t_2,\\ \vdots & \\ f_{k-1}(t), & t_{k-1}\leqslant t<t_k,\\ f_k(t), & t\geqslant t_k,\end{cases}$$

where f_m is continuous on $[t_m,\infty)$ for $m=0,\dots,k$ (let $t_0=0$), and define

$$g_m(t)=f_m(t+t_m)-f_{m-1}(t+t_m),\ m=1,\dots,k.$$

Extend the results of Exercise 12 to show that the solution of

$$ay''+by'+cy=f(t),\quad y(0)=0,\quad y'(0)=0$$

is

$$y(t)=\int_0^t w(t-\tau)f_0(\tau)\,d\tau+\sum_{m=1}^{k} u(t-t_m)\int_0^{t-t_m} w(t-t_m-\tau)g_m(\tau)\,d\tau.$$

14. Let $\{t_m\}_{m=0}^{\infty}$ be a sequence of points such that $t_0=0$, $t_{m+1}>t_m$, and $\lim_{m\to\infty}t_m=\infty$. For each nonegative integer m let f_m be continuous on $[t_m,\infty)$, and let f be defined on $[0,\infty)$ by

$$f(t)=f_m(t),\quad t_m\leqslant t<t_{m+1}\quad m=0,1,2\dots.$$

Let

$$g_m(t)=f_m(t+t_m)-f_{m-1}(t+t_m),\quad m=1,\dots,k.$$

Extend the results of Exercise 13 to show that the solution of

$$ay''+by'+cy=f(t),\quad y(0)=0,\quad y'(0)=0$$

is

$$y(t) = \int_0^t w(t-\tau)f_0(\tau)\,d\tau + \sum_{m=1}^{\infty} u(t-t_m)\int_0^{t-t_m} w(t-t_m-\tau)g_m(\tau)\,d\tau.$$

HINT: *See Exercise*30.

5.7 CONSTANT COEFFICIENT EQUATIONS WITH IMPULSES

So far in this chapter, we've considered initial value problems for the constant coefficient equation

$$ay'' + by' + cy = f(t),$$

where f is continuous or piecewise continuous on $[0,\infty)$. In this section we consider initial value problems where f represents a force that's very large for a short time and zero otherwise. We say that such forces are *impulsive*. Impulsive forces occur, for example, when two objects collide. Since it isn't feasible to represent such forces as continuous or piecewise continuous functions, we must construct a different mathematical model to deal with them.

If f is an integrable function and $f(t) = 0$ for t outside of the interval $[t_0, t_0+h]$, then $\int_{t_0}^{t_0+h} f(t)\,dt$ is called the *total impulse* of f. We're interested in the idealized situation where h is so small that the total impulse can be assumed to be applied instantaneously at $t = t_0$. We say in this case that f is an *impulse function*. In particular, we denote by $\delta(t-t_0)$ the impulse function with total impulse equal to one, applied at $t = t_0$. (The impulse function $\delta(t)$ obtained by setting $t_0 = 0$ is the *Dirac δ function*.) It must be understood, however, that $\delta(t-t_0)$ isn't a function in the standard sense, since our "definition" implies that $\delta(t-t_0) = 0$ if $t \neq t_0$, while

$$\int_{t_0}^{t_0} \delta(t-t_0)\,dt = 1.$$

From calculus we know that no function can have these properties; nevertheless, there's a branch of mathematics known as the *theory of distributions* where the definition can be made rigorous. Since the theory of distributions is beyond the scope of this book, we'll take an intuitive approach to impulse functions.

Our first task is to define what we mean by the solution of the initial value problem

$$ay'' + by' + cy = \delta(t-t_0), \quad y(0) = 0, \quad y'(0) = 0,$$

where t_0 is a fixed nonnegative number. The next theorem will motivate our definition.

Theorem 5.7.1 *Suppose $t_0 \geqslant 0$. For each positive number h, let y_h be the solution of the initial value problem*

$$ay_h'' + by_h' + cy_h = f_h(t), \quad y_h(0) = 0, \quad y_h'(0) = 0, \tag{5.7.1}$$

Figure 5.1 $y = f_h(t)$

where

$$f_h(t) = \begin{cases} 0, & 0 \leqslant t < t_0, \\ 1/h, & t_0 \leqslant t < t_0 + h, \\ 0, & t \geqslant t_0 + h, \end{cases} \tag{5.7.2}$$

so f_h *has unit total impulse equal to the area of the shaded rectangle in Figure* 5.1. *Then*

$$\lim_{h \to 0+} y_h(t) = u(t - t_0) w(t - t_0), \tag{5.7.3}$$

where

$$w = \mathcal{L}^{-1}\left(\frac{1}{as^2 + bs + c}\right).$$

Proof Taking Laplace transforms in (5.7.1) yields

$$(as^2 + bs + c)Y_h(s) = F_h(s),$$

so

$$Y_h(s) = \frac{F_h(s)}{as^2 + bs + c}.$$

The convolution theorem implies that

$$y_h(t) = \int_0^t w(t - \tau) f_h(\tau)\, d\tau.$$

Therefore, (5.7.2) implies that

$$y_h(t) = \begin{cases} 0, & 0 \leqslant t < t_0, \\ \displaystyle\frac{1}{h}\int_{t_0}^{t} w(t-\tau)\, d\tau, & t_0 \leqslant t \leqslant t_0 + h, \\ \displaystyle\frac{1}{h}\int_{t_0}^{t_0+h} w(t-\tau)\, d\tau, & t > t_0 + h. \end{cases} \tag{5.7.4}$$

Since $y_h(t) = 0$ for all h if $0 \leqslant t \leqslant t_0$, it follows that

$$\lim_{h \to 0+} y_h(t) = 0 \quad \text{if} \quad 0 \leqslant t \leqslant t_0. \tag{5.7.5}$$

We'll now show that

$$\lim_{h \to 0+} y_h(t) = w(t - t_0) \quad \text{if} \quad t > t_0. \tag{5.7.6}$$

Suppose t is fixed and $t > t_0$. From (5.7.4),

$$y_h(t) = \frac{1}{h}\int_{t_0}^{t_0+h} w(t-\tau) d\tau \quad \text{if} \quad h < t - t_0. \tag{5.7.7}$$

Since

$$\frac{1}{h}\int_{t_0}^{t_0+h} d\tau = 1, \tag{5.7.8}$$

we can write

$$w(t-t_0) = \frac{1}{h}w(t-t_0)\int_{t_0}^{t_0+h} d\tau = \frac{1}{h}\int_{t_0}^{t_0+h} w(t-t_0)\, d\tau.$$

From this and (5.7.7),

$$y_h(t) - w(t-t_0) = \frac{1}{h}\int_{t_0}^{t_0+h} \left(w(t-\tau) - w(t-t_0)\right)\, d\tau.$$

Therefore

$$|y_h(t) - w(t-t_0)| \leqslant \frac{1}{h}\int_{t_0}^{t_0+h} |w(t-\tau) - w(t-t_0)|\, d\tau. \tag{5.7.9}$$

Now let M_h be the maximum value of $|w(t-\tau) - w(t-t_0)|$ as τ varies over the interval $[t_0, t_0+h]$. (Remember that t and t_0 are fixed.) Then (5.7.8) and (5.7.9) imply that

$$|y_h(t) - w(t-t_0)| \leqslant \frac{1}{h}M_h\int_{t_0}^{t_0+h} d\tau = M_h. \tag{5.7.10}$$

But $\lim_{h\to 0+} M_h = 0$, since w is continuous. Therefore (5.7.10) implies (5.7.6). This and (5.7.5) imply (5.7.3). ■

Theorem 5.7.1 motivates the next definition.

Definition 5.7.2 If $t_0 > 0$, then the solution of the initial value problem

$$ay'' + by' + cy = \delta(t-t_0), \quad y(0) = 0, \quad y'(0) = 0, \tag{5.7.11}$$

is defined to be

$$y = u(t-t_0)w(t-t_0),$$

where

$$w = \mathcal{L}^{-1}\left(\frac{1}{as^2+bs+c}\right).$$

In physical applications where the input f and the output y of a device are related by the differential equation

$$ay'' + by' + cy = f(t),$$

w is called the *impulse response* of the device. Note that w is the solution of the initial value problem

$$aw'' + bw' + cw = 0, \quad w(0) = 0, \quad w'(0) = 1/a, \tag{5.7.12}$$

Figure 5.2 An illustration of Theorem 5.7.1

as can be seen by using the Laplace transform to solve this problem. (Verify.) On the other hand, we can solve (5.7.12) by the methods of Section 5.2 and show that w is defined on $(-\infty,\infty)$ by

$$w=\frac{e^{r_2t}-e^{r_1t}}{a(r_2-r_1)},\quad w=\frac{1}{a}te^{r_1t},\quad \text{or}\quad w=\frac{1}{a\omega}e^{\lambda t}\sin\omega t, \tag{5.7.13}$$

depending upon whether the polynomial $p(r)=ar^2+br+c$ has distinct real zeros r_1 and r_2, a repeated zero r_1, or complex conjugate zeros $\lambda\pm i\omega$. (In most physical applications, the zeros of the characteristic polynomial have negative real parts, so $\lim_{t\to\infty}w(t)=0$.) This means that $y=u(t-t_0)w(t-t_0)$ is defined on $(-\infty,\infty)$ and has the following properties:

$$y(t)=0,\quad t<t_0,$$

$$ay''+by'+cy=0\quad\text{on}\quad(-\infty,t_0)\quad\text{and}\quad(t_0,\infty),$$

and

$$y'_-(t_0)=0,\quad y'_+(t_0)=1/a \tag{5.7.14}$$

(remember that $y'_-(t_0)$ and $y'_+(t_0)$ are derivatives from the right and left, respectively) and $y'(t_0)$ does not exist. Thus, even though we defined $y=u(t-t_0)w(t-t_0)$ to be the solution of (5.7.11), this function *doesn't satisfy* the differential equation in (5.7.11) *at* t_0, since it isn't differentiable there; in fact (5.7.14) indicates that an impulse causes a jump discontinuity in velocity. (To see that this is reasonable, think of what happens when you hit a ball with a bat.) This means that the initial value problem (5.7.11) doesn't make sense if $t_0=0$, since $y'(0)$ doesn't exist in this case. However $y=u(t)w(t)$ can be defined to be the solution of the modified initial value problem

$$ay''+by'+cy=\delta(t),\quad y(0)=0,\quad y'_-(0)=0,$$

where the condition on the derivative at $t=0$ has been replaced by a condition on the derivative from the left.

Figure 5.2 illustrates Theorem 5.7.1 for the case where the impulse response w is the first expression in (5.7.13) and r_1 and r_2 are distinct and both negative. The solid curve in the figure is the graph of w. The dashed curves are solutions of (5.7.1) for various values of h. As h decreases the graph of y_h moves to the left toward the graph of w.

Example 5.7.1 Find the solution of the initial value problem

$$y''-2y'+y=\delta(t-t_0),\quad y(0)=0,\quad y'(0)=0, \tag{5.7.15}$$

where $t_0>0$. Then interpret the solution for the case where $t_0=0$.

Solution Here

$$w=\mathcal{L}^{-1}\left(\frac{1}{s^2-2s+1}\right)=\mathcal{L}^{-1}\left(\frac{1}{(s-1)^2}\right)=te^{-t},$$

Figure 5.3 $y = u(t - t_0)(t - t_0)e^{-(t-t_0)}$

so Definition 5.7.2 yields

$$y = u(t - t_0)(t - t_0)e^{-(t-t_0)}$$

as the solution of (5.7.15) if $t_0 > 0$. If $t_0 = 0$, then (5.7.15) doesn't have a solution; however, $y = u(t)te^{-t}$ (which we would usually write simply as $y = te^{-t}$) is the solution of the modified initial value problem

$$y'' - 2y' + y = \delta(t), \quad y(0) = 0, \quad y'_-(0) = 0.$$

The graph of $y = u(t - t_0)(t - t_0)e^{-(t-t_0)}$ is shown in Figure 5.3 ■

Definition 5.7.2 and the principle of superposition motivate the next definition.

Definition 5.7.3 Suppose α is a nonzero constant and f is piecewise continuous on $[0, \infty)$. If $t_0 > 0$, then the solution of the initial value problem

$$ay'' + by' + cy = f(t) + \alpha\delta(t - t_0), \quad y(0) = k_0, \quad y'(0) = k_1$$

is defined to be

$$y(t) = \hat{y}(t) + \alpha u(t - t_0)w(t - t_0),$$

where $\hat{y}$ is the solution of

$$ay'' + by' + cy = f(t), \quad y(0) = k_0, \quad y'(0) = k_1.$$

This definition also applies if $t_0 = 0$, provided that the initial condition $y'(0) = k_1$ is replaced by $y'_-(0) = k_1$.

Example 5.7.2 Solve the initial value problem

$$y'' + 6y' + 5y = 3e^{-2t} + 2\delta(t - 1), \quad y(0) = -3, \quad y'(0) = 2. \tag{5.7.16}$$

Solution We leave it to you to show that the solution of

$$y'' + 6y' + 5y = 3e^{-2t}, \quad y(0) = -3, \; y'(0) = 2$$

is

$$\hat{y} = -e^{-2t} + \frac{1}{2}e^{-5t} - \frac{5}{2}e^{-t}.$$

Since

$$\begin{aligned} w(t) &= \mathcal{L}^{-1}\left(\frac{1}{s^2 + 6s + 5}\right) &= \mathcal{L}^{-1}\left(\frac{1}{(s+1)(s+5)}\right) \\ &= \frac{1}{4}\mathcal{L}^{-1}\left(\frac{1}{s+1} - \frac{1}{s+5}\right) &= \frac{e^{-t} - e^{-5t}}{4}, \end{aligned}$$

Figure 5.4 Graph of (5.7.17)

Figure 5.5 Graph of (5.7.19)

the solution of (5.7.16) is

$$y = -e^{-2t} + \frac{1}{2}e^{-5t} - \frac{5}{2}e^{-t} + u(t-1)\frac{e^{-(t-1)} - e^{-5(t-1)}}{2} \tag{5.7.17}$$

(Figure 5.4) ■.

Definition 5.7.3 can be extended in the obvious way to cover the case where the forcing function contains more than one impulse.

Example 5.7.3 Solve the initial value problem

$$y'' + y = 1 + 2\delta(t-\pi) - 3\delta(t-2\pi), \quad y(0) = -1,\ y'(0) = 2. \tag{5.7.18}$$

Solution We leave it to you to show that

$$\hat{y} = 1 - 2\cos t + 2\sin t$$

is the solution of

$$y'' + y = 1, \quad y(0) = -1, \quad y'(0) = 2.$$

Since

$$w = \mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) = \sin t,$$

the solution of (5.7.18) is

$$\begin{aligned} y &= 1 - 2\cos t + 2\sin t + 2u(t-\pi)\sin(t-\pi) - 3u(t-2\pi)\sin(t-2\pi) \\ &= 1 - 2\cos t + 2\sin t - 2u(t-\pi)\sin t - 3u(t-2\pi)\sin t, \end{aligned}$$

or

$$y = \begin{cases} 1 - 2\cos t + 2\sin t, & 0 \leqslant t < \pi, \\ 1 - 2\cos t, & \pi \leqslant t < 2\pi, \\ 1 - 2\cos t - 3\sin t, & t \geqslant 2\pi \end{cases} \tag{5.7.19}$$

(Figure 5.5).

5.7 Exercises

In Exercises 1–20 solve the initial value problem. Where indicated by C/G , graph the solution.

1. $y'' + 3y' + 2y = 6e^{2t} + 2\delta(t-1), \quad y(0) = 2, \quad y'(0) = -6$

2. C/G $y'' + y' - 2y = -10e^{-t} + 5\delta(t-1), \quad y(0) = 7, \quad y'(0) = -9$

3. $y'' - 4y = 2e^{-t} + 5\delta(t-1), \quad y(0) = -1, \quad y'(0) = 2$

4. C/G $y'' + y = \sin 3t + 2\delta(t-\pi/2), \quad y(0) = 1, \quad y'(0) = -1$

5. $y'' + 4y = 4 + \delta(t-3\pi), \quad y(0) = 0, \quad y'(0) = 1$

6. $y'' - y = 8 + 2\delta(t-2), \quad y(0) = -1, \quad y'(0) = 1$

7. $y'' + y' = e^t + 3\delta(t-6), \quad y(0) = -1, \quad y'(0) = 4$

8. $y'' + 4y = 8e^{2t} + \delta(t-\pi/2), \quad y(0) = 8, \quad y'(0) = 0$

9. C/G $y'' + 3y' + 2y = 1 + \delta(t-1), \quad y(0) = 1, \quad y'(0) = -1$

10. $y'' + 2y' + y = e^t + 2\delta(t-2), \quad y(0) = -1, \quad y'(0) = 2$

11. C/G $y'' + 4y = \sin t + \delta(t-\pi/2), \quad y(0) = 0, \quad y'(0) = 2$

12. $y'' + 2y' + 2y = \delta(t-\pi) - 3\delta(t-2\pi), \quad y(0) = -1, \quad y'(0) = 2$

13. $y'' + 4y' + 13y = \delta(t-\pi/6) + 2\delta(t-\pi/3), \quad y(0) = 1, \quad y'(0) = 2$

14. $2y'' - 3y' - 2y = 1 + \delta(t-2), \quad y(0) = -1, \quad y'(0) = 2$

15. $4y'' - 4y' + 5y = 4\sin t - 4\cos t + \delta(t-\pi/2) - \delta(t-\pi), \quad y(0) = 1, \quad y'(0) = 1$

16. $y'' + y = \cos 2t + 2\delta(t-\pi/2) - 3\delta(t-\pi), \quad y(0) = 0, \quad y'(0) = -1$

17. C/G $y'' - y = 4e^{-t} - 5\delta(t-1) + 3\delta(t-2), \quad y(0) = 0, \quad y'(0) = 0$

18. $y'' + 2y' + y = e^t - \delta(t-1) + 2\delta(t-2), \quad y(0) = 0, \quad y'(0) = -1$

19. $y'' + y = f(t) + \delta(t-2\pi), \quad y(0) = 0, \quad y'(0) = 1$, and

$$f(t) = \begin{cases} \sin 2t, & 0 \leqslant t < \pi, \\ 0, & t \geqslant \pi. \end{cases}$$

20. $y'' + 4y = f(t) + \delta(t-\pi) - 3\delta(t-3\pi/2), \quad y(0) = 1, \quad y'(0) = -1$, and

$$f(t) = \begin{cases} 1, & 0 \leqslant t < \pi/2, \\ 2, & t \geqslant \pi/2 \end{cases}$$

21. $y'' + y = \delta(t), \quad y(0) = 1, \quad y'_-(0) = -2$

22. $y'' - 4y = 3\delta(t), \quad y(0) = -1, \quad y'_-(0) = 7$

23. $y'' + 3y' + 2y = -5\delta(t), \quad y(0) = 0, \quad y'_-(0) = 0$

24. $y'' + 4y' + 4y = -\delta(t), \quad y(0) = 1, \quad y'_-(0) = 5$

25. $4y'' + 4y' + y = 3\delta(t), \quad y(0) = 1, \quad y'_-(0) = -6$

In Exercises 26-28*, solve the initial value problem*

$$ay_h'' + by_h' + cy_h = \begin{cases} 0, & 0 \leqslant t < t_0, \\ 1/h, & t_0 \leqslant t < t_0 + h, \\ 0, & t \geqslant t_0 + h, \end{cases} \qquad y_h(0) = 0, \quad y_h'(0) = 0,$$

where $t_0 > 0$ *and* $h > 0$*. Then find*

$$w = \mathcal{L}^{-1}\left(\frac{1}{as^2 + bs + c}\right)$$

and verify Theorem 5.7.1 *by graphing* w *and* y_h *on the same axes, for small positive values of* h.

26. L $y'' + 2y' + 2y = f_h(t), \quad y(0) = 0, \quad y'(0) = 0$

27. L $y'' + 2y' + y = f_h(t), \quad y(0) = 0, \quad y'(0) = 0$

28. L $y'' + 3y' + 2y = f_h(t), \quad y(0) = 0, \quad y'(0) = 0$

29. Recall from Section 6.2 that the displacement of an object of mass m in a spring–mass system in free damped oscillation is

$$my'' + cy' + ky = 0, \quad y(0) = y_0, \quad y'(0) = v_0,$$

and that y can be written as

$$y = Re^{-ct/2m}\cos(\omega_1 t - \phi)$$

if the motion is underdamped. Suppose $y(\tau) = 0$. Find the impulse that would have to be applied to the object at $t = \tau$ to put it in equilibrium.

30. Solve the initial value problem. Find a formula that does not involve step functions and represents y on each subinterval of $[0, \infty)$ on which the forcing function is zero.

(a) $y'' - y = \sum_{k=1}^{\infty} \delta(t - k), \quad y(0) = 0, \quad y'(0) = 1$

(b) $y'' + y = \sum_{k=1}^{\infty} \delta(t - 2k\pi), \quad y(0) = 0, \quad y'(0) = 1$

(c) $y'' - 3y' + 2y = \sum_{k=1}^{\infty} \delta(t - k), \quad y(0) = 0, \quad y'(0) = 1$

(d) $y'' + y = \sum_{k=1}^{\infty} \delta(t - k\pi), \quad y(0) = 0, \quad y'(0) = 0$

5.8 A BRIEF TABLE OF LAPLACE TRANSFORMS

$f(t)$	$F(s)$	
1	$\frac{1}{s}$	$(s > 0)$
t^n ($n =$ integer > 0)	$\frac{n!}{s^{n+1}}$	$(s > 0)$
$t^p,\ p > -1$	$\frac{\Gamma(p+1)}{s^{(p+1)}}$	$(s > 0)$
e^{at}	$\frac{1}{s-a}$	$(s > a)$
$t^n e^{at}$ ($n =$ integer > 0)	$\frac{n!}{(s-a)^{n+1}}$	$(s > 0)$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$	$(s > 0)$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$	$(s > 0)$
$e^{\lambda t} \cos \omega t$	$\frac{s-\lambda}{(s-\lambda)^2+\omega^2}$	$(s > \lambda)$
$e^{\lambda t} \sin \omega t$	$\frac{\omega}{(s-\lambda)^2+\omega^2}$	$(s > \lambda)$
$\cosh bt$	$\frac{s}{s^2-b^2}$	$(s > \|b\|)$
$\sinh bt$	$\frac{b}{s^2-b^2}$	$(s > \|b\|)$
$t \cos \omega t$	$\frac{s^2-\omega^2}{(s^2+\omega^2)^2}$	$(s > 0)$

$t \sin \omega t$	$\dfrac{2\omega s}{(s^2+\omega^2)^2}$	$(s > 0)$
$\sin \omega t - \omega t \cos \omega t$	$\dfrac{2\omega^3}{(s^2+\omega^2)^2}$	$(s > 0)$
$\omega t - \sin \omega t$	$\dfrac{\omega^3}{s^2(s^2+\omega^2)^2}$	$(s > 0)$
$\dfrac{1}{t} \sin \omega t$	$\arctan\left(\dfrac{\omega}{s}\right)$	$(s > 0)$
$e^{at} f(t)$	$F(s-a)$	
$t^k f(t)$	$(-1)^k F^{(k)}(s)$	
$f(\omega t)$	$\dfrac{1}{\omega} F\left(\dfrac{s}{\omega}\right), \quad \omega > 0$	
$u(t-\tau)$	$\dfrac{e^{-\tau s}}{s}$	$(s > 0)$
$u(t-\tau) f(t-\tau)\ (\tau > 0)$	$e^{-\tau s} F(s)$	
$\int_0^t f(\tau) g(t-\tau)\, d\tau$	$F(s) \cdot G(s)$	
$\delta(t-a)$	e^{-as}	$(s > 0)$

CHAPTER 6

LINEAR SYSTEMS OF DIFFERENTIAL EQUATIONS

IN THIS CHAPTER we consider systems of differential equations involving more than one unknown function. Such systems arise in many physical applications.

SECTION 10.1 presents examples of physical situations that lead to systems of differential equations.

SECTION 10.2 discusses linear systems of differential equations.

SECTION 10.3 deals with the basic theory of homogeneous linear systems.

SECTIONS 10.4, 10.5, AND 10.6 present the theory of constant coefficient homogeneous systems.

SECTION 10.7 presents the method of variation of parameters for nonhomogeneous linear systems.

6.1 INTRODUCTION TO SYSTEMS OF DIFFERENTIAL EQUATIONS

Many physical situations are modelled by systems of n differential equations in n unknown functions, where $n \geqslant 2$. The next three examples illustrate physical problems that lead to systems of differential equations. In these examples and throughout this chapter we'll denote the independent variable by t.

Example 6.1.1 Tanks T_1 and T_2 contain 100 gallons and 300 gallons of salt solutions, respectively. Salt solutions are simultaneously added to both tanks from external sources, pumped from each tank to the other, and drained from both tanks (Figure 6.1). A solution with 1 pound of salt per gallon is pumped into T_1 from an external source at 5 gal/min, and a solution with 2 pounds of salt per gallon is pumped into T_2 from an external source at 4 gal/min. The solution from T_1 is pumped into T_2 at 2 gal/min, and the solution from T_2 is pumped into T_1 at 3 gal/min. T_1 is drained at 6 gal/min and T_2 is drained at 3 gal/min. Let $Q_1(t)$ and $Q_2(t)$ be the number of pounds of salt in T_1 and T_2, respectively, at time $t > 0$. Derive a system of differential equations for Q_1 and Q_2. Assume that both mixtures are well stirred.

Figure 6.1

Solution As in Section 4.2, let *rate in* and *rate out* denote the rates (lb/min) at which salt enters and leaves a tank; thus,

$$\begin{aligned} Q_1' &= (\text{rate in})_1 - (\text{rate out})_1, \\ Q_2' &= (\text{rate in})_2 - (\text{rate out})_2. \end{aligned}$$

Note that the volumes of the solutions in T_1 and T_2 remain constant at 100 gallons and 300 gallons, respectively.

T_1 receives salt from the external source at the rate of

$$(1 \text{ lb/gal}) \times (5 \text{ gal/min}) = 5 \text{ lb/min},$$

and from T_2 at the rate of

$$(\text{lb/gal in } T_2) \times (3 \text{ gal/min}) = \frac{1}{300}Q_2 \times 3 = \frac{1}{100}Q_2 \text{ lb/min}.$$

Therefore

$$(\text{rate in})_1 = 5 + \frac{1}{100}Q_2. \tag{6.1.1}$$

Solution leaves T_1 at the rate of 8 gal/min, since 6 gal/min are drained and 2 gal/min are pumped to T_2; hence,

$$(\text{rate out})_1 = (\text{ lb/gal in } T_1) \times (8 \text{ gal/min}) = \frac{1}{100}Q_1 \times 8 = \frac{2}{25}Q_1. \tag{6.1.2}$$

Eqns. (6.1.1) and (6.1.2) imply that

$$Q_1' = 5 + \frac{1}{100}Q_2 - \frac{2}{25}Q_1. \tag{6.1.3}$$

T_2 receives salt from the external source at the rate of

$$(2 \text{ lb/gal}) \times (4 \text{ gal/min}) = 8 \text{ lb/min},$$

and from T_1 at the rate of

$$(\text{lb/gal in } T_1) \times (2 \text{ gal/min}) = \frac{1}{100}Q_1 \times 2 = \frac{1}{50}Q_1 \text{ lb/min}.$$

Figure 6.2

Therefore

$$(\text{rate in})_2 = 8 + \frac{1}{50}Q_1. \tag{6.1.4}$$

Solution leaves T_2 at the rate of 6 gal/min, since 3 gal/min are drained and 3 gal/min are pumped to T_1; hence,

$$(\text{rate out})_2 = (\text{ lb/gal in } T_2) \times (6 \text{ gal/min}) = \frac{1}{300}Q_2 \times 6 = \frac{1}{50}Q_2. \tag{6.1.5}$$

Eqns. (6.1.4) and (6.1.5) imply that

$$Q_2' = 8 + \frac{1}{50}Q_1 - \frac{1}{50}Q_2. \tag{6.1.6}$$

We say that (6.1.3) and (6.1.6) form a *system of two first order equations in two unknowns*, and write them together as

$$\begin{aligned} Q_1' &= 5 - \frac{2}{25}Q_1 + \frac{1}{100}Q_2 \\ Q_2' &= 8 + \frac{1}{50}Q_1 - \frac{1}{50}Q_2. \quad \blacksquare \end{aligned}$$

Example 6.1.2 A mass m_1 is suspended from a rigid support on a spring S_1 and a second mass m_2 is suspended from the first on a spring S_2 (Figure 6.2). The springs obey Hooke's law, with spring constants k_1 and k_2. Internal friction causes the springs to exert damping forces proportional to the rates of change of their lengths, with damping constants c_1 and c_2. Let $y_1 = y_1(t)$ and $y_2 = y_2(t)$ be the displacements of the two masses from their equilibrium positions at time t, measured positive upward. Derive a system of differential equations for y_1 and y_2, assuming that the masses of the springs are negligible and that vertical external forces F_1 and F_2 also act on the objects.

Solution In equilibrium, S_1 supports both m_1 and m_2 and S_2 supports only m_2. Therefore, if $\Delta\ell_1$ and $\Delta\ell_2$ are the elongations of the springs in equilibrium then

$$(m_1 + m_2)g = k_1\Delta\ell_1 \quad \text{and} \quad m_2 g = k_2\Delta\ell_2. \tag{6.1.7}$$

Let H_1 be the Hooke's law force acting on m_1, and let D_1 be the damping force on m_1. Similarly, let H_2 and D_2 be the Hooke's law and damping forces acting on m_2. According to Newton's second law of motion,

$$\begin{aligned} m_1 y_1'' &= -m_1 g + H_1 + D_1 + F_1, \\ m_2 y_2'' &= -m_2 g + H_2 + D_2 + F_2. \end{aligned} \tag{6.1.8}$$

When the displacements are y_1 and y_2, the change in length of S_1 is $-y_1 + \Delta\ell_1$ and the change in length of S_2 is $-y_2 + y_1 + \Delta\ell_2$. Both springs exert Hooke's law forces on m_1, while only S_2 exerts a Hooke's law force on m_2. These forces are in directions that tend to restore the springs to their natural lengths. Therefore

$$H_1 = k_1(-y_1 + \Delta\ell_1) - k_2(-y_2 + y_1 + \Delta\ell_2) \quad \text{and} \quad H_2 = k_2(-y_2 + y_1 + \Delta\ell_2). \quad (6.1.9)$$

When the velocities are y_1' and y_2', S_1 and S_2 are changing length at the rates $-y_1'$ and $-y_2' + y_1'$, respectively. Both springs exert damping forces on m_1, while only S_2 exerts a damping force on m_2. Since the force due to damping exerted by a spring is proportional to the rate of change of length of the spring and in a direction that opposes the change, it follows that

$$D_1 = -c_1y_1' + c_2(y_2' - y_1') \quad \text{and} \quad D_2 = -c_2(y_2' - y_1'). \quad (6.1.10)$$

From (6.1.8), (6.1.9), and (6.1.10),

$$\begin{aligned} m_1y_1'' &= -m_1g + k_1(-y_1 + \Delta\ell_1) - k_2(-y_2 + y_1 + \Delta\ell_2) \\ &\quad -c_1y_1' + c_2(y_2' - y_1') + F_1 \\ &= -(m_1g - k_1\Delta\ell_1 + k_2\Delta\ell_2) - k_1y_1 + k_2(y_2 - y_1) \\ &\quad -c_1y_1' + c_2(y_2' - y_1') + F_1 \end{aligned} \quad (6.1.11)$$

and

$$\begin{aligned} m_2y_2'' &= -m_2g + k_2(-y_2 + y_1 + \Delta\ell_2) - c_2(y_2' - y_1') + F_2 \\ &= -(m_2g - k_2\Delta\ell_2) - k_2(y_2 - y_1) - c_2(y_2' - y_1') + F_2. \end{aligned} \quad (6.1.12)$$

From (6.1.7),

$$m_1g - k_1\Delta\ell_1 + k_2\Delta\ell_2 = -m_2g + k_2\Delta\ell_2 = 0.$$

Therefore we can rewrite (6.1.11) and (6.1.12) as

$$\begin{aligned} m_1y_1'' &= -(c_1 + c_2)y_1' + c_2y_2' - (k_1 + k_2)y_1 + k_2y_2 + F_1 \\ m_2y_2'' &= c_2y_1' - c_2y_2' + k_2y_1 - k_2y_2 + F_2. \quad \blacksquare \end{aligned}$$

Example 6.1.3 Let $\mathbf{X} = \mathbf{X}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$ be the position vector at time t of an object with mass m, relative to a rectangular coordinate system with origin at Earth's center (Figure 6.3). According to Newton's law of gravitation, Earth's gravitational force $\mathbf{F} = \mathbf{F}(x, y, z)$ on the object is inversely proportional to the square of the distance of the object from Earth's center, and directed toward the center; thus,

$$\mathbf{F} = \frac{K}{\|\mathbf{X}\|^2}\left(-\frac{\mathbf{X}}{\|\mathbf{X}\|}\right) = -K\frac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{(x^2 + y^2 + z^2)^{3/2}}, \quad (6.1.13)$$

where K is a constant. To determine K, we observe that the magnitude of $\mathbf{F}$ is

$$\|\mathbf{F}\| = K\frac{\|\mathbf{X}\|}{\|\mathbf{X}\|^3} = \frac{K}{\|\mathbf{X}\|^2} = \frac{K}{(x^2 + y^2 + z^2)}.$$

Figure 6.3

Let R be Earth's radius. Since $\|\mathbf{F}\| = mg$ when the object is at Earth's surface,

$$mg = \frac{K}{R^2}, \quad \text{so} \quad K = mgR^2.$$

Therefore we can rewrite (6.1.13) as

$$\mathbf{F} = -mgR^2 \frac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{(x^2 + y^2 + z^2)^{3/2}}.$$

Now suppose $\mathbf{F}$ is the only force acting on the object. According to Newton's second law of motion, $\mathbf{F} = m\mathbf{X}''$; that is,

$$m(x''\,\mathbf{i} + y''\,\mathbf{j} + z''\,\mathbf{k}) = -mgR^2 \frac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{(x^2 + y^2 + z^2)^{3/2}}.$$

Cancelling the common factor m and equating components on the two sides of this equation yields the system

$$\begin{aligned} x'' &= -\frac{gR^2 x}{(x^2 + y^2 + z^2)^{3/2}} \\ y'' &= -\frac{gR^2 y}{(x^2 + y^2 + z^2)^{3/2}} \\ z'' &= -\frac{gR^2 z}{(x^2 + y^2 + z^2)^{3/2}}. \end{aligned} \tag{6.1.14}$$

Rewriting Higher Order Systems as First Order Systems

A system of the form

$$\begin{aligned} y_1' &= g_1(t, y_1, y_2, \dots, y_n) \\ y_2' &= g_2(t, y_1, y_2, \dots, y_n) \\ &\vdots \\ y_n' &= g_n(t, y_1, y_2, \dots, y_n) \end{aligned} \tag{6.1.15}$$

is called a *first order system*, since the only derivatives occurring in it are first derivatives. The derivative of each of the unknowns may depend upon the independent variable and all the unknowns, but not on the derivatives of other unknowns. When we wish to emphasize the number of unknown functions in (6.1.15) we will say that (6.1.15) is an $n \times n$ system.

Systems involving higher order derivatives can often be reformulated as first order systems by introducing additional unknowns. The next two examples illustrate this.

Example 6.1.4 Rewrite the system

$$\begin{aligned} m_1 y_1'' &= -(c_1 + c_2)y_1' + c_2 y_2' - (k_1 + k_2)y_1 + k_2 y_2 + F_1 \\ m_2 y_2'' &= c_2 y_1' - c_2 y_2' + k_2 y_1 - k_2 y_2 + F_2. \end{aligned} \tag{6.1.16}$$

derived in Example 6.1.2 as a system of first order equations.

Solution If we define $v_1 = y_1'$ and $v_2 = y_2'$, then $v_1' = y_1''$ and $v_2' = y_2''$, so (6.1.16) becomes

$$\begin{aligned} m_1v_1' &= -(c_1+c_2)v_1+c_2v_2-(k_1+k_2)y_1+k_2y_2+F_1 \\ m_2v_2' &= c_2v_1-c_2v_2+k_2y_1-k_2y_2+F_2. \end{aligned}$$

Therefore $\{y_1, y_2, v_1, v_2\}$ satisfies the 4×4 first order system

$$\begin{aligned} y_1' &= v_1 \\ y_2' &= v_2 \\ v_1' &= \frac{1}{m_1}\left[-(c_1+c_2)v_1+c_2v_2-(k_1+k_2)y_1+k_2y_2+F_1\right] \\ v_2' &= \frac{1}{m_2}\left[c_2v_1-c_2v_2+k_2y_1-k_2y_2+F_2\right]. \end{aligned} \tag{6.1.17}$$

REMARK: The difference in form between (6.1.15) and (6.1.17), due to the way in which the unknowns are *denoted* in the two systems, isn't important; (6.1.17) is a first order system, in that each equation in (6.1.17) expresses the first derivative of one of the unknown functions in a way that does not involve derivatives of any of the other unknowns.

Example 6.1.5 Rewrite the system

$$\begin{aligned} x'' &= f(t,x,x',y,y',y'') \\ y''' &= g(t,x,x',y,y'y'') \end{aligned}$$

as a first order system.

Solution We regard x, x', y, y', and y'' as unknown functions, and rename them

$$x = x_1,\ x' = x_2, \quad y = y_1, \quad y' = y_2, \quad y'' = y_3.$$

These unknowns satisfy the system

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= f(t,x_1,x_2,y_1,y_2,y_3) \\ y_1' &= y_2 \\ y_2' &= y_3 \\ y_3' &= g(t,x_1,x_2,y_1,y_2,y_3). \end{aligned}$$

Rewriting Scalar Differential Equations as Systems

In this chapter we'll refer to differential equations involving only one unknown function as *scalar* differential equations. Scalar differential equations can be rewritten as systems of first order equations by the method illustrated in the next two examples.

Example 6.1.6 Rewrite the equation

$$y^{(4)} + 4y''' + 6y'' + 4y' + y = 0 \tag{6.1.18}$$

as a 4×4 first order system.

Solution We regard y, y', y'', and y''' as unknowns and rename them

$$y = y_1, \quad y' = y_2, \quad y'' = y_3, \quad \text{and} \quad y''' = y_4.$$

Then $y^{(4)} = y_4'$, so (6.1.18) can be written as

$$y_4' + 4y_4 + 6y_3 + 4y_2 + y_1 = 0.$$

Therefore $\{y_1, y_2, y_3, y_4\}$ satisfies the system

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ y_3' &= y_4 \\ y_4' &= -4y_4 - 6y_3 - 4y_2 - y_1. \ \blacksquare \end{aligned}$$

Example 6.1.7 Rewrite

$$x''' = f(t, x, x', x'')$$

as a system of first order equations.

Solution We regard x, x', and x'' as unknowns and rename them

$$x = y_1, \quad x' = y_2, \quad \text{and} \quad x'' = y_3.$$

Then

$$y_1' = x' = y_2, \quad y_2' = x'' = y_3, \quad \text{and} \quad y_3' = x'''.$$

Therefore $\{y_1, y_2, y_3\}$ satisfies the first order system

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ y_3' &= f(t, y_1, y_2, y_3). \end{aligned}$$

Since systems of differential equations involving higher derivatives can be rewritten as first order systems by the method used in Examples 6.1.5 –6.1.7 , we'll consider only first order systems.

Numerical Solution of Systems

The numerical methods that we studied in Chapter 3 can be extended to systems, and most differential equation software packages include programs to solve systems of equations. We won't go into detail on numerical methods for systems; however, for illustrative purposes we'll describe the Runge-Kutta method for the numerical solution of the initial value problem

$$\begin{aligned} y_1' &= g_1(t, y_1, y_2), \quad y_1(t_0) = y_{10}, \\ y_2' &= g_2(t, y_1, y_2), \quad y_2(t_0) = y_{20} \end{aligned}$$

at equally spaced points $t_0, t_1, \dots, t_n = b$ in an interval $[t_0, b]$. Thus,

$$t_i = t_0 + ih, \quad i = 0, 1, \dots, n,$$

where

$$h = \frac{b - t_0}{n}.$$

We'll denote the approximate values of y_1 and y_2 at these points by $y_{10}, y_{11}, \dots, y_{1n}$ and $y_{20}, y_{21}, \dots, y_{2n}$. The Runge-Kutta method computes these approximate values as follows: given y_{1i} and y_{2i}, compute

$$\begin{aligned} I_{1i} &= g_1(t_i, y_{1i}, y_{2i}), \\ J_{1i} &= g_2(t_i, y_{1i}, y_{2i}), \\ I_{2i} &= g_1\left(t_i + \frac{h}{2}, y_{1i} + \frac{h}{2}I_{1i}, y_{2i} + \frac{h}{2}J_{1i}\right), \\ J_{2i} &= g_2\left(t_i + \frac{h}{2}, y_{1i} + \frac{h}{2}I_{1i}, y_{2i} + \frac{h}{2}J_{1i}\right), \\ I_{3i} &= g_1\left(t_i + \frac{h}{2}, y_{1i} + \frac{h}{2}I_{2i}, y_{2i} + \frac{h}{2}J_{2i}\right), \\ J_{3i} &= g_2\left(t_i + \frac{h}{2}, y_{1i} + \frac{h}{2}I_{2i}, y_{2i} + \frac{h}{2}J_{2i}\right), \\ I_{4i} &= g_1(t_i + h, y_{1i} + hI_{3i}, y_{2i} + hJ_{3i}), \\ J_{4i} &= g_2(t_i + h, y_{1i} + hI_{3i}, y_{2i} + hJ_{3i}), \end{aligned}$$

and

$$\begin{aligned} y_{1,i+1} &= y_{1i} + \frac{h}{6}(I_{1i} + 2I_{2i} + 2I_{3i} + I_{4i}), \\ y_{2,i+1} &= y_{2i} + \frac{h}{6}(J_{1i} + 2J_{2i} + 2J_{3i} + J_{4i}) \end{aligned}$$

for $i = 0, \dots, n-1$. Under appropriate conditions on g_1 and g_2, it can be shown that the global truncation error for the Runge-Kutta method is $O(h^4)$, as in the scalar case considered in Section 3.3.

6.1 Exercises

1. Tanks T_1 and T_2 contain 50 gallons and 100 gallons of salt solutions, respectively. A solution with 2 pounds of salt per gallon is pumped into T_1 from an external source at 1 gal/min, and a solution with 3 pounds of salt per gallon is pumped into T_2 from an external source at 2 gal/min. The solution from T_1 is pumped into T_2 at 3 gal/min, and the solution from T_2 is pumped into T_1 at 4 gal/min. T_1 is drained at 2 gal/min and T_2 is drained at 1 gal/min. Let $Q_1(t)$ and $Q_2(t)$ be the number of pounds of salt in T_1 and T_2, respectively, at time $t > 0$. Derive a system of differential equations for Q_1 and Q_2. Assume that both mixtures are well stirred.

2. Two 500 gallon tanks T_1 and T_2 initially contain 100 gallons each of salt solution. A solution with 2 pounds of salt per gallon is pumped into T_1 from an external source at 6 gal/min, and a solution with 1 pound of salt per gallon is pumped into T_2 from an external source at 5 gal/min. The solution from T_1 is pumped into T_2 at 2 gal/min, and the solution from T_2 is pumped into T_1 at 1 gal/min. Both tanks are drained at 3 gal/min. Let $Q_1(t)$ and $Q_2(t)$ be the number of pounds of salt in T_1 and T_2, respectively, at time $t > 0$. Derive a system of differential equations for Q_1 and Q_2 that's valid until a tank is about to overflow. Assume that both mixtures are well stirred.

3. A mass m_1 is suspended from a rigid support on a spring S_1 with spring constant k_1 and damping constant c_1. A second mass m_2 is suspended from the first on a spring S_2 with spring constant k_2 and damping constant c_2, and a third mass m_3 is suspended from the second on a spring S_3 with spring constant k_3 and damping constant c_3. Let $y_1 = y_1(t)$, $y_2 = y_2(t)$, and $y_3 = y_3(t)$ be the displacements of the three masses from their equilibrium positions at time t, measured positive upward. Derive a system of differential equations for y_1, y_2 and y_3, assuming that the masses of the springs are negligible and that vertical external forces F_1, F_2, and F_3 also act on the masses.

4. Let $\mathbf{X} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ be the position vector of an object with mass m, expressed in terms of a rectangular coordinate system with origin at Earth's center (Figure 6.3). Derive a system of differential equations for x, y, and z, assuming that the object moves under Earth's gravitational force (given by Newton's law of gravitation, as in Example 6.1.3) and a resistive force proportional to the speed of the object. Let α be the constant of proportionality.

5. Rewrite the given system as a first order system.

(a) $$\begin{aligned} x''' &= f(t,x,y,y') \\ y'' &= g(t,y,y') \end{aligned}$$

(b) $$\begin{aligned} u' &= f(t,u,v,v',w') \\ v'' &= g(t,u,v,v',w) \\ w'' &= h(t,u,v,v',w,w') \end{aligned}$$

(c) $y''' = f(t,y,y',y'')$

(d) $y^{(4)} = f(t,y)$

(e) $$\begin{aligned} x'' &= f(t,x,y) \\ y'' &= g(t,x,y) \end{aligned}$$

6. Rewrite the system (6.1.14) of differential equations derived in Example 6.1.3 as a first order system.

7. Formulate a version of Euler's method (Section 3.1) for the numerical solution of the initial value problem

$$\begin{aligned} y_1' &= g_1(t,y_1,y_2), & y_1(t_0) &= y_{10}, \\ y_2' &= g_2(t,y_1,y_2), & y_2(t_0) &= y_{20}, \end{aligned}$$

on an interval $[t_0,b]$.

8. Formulate a version of the improved Euler method (Section 3.2) for the numerical solution of the initial value problem

$$\begin{aligned} y_1' &= g_1(t,y_1,y_2), & y_1(t_0) &= y_{10}, \\ y_2' &= g_2(t,y_1,y_2), & y_2(t_0) &= y_{20}, \end{aligned}$$

on an interval $[t_0,b]$.

6.2 LINEAR SYSTEMS OF DIFFERENTIAL EQUATIONS

A first order system of differential equations that can be written in the form

$$\begin{aligned} y_1' &= a_{11}(t)y_1 + a_{12}(t)y_2 + \cdots + a_{1n}(t)y_n + f_1(t) \\ y_2' &= a_{21}(t)y_1 + a_{22}(t)y_2 + \cdots + a_{2n}(t)y_n + f_2(t) \\ &\vdots \\ y_n' &= a_{n1}(t)y_1 + a_{n2}(t)y_2 + \cdots + a_{nn}(t)y_n + f_n(t) \end{aligned} \tag{6.2.1}$$

is called a *linear system*.

The linear system (6.2.1) can be written in matrix form as

$$\begin{bmatrix} y_1' \\ y_2' \\ \vdots \\ y_n' \end{bmatrix} = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} + \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{bmatrix},$$

or more briefly as

$$\mathbf{y}' = A(t)\mathbf{y} + \mathbf{f}(t), \tag{6.2.2}$$

where

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \quad A(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{bmatrix}, \quad \text{and} \quad \mathbf{f}(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{bmatrix}.$$

We call A the *coefficient matrix* of (6.2.2) and $\mathbf{f}$ the *forcing function*. We'll say that A and $\mathbf{f}$ are *continuous* if their entries are continuous. If $\mathbf{f} = \mathbf{0}$, then (6.2.2) is *homogeneous*; otherwise, (6.2.2) is *nonhomogeneous*.

An initial value problem for (6.2.2) consists of finding a solution of (6.2.2) that equals a given constant vector

$$\mathbf{k} = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix}.$$

at some initial point t_0. We write this initial value problem as

$$\mathbf{y}' = A(t)\mathbf{y} + \mathbf{f}(t), \quad \mathbf{y}(t_0) = \mathbf{k}.$$

The next theorem gives sufficient conditions for the existence of solutions of initial value problems for (6.2.2). We omit the proof.

Theorem 6.2.1 *Suppose the coefficient matrix A and the forcing function $\mathbf{f}$ are continuous on (a, b), let t_0 be in (a, b), and let $\mathbf{k}$ be an arbitrary constant n-vector. Then the initial value problem*

$$\mathbf{y}' = A(t)\mathbf{y} + \mathbf{f}(t), \quad \mathbf{y}(t_0) = \mathbf{k}$$

has a unique solution on (a, b).

Example 6.2.1

(a) Write the system

$$\begin{aligned} y_1' &= y_1 + 2y_2 + 2e^{4t} \\ y_2' &= 2y_1 + y_2 + e^{4t} \end{aligned} \tag{6.2.3}$$

in matrix form and conclude from Theorem 6.2.1 that every initial value problem for (6.2.3) has a unique solution on $(-\infty, \infty)$.

(b) Verify that

$$\mathbf{y} = \frac{1}{5}\begin{bmatrix} 8 \\ 7 \end{bmatrix} e^{4t} + c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} \tag{6.2.4}$$

is a solution of (6.2.3) for all values of the constants c_1 and c_2.

(c) Find the solution of the initial value problem

$$\mathbf{y}' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{4t}, \quad \mathbf{y}(0) = \frac{1}{5} \begin{bmatrix} 3 \\ 22 \end{bmatrix}. \tag{6.2.5}$$

SOLUTION(**a**) The system (6.2.3) can be written in matrix form as

$$\mathbf{y}' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{4t}.$$

An initial value problem for (6.2.3) can be written as

$$\mathbf{y}' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{4t}, \quad y(t_0) = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}.$$

Since the coefficient matrix and the forcing function are both continuous on $(-\infty, \infty)$, Theorem 6.2.1 implies that this problem has a unique solution on $(-\infty, \infty)$.

SOLUTION(**b**) If $\mathbf{y}$ is given by (6.2.4), then

$$\begin{aligned} A\mathbf{y} + \mathbf{f} &= \frac{1}{5} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 8 \\ 7 \end{bmatrix} e^{4t} + c_1 \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} \\ &\quad + c_2 \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{4t} \\ &= \frac{1}{5} \begin{bmatrix} 22 \\ 23 \end{bmatrix} e^{4t} + c_1 \begin{bmatrix} 3 \\ 3 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{4t} \\ &= \frac{1}{5} \begin{bmatrix} 32 \\ 28 \end{bmatrix} e^{4t} + 3c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} - c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} = \mathbf{y}'. \end{aligned}$$

SOLUTION(**c**) We must choose c_1 and c_2 in (6.2.4) so that

$$\frac{1}{5} \begin{bmatrix} 8 \\ 7 \end{bmatrix} + c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 3 \\ 22 \end{bmatrix},$$

which is equivalent to

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}.$$

Solving this system yields $c_1 = 1$, $c_2 = -2$, so

$$\mathbf{y} = \frac{1}{5} \begin{bmatrix} 8 \\ 7 \end{bmatrix} e^{4t} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} - 2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}$$

is the solution of (6.2.5).

REMARK: The theory of $n \times n$ linear systems of differential equations is analogous to the theory of the scalar n-th order equation

$$P_0(t)y^{(n)} + P_1(t)y^{(n-1)} + \cdots + P_n(t)y = F(t), \tag{6.2.6}$$

as developed in Sections 9.1. For example, by rewriting (6.2.6) as an equivalent linear system it can be shown that Theorem 6.2.1 implies Theorem **??** (Exercise 12).

6.2 Exercises

1. Rewrite the system in matrix form and verify that the given vector function satisfies the system for any choice of the constants c_1 and c_2.

(a) $\begin{aligned} y_1' &= 2y_1 + 4y_2 \\ y_2' &= 4y_1 + 2y_2; \end{aligned} \qquad \mathbf{y} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{6t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2t}$

(b) $\begin{aligned} y_1' &= -2y_1 - 2y_2 \\ y_2' &= -5y_1 + y_2; \end{aligned} \qquad \mathbf{y} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-4t} + c_2 \begin{bmatrix} -2 \\ 5 \end{bmatrix} e^{3t}$

(c) $\begin{aligned} y_1' &= -4y_1 - 10y_2 \\ y_2' &= 3y_1 + 7y_2; \end{aligned} \qquad \mathbf{y} = c_1 \begin{bmatrix} -5 \\ 3 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 2 \\ -1 \end{bmatrix} e^{t}$

(d) $\begin{aligned} y_1' &= 2y_1 + y_2 \\ y_2' &= y_1 + 2y_2; \end{aligned} \qquad \mathbf{y} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{t}$

2. Rewrite the system in matrix form and verify that the given vector function satisfies the system for any choice of the constants c_1, c_2, and c_3.

(a)
$$\begin{aligned} y_1' &= -y_1 + 2y_2 + 3y_3 \\ y_2' &= y_2 + 6y_3 \\ y_3' &= -2y_3; \end{aligned}$$
$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{-t} + c_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} e^{-2t}$$

(b)
$$\begin{aligned} y_1' &= 2y_2 + 2y_3 \\ y_2' &= 2y_1 + 2y_3 \\ y_3' &= 2y_1 + 2y_2; \end{aligned}$$
$$\mathbf{y} = c_1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} e^{-2t} + c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{4t}$$

(c)
$$\begin{aligned} y_1' &= -y_1 + 2y_2 + 2y_3 \\ y_2' &= 2y_1 - y_2 + 2y_3 \\ y_3' &= 2y_1 + 2y_2 - y_3; \end{aligned}$$
$$\mathbf{y} = c_1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} e^{-3t} + c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{3t}$$

(d)
$$\begin{aligned} y_1' &= 3y_1 - y_2 - y_3 \\ y_2' &= -2y_1 + 3y_2 + 2y_3 \\ y_3' &= 4y_1 - y_2 - 2y_3; \end{aligned}$$
$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} e^{3t} + c_3 \begin{bmatrix} 1 \\ -3 \\ 7 \end{bmatrix} e^{-t}$$

3. Rewrite the initial value problem in matrix form and verify that the given vector function is a solution.

(a) $$\begin{aligned} y_1' &= y_1 + y_2 & y_1(0) &= 1 \\ y_2' &= -2y_1 + 4y_2, & y_2(0) &= 0; \end{aligned} \qquad \mathbf{y} = 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} - \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t}$$

(b) $$\begin{aligned} y_1' &= 5y_1 + 3y_2 & y_1(0) &= 12 \\ y_2' &= -y_1 + y_2, & y_2(0) &= -6; \end{aligned} \qquad \mathbf{y} = 3\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t} + 3\begin{bmatrix} 3 \\ -1 \end{bmatrix} e^{4t}$$

4. Rewrite the initial value problem in matrix form and verify that the given vector function is a solution.

(a) $$\begin{aligned} y_1' &= 6y_1 + 4y_2 + 4y_3 & y_1(0) &= 3 \\ y_2' &= -7y_1 - 2y_2 - y_3, , & y_2(0) &= -6 \\ y_3' &= 7y_1 + 4y_2 + 3y_3 & y_3(0) &= 4 \end{aligned}$$

$$\mathbf{y} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} e^{6t} + 2\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} e^{2t} + \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} e^{-t}$$

(b) $$\begin{aligned} y_1' &= 8y_1 + 7y_2 + 7y_3 & y_1(0) &= 2 \\ y_2' &= -5y_1 - 6y_2 - 9y_3, & y_2(0) &= -4 \\ y_3' &= 5y_1 + 7y_2 + 10y_3, & y_3(0) &= 3 \end{aligned}$$

$$\mathbf{y} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} e^{8t} + \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} e^{3t} + \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} e^{t}$$

5. Rewrite the system in matrix form and verify that the given vector function satisfies the system for any choice of the constants c_1 and c_2.

(a) $$\begin{aligned} y_1' &= -3y_1 + 2y_2 + 3 - 2t \\ y_2' &= -5y_1 + 3y_2 + 6 - 3t \end{aligned}$$

$$\mathbf{y} = c_1\begin{bmatrix} 2\cos t \\ 3\cos t - \sin t \end{bmatrix} + c_2\begin{bmatrix} 2\sin t \\ 3\sin t + \cos t \end{bmatrix} + \begin{bmatrix} 1 \\ t \end{bmatrix}$$

(b) $$\begin{aligned} y_1' &= 3y_1 + y_2 - 5e^t \\ y_2' &= -y_1 + y_2 + e^t \end{aligned}$$

$$\mathbf{y} = c_1\begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{2t} + c_2\begin{bmatrix} 1+t \\ -t \end{bmatrix} e^{2t} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{t}$$

(c) $$\begin{aligned} y_1' &= -y_1 - 4y_2 + 4e^t + 8te^t \\ y_2' &= -y_1 - y_2 + e^{3t} + (4t+2)e^t \end{aligned}$$

$$\mathbf{y} = c_1\begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-3t} + c_2\begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{t} + \begin{bmatrix} e^{3t} \\ 2te^t \end{bmatrix}$$

(d) $$\begin{aligned} y_1' &= -6y_1 - 3y_2 + 14e^{2t} + 12e^t \\ y_2' &= y_1 - 2y_2 + 7e^{2t} - 12e^t \end{aligned}$$

$$\mathbf{y} = c_1\begin{bmatrix} -3 \\ 1 \end{bmatrix} e^{-5t} + c_2\begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-3t} + \begin{bmatrix} e^{2t} + 3e^t \\ 2e^{2t} - 3e^t \end{bmatrix}$$

6. Convert the linear scalar equation

$$P_0(t)y^{(n)} + P_1(t)y^{(n-1)} + \cdots + P_n(t)y(t) = F(t) \qquad \text{(A)}$$

into an equivalent $n \times n$ system

$$\mathbf{y}' = A(t)\mathbf{y} + \mathbf{f}(t),$$

and show that A and $\mathbf{f}$ are continuous on an interval (a, b) if and only if (A) is normal on (a, b).

7. A matrix function

$$Q(t) = \begin{bmatrix} q_{11}(t) & q_{12}(t) & \cdots & q_{1s}(t) \\ q_{21}(t) & q_{22}(t) & \cdots & q_{2s}(t) \\ \vdots & \vdots & \ddots & \vdots \\ q_{r1}(t) & q_{r2}(t) & \cdots & q_{rs}(t) \end{bmatrix}$$

is said to be *differentiable* if its entries $\{q_{ij}\}$ are differentiable. Then the *derivative* Q' is defined by

$$Q'(t) = \begin{bmatrix} q_{11}'(t) & q_{12}'(t) & \cdots & q_{1s}'(t) \\ q_{21}'(t) & q_{22}'(t) & \cdots & q_{2s}'(t) \\ \vdots & \vdots & \ddots & \vdots \\ q_{r1}'(t) & q_{r2}'(t) & \cdots & q_{rs}'(t) \end{bmatrix}.$$

(a) Prove: If P and Q are differentiable matrices such that $P + Q$ is defined and if c_1 and c_2 are constants, then

$$(c_1P + c_2Q)' = c_1P' + c_2Q'.$$

(b) Prove: If P and Q are differentiable matrices such that PQ is defined, then

$$(PQ)' = P'Q + PQ'.$$

8. Verify that $Y' = AY$.

(a) $Y = \begin{bmatrix} e^{6t} & e^{-2t} \\ e^{6t} & -e^{-2t} \end{bmatrix}, \quad A = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}$

(b) $Y = \begin{bmatrix} e^{-4t} & -2e^{3t} \\ e^{-4t} & 5e^{3t} \end{bmatrix}, \quad A = \begin{bmatrix} -2 & -2 \\ -5 & 1 \end{bmatrix}$

(c) $Y = \begin{bmatrix} -5e^{2t} & 2e^{t} \\ 3e^{2t} & -e^{t} \end{bmatrix}, \quad A = \begin{bmatrix} -4 & -10 \\ 3 & 7 \end{bmatrix}$

(d) $Y = \begin{bmatrix} e^{3t} & e^{t} \\ e^{3t} & -e^{t} \end{bmatrix}, \quad A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

(e) $Y = \begin{bmatrix} e^{t} & e^{-t} & e^{-2t} \\ e^{t} & 0 & -2e^{-2t} \\ 0 & 0 & e^{-2t} \end{bmatrix}, \quad A = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 1 & 6 \\ 0 & 0 & -2 \end{bmatrix}$

(f) $Y = \begin{bmatrix} -e^{-2t} & -e^{-2t} & e^{4t} \\ 0 & e^{-2t} & e^{4t} \\ e^{-2t} & 0 & e^{4t} \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix}$

(g) $Y = \begin{bmatrix} e^{3t} & e^{-3t} & 0 \\ e^{3t} & 0 & -e^{-3t} \\ e^{3t} & e^{-3t} & e^{-3t} \end{bmatrix}, \quad A = \begin{bmatrix} -9 & 6 & 6 \\ -6 & 3 & 6 \\ -6 & 6 & 3 \end{bmatrix}$

(h) $Y = \begin{bmatrix} e^{2t} & e^{3t} & e^{-t} \\ 0 & -e^{3t} & -3e^{-t} \\ e^{2t} & e^{3t} & 7e^{-t} \end{bmatrix}, \quad A = \begin{bmatrix} 3 & -1 & -1 \\ -2 & 3 & 2 \\ 4 & -1 & -2 \end{bmatrix}$

9. Suppose

$$\mathbf{y}_1 = \begin{bmatrix} y_{11} \\ y_{21} \end{bmatrix} \quad \text{and} \quad \mathbf{y}_2 = \begin{bmatrix} y_{12} \\ y_{22} \end{bmatrix}$$

are solutions of the homogeneous system

$$\mathbf{y}' = A(t)\mathbf{y}, \tag{A}$$

and define

$$Y = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix}.$$

(a) Show that $Y' = AY$.

(b) Show that if $\mathbf{c}$ is a constant vector then $\mathbf{y} = Y\mathbf{c}$ is a solution of (A).

(c) State generalizations of **(a)** and **(b)** for $n \times n$ systems.

10. Suppose Y is a differentiable square matrix.

(a) Find a formula for the derivative of Y^2.

(b) Find a formula for the derivative of Y^n, where n is any positive integer.

(c) State how the results obtained in **(a)** and **(b)** are analogous to results from calculus concerning scalar functions.

11. It can be shown that if Y is a differentiable and invertible square matrix function, then Y^{-1} is differentiable.

(a) Show that $(Y^{-1})' = -Y^{-1}Y'Y^{-1}$. (Hint: Differentiate the identity $Y^{-1}Y = I$.)

(b) Find the derivative of $Y^{-n} = \left(Y^{-1}\right)^n$, where n is a positive integer.

(c) State how the results obtained in **(a)** and **(b)** are analogous to results from calculus concerning scalar functions.

12. Show that Theorem 6.2.1 implies Theorem **??**. HINT: *Write the scalar equation*

$$P_0(x)y^{(n)} + P_1(x)y^{(n-1)} + \cdots + P_n(x)y = F(x)$$

as an $n \times n$ system of linear equations.

13. Suppose $\mathbf{y}$ is a solution of the $n \times n$ system $\mathbf{y}' = A(t)\mathbf{y}$ on (a, b), and that the $n \times n$ matrix P is invertible and differentiable on (a, b). Find a matrix B such that the function $\mathbf{x} = P\mathbf{y}$ is a solution of $\mathbf{x}' = B\mathbf{x}$ on (a, b).

6.3 BASIC THEORY OF HOMOGENEOUS LINEAR SYSTEMS

In this section we consider homogeneous linear systems $\mathbf{y}' = A(t)\mathbf{y}$, where $A = A(t)$ is a continuous $n \times n$ matrix function on an interval (a, b). The theory of linear homogeneous systems has much in common with the theory of linear homogeneous scalar equations, which we considered in Sections 2.1, 5.1, and 9.1.

Whenever we refer to solutions of $\mathbf{y}' = A(t)\mathbf{y}$ we'll mean solutions on (a, b). Since $\mathbf{y} \equiv \mathbf{0}$ is obviously a solution of $\mathbf{y}' = A(t)\mathbf{y}$, we call it the *trivial* solution. Any other solution is *nontrivial*.

If $\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n$ are vector functions defined on an interval (a, b) and $c_1, c_2, \dots, c_n$ are constants, then

$$\mathbf{y} = c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + \cdots + c_n\mathbf{y}_n \tag{6.3.1}$$

is a *linear combination of* $\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n$. It's easy show that if $\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n$ are solutions of $\mathbf{y}' = A(t)\mathbf{y}$ on (a, b), then so is any linear combination of $\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n$ (Exercise 1). We say that $\{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n\}$ is a *fundamental set of solutions of* $\mathbf{y}' = A(t)\mathbf{y}$ *on* (a, b) on if every solution of $\mathbf{y}' = A(t)\mathbf{y}$ on (a, b) can be written as a linear combination of $\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n$, as in (6.3.1). In this case we say that (6.3.1) is the *general solution of* $\mathbf{y}' = A(t)\mathbf{y}$ *on* (a, b).

It can be shown that if A is continuous on (a, b) then $\mathbf{y}' = A(t)\mathbf{y}$ has infinitely many fundamental sets of solutions on (a, b) (Exercises 15 and 16). The next definition will help to characterize fundamental sets of solutions of $\mathbf{y}' = A(t)\mathbf{y}$.

We say that a set $\{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n\}$ of n-vector functions is *linearly independent* on (a, b) if the only constants $c_1, c_2, \dots, c_n$ such that

$$c_1\mathbf{y}_1(t) + c_2\mathbf{y}_2(t) + \cdots + c_n\mathbf{y}_n(t) = 0, \quad a < t < b, \tag{6.3.2}$$

are $c_1 = c_2 = \cdots = c_n = 0$. If (6.3.2) holds for some set of constants $c_1, c_2, \dots, c_n$ that are not all zero, then $\{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n\}$ is *linearly dependent* on (a, b)

The next theorem is analogous to Theorems **??** and **??**.

Theorem 6.3.1 *Suppose the $n \times n$ matrix $A = A(t)$ is continuous on (a, b). Then a set $\{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n\}$ of n solutions of $\mathbf{y}' = A(t)\mathbf{y}$ on (a, b) is a fundamental set if and only if it's linearly independent on (a, b).*

Example 6.3.1 Show that the vector functions

$$\mathbf{y}_1 = \begin{bmatrix} e^t \\ 0 \\ e^{-t} \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} 0 \\ e^{3t} \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{y}_3 = \begin{bmatrix} e^{2t} \\ e^{3t} \\ 0 \end{bmatrix}$$

are linearly independent on every interval (a, b).

Solution Suppose

$$c_1 \begin{bmatrix} e^t \\ 0 \\ e^{-t} \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ e^{3t} \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} e^{2t} \\ e^{3t} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad a < t < b.$$

We must show that $c_1 = c_2 = c_3 = 0$. Rewriting this equation in matrix form yields

$$\begin{bmatrix} e^t & 0 & e^{2t} \\ 0 & e^{3t} & e^{3t} \\ e^{-t} & 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad a < t < b.$$

Expanding the determinant of this system in cofactors of the entries of the first row yields

$$\begin{aligned} \begin{vmatrix} e^t & 0 & e^{2t} \\ 0 & e^{3t} & e^{3t} \\ e^{-t} & 1 & 0 \end{vmatrix} &= e^t \begin{vmatrix} e^{3t} & e^{3t} \\ 1 & 0 \end{vmatrix} - 0 \begin{vmatrix} 0 & e^{3t} \\ e^{-t} & 0 \end{vmatrix} + e^{2t} \begin{vmatrix} 0 & e^{3t} \\ e^{-t} & 1 \end{vmatrix} \\ &= e^t(-e^{3t}) + e^{2t}(-e^{2t}) = -2e^{4t}. \end{aligned}$$

Since this determinant is never zero, $c_1 = c_2 = c_3 = 0$. ■

We can use the method in Example 6.3.1 to test n solutions $\{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n\}$ of any $n \times n$ system $\mathbf{y}' = A(t)\mathbf{y}$ for linear independence on an interval (a, b) on which A is continuous. To explain this (and for other purposes later), it's useful to write a linear combination of $\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n$ in a different way. We first write the vector functions in terms of their components as

$$\mathbf{y}_1 = \begin{bmatrix} y_{11} \\ y_{21} \\ \vdots \\ y_{n1} \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} y_{12} \\ y_{22} \\ \vdots \\ y_{n2} \end{bmatrix}, \dots, \quad \mathbf{y}_n = \begin{bmatrix} y_{1n} \\ y_{2n} \\ \vdots \\ y_{nn} \end{bmatrix}.$$

If

$$\mathbf{y} = c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + \cdots + c_n\mathbf{y}_n$$

then

$$\begin{aligned} \mathbf{y} &= c_1 \begin{bmatrix} y_{11} \\ y_{21} \\ \vdots \\ y_{n1} \end{bmatrix} + c_2 \begin{bmatrix} y_{12} \\ y_{22} \\ \vdots \\ y_{n2} \end{bmatrix} + \cdots + c_n \begin{bmatrix} y_{1n} \\ y_{2n} \\ \vdots \\ y_{nn} \end{bmatrix} \\ &= \begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}. \end{aligned}$$

This shows that

$$c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + \cdots + c_n\mathbf{y}_n = Y\mathbf{c}, \tag{6.3.3}$$

where

$$\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

and

$$Y = [\mathbf{y}_1\ \mathbf{y}_2\ \cdots\ \mathbf{y}_n] = \begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{bmatrix}; \tag{6.3.4}$$

that is, the columns of Y are the vector functions $\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n$.

For reference below, note that

$$\begin{aligned} Y' &= [\mathbf{y}_1'\ \mathbf{y}_2'\ \cdots\ \mathbf{y}_n'] \\ &= [A\mathbf{y}_1\ A\mathbf{y}_2\ \cdots\ A\mathbf{y}_n] \\ &= A[\mathbf{y}_1\ \mathbf{y}_2\ \cdots\ \mathbf{y}_n] = AY; \end{aligned}$$

that is, Y satisfies the matrix differential equation

$$Y' = AY.$$

The determinant of Y,

$$W = \begin{vmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{vmatrix} \tag{6.3.5}$$

is called the *Wronskian* of $\{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n\}$. It can be shown (Exercises 2 and 3) that this definition is analogous to definitions of the Wronskian of scalar functions given in Sections 5.1 and 9.1. The next theorem is analogous to Theorems **??** and **??**. The proof is sketched in Exercise 4 for $n = 2$ and in Exercise 5 for general n.

Theorem 6.3.2 *[Abel's Formula] Suppose the $n \times n$ matrix $A = A(t)$ is continuous on (a, b), let $\mathbf{y}_1$, $\mathbf{y}_2$, ..., $\mathbf{y}_n$ be solutions of $\mathbf{y}' = A(t)\mathbf{y}$ on (a, b), and let t_0 be in (a, b). Then the Wronskian of $\{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n\}$ is given by*

$$W(t) = W(t_0) \exp\left(\int_{t_0}^{t} [a_{11}(s) + a_{22}(s) + \cdots + a_{nn}(s)]\, ds \right), \quad a < t < b. \tag{6.3.6}$$

Therefore, either W has no zeros in (a, b) or $W \equiv 0$ on (a, b).

REMARK: The sum of the diagonal entries of a square matrix A is called the *trace* of A, denoted by $\mathrm{tr}(A)$. Thus, for an $n \times n$ matrix A,

$$\mathrm{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn},$$

and (6.3.6) can be written as

$$W(t) = W(t_0) \exp\left(\int_{t_0}^{t} \mathrm{tr}(A(s))\, ds \right), \quad a < t < b.$$

The next theorem is analogous to Theorems **??** and **??**.

Theorem 6.3.3 *Suppose the $n \times n$ matrix $A = A(t)$ is continuous on (a, b) and let $\mathbf{y}_1$, $\mathbf{y}_2$, …,$\mathbf{y}_n$ be solutions of $\mathbf{y}' = A(t)\mathbf{y}$ on (a, b). Then the following statements are equivalent; that is, they are either all true or all false:*

(a) *The general solution of $\mathbf{y}' = A(t)\mathbf{y}$ on (a, b) is $\mathbf{y} = c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + \cdots + c_n\mathbf{y}_n$, where c_1, c_2, …, c_n are arbitrary constants.*

(b) *$\{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n\}$ is a fundamental set of solutions of $\mathbf{y}' = A(t)\mathbf{y}$ on (a, b).*

(c) *$\{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n\}$ is linearly independent on (a, b).*

(d) *The Wronskian of $\{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n\}$ is nonzero at some point in (a, b).*

(e) *The Wronskian of $\{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n\}$ is nonzero at all points in (a, b).*

We say that Y in (6.3.4) is a *fundamental matrix* for $\mathbf{y}' = A(t)\mathbf{y}$ if any (and therefore all) of the statements (**a**)-(**e**) of Theorem 6.3.2 are true for the columns of Y. In this case, (6.3.3) implies that the general solution of $\mathbf{y}' = A(t)\mathbf{y}$ can be written as $\mathbf{y} = Y\mathbf{c}$, where $\mathbf{c}$ is an arbitrary constant n-vector.

Example 6.3.2 The vector functions

$$\mathbf{y}_1 = \begin{bmatrix} -e^{2t} \\ 2e^{2t} \end{bmatrix} \quad \text{and} \quad \mathbf{y}_2 = \begin{bmatrix} -e^{-t} \\ e^{-t} \end{bmatrix}$$

are solutions of the constant coefficient system

$$\mathbf{y}' = \begin{bmatrix} -4 & -3 \\ 6 & 5 \end{bmatrix} \mathbf{y} \tag{6.3.7}$$

on $(-\infty, \infty)$. (Verify.)

(a) Compute the Wronskian of $\{\mathbf{y}_1, \mathbf{y}_2\}$ directly from the definition (6.3.5)

(b) Verify Abel's formula (6.3.6) for the Wronskian of $\{\mathbf{y}_1, \mathbf{y}_2\}$.

(c) Find the general solution of (6.3.7).

(d) Solve the initial value problem

$$\mathbf{y}' = \begin{bmatrix} -4 & -3 \\ 6 & 5 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 4 \\ -5 \end{bmatrix}. \tag{6.3.8}$$

SOLUTION(**a**) From (6.3.5)

$$W(t) = \begin{vmatrix} -e^{2t} & -e^{-t} \\ 2e^{2t} & e^{-t} \end{vmatrix} = e^{2t}e^{-t} \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix} = e^t. \tag{6.3.9}$$

SOLUTION(**b**) Here

$$A = \begin{bmatrix} -4 & -3 \\ 6 & 5 \end{bmatrix},$$

so $\text{tr}(A) = -4 + 5 = 1$. If t_0 is an arbitrary real number then (6.3.6) implies that

$$W(t) = W(t_0)\exp\left(\int_{t_0}^{t} 1\,ds\right) = \begin{vmatrix} -e^{2t_0} & -e^{-t_0} \\ 2e^{2t_0} & e^{-t_0} \end{vmatrix} e^{(t-t_0)} = e^{t_0}e^{t-t_0} = e^{t},$$

which is consistent with (6.3.9).

SOLUTION(c) Since $W(t) \neq 0$, Theorem 6.3.3 implies that $\{\mathbf{y}_1, \mathbf{y}_2\}$ is a fundamental set of solutions of (6.3.7) and

$$Y = \begin{bmatrix} -e^{2t} & -e^{-t} \\ 2e^{2t} & e^{-t} \end{bmatrix}$$

is a fundamental matrix for (6.3.7). Therefore the general solution of (6.3.7) is

$$\mathbf{y} = c_1\mathbf{y}_1 + c_2\mathbf{y}_2 = c_1\begin{bmatrix} -e^{2t} \\ 2e^{2t} \end{bmatrix} + c_2\begin{bmatrix} -e^{-t} \\ e^{-t} \end{bmatrix} = \begin{bmatrix} -e^{2t} & -e^{-t} \\ 2e^{2t} & e^{-t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}. \qquad (6.3.10)$$

SOLUTION(d) Setting $t = 0$ in (6.3.10) and imposing the initial condition in (6.3.8) yields

$$c_1\begin{bmatrix} -1 \\ 2 \end{bmatrix} + c_2\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -5 \end{bmatrix}.$$

Thus,

$$\begin{aligned} -c_1 - c_2 &= 4 \\ 2c_1 + c_2 &= -5. \end{aligned}$$

The solution of this system is $c_1 = -1$, $c_2 = -3$. Substituting these values into (6.3.10) yields

$$\mathbf{y} = -\begin{bmatrix} -e^{2t} \\ 2e^{2t} \end{bmatrix} - 3\begin{bmatrix} -e^{-t} \\ e^{-t} \end{bmatrix} = \begin{bmatrix} e^{2t} + 3e^{-t} \\ -2e^{2t} - 3e^{-t} \end{bmatrix}$$

as the solution of (6.3.8).

6.3 Exercises

1. Prove: If $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n$ are solutions of $\mathbf{y}' = A(t)\mathbf{y}$ on (a, b), then any linear combination of $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n$ is also a solution of $\mathbf{y}' = A(t)\mathbf{y}$ on (a, b).

2. In Section 5.1 the Wronskian of two solutions y_1 and y_2 of the scalar second order equation

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0 \qquad (A)$$

was defined to be

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}.$$

(a) Rewrite (A) as a system of first order equations and show that W is the Wronskian (as defined in this section) of two solutions of this system.

(b) Apply Eqn. (6.3.6) to the system derived in **(a)**, and show that

$$W(x) = W(x_0)\exp\left\{-\int_{x_0}^{x} \frac{P_1(s)}{P_0(s)}\,ds\right\},$$

which is the form of Abel's formula given in Theorem 9.1.3.

3. In Section 9.1 the Wronskian of n solutions $y_1, y_2, \dots, y_n$ of the n–th order equation

$$P_0(x)y^{(n)} + P_1(x)y^{(n-1)} + \cdots + P_n(x)y = 0 \tag{A}$$

was defined to be

$$W = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}.$$

(a) Rewrite (A) as a system of first order equations and show that W is the Wronskian (as defined in this section) of n solutions of this system.

(b) Apply Eqn. (6.3.6) to the system derived in **(a)**, and show that

$$W(x) = W(x_0) \exp\left\{-\int_{x_0}^{x} \frac{P_1(s)}{P_0(s)}\, ds\right\},$$

which is the form of Abel's formula given in Theorem 9.1.3.

4. Suppose

$$\mathbf{y}_1 = \begin{bmatrix} y_{11} \\ y_{21} \end{bmatrix} \quad \text{and} \quad \mathbf{y}_2 = \begin{bmatrix} y_{12} \\ y_{22} \end{bmatrix}$$

are solutions of the 2×2 system $\mathbf{y}' = A\mathbf{y}$ on (a, b), and let

$$Y = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \quad \text{and} \quad W = \begin{vmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{vmatrix};$$

thus, W is the Wronskian of $\{\mathbf{y}_1, \mathbf{y}_2\}$.

(a) Deduce from the definition of determinant that

$$W' = \begin{vmatrix} y_{11}' & y_{12}' \\ y_{21} & y_{22} \end{vmatrix} + \begin{vmatrix} y_{11} & y_{12} \\ y_{21}' & y_{22}' \end{vmatrix}.$$

(b) Use the equation $Y' = A(t)Y$ and the definition of matrix multiplication to show that

$$[y_{11}' \quad y_{12}'] = a_{11}[y_{11} \quad y_{12}] + a_{12}[y_{21} \quad y_{22}]$$

and

$$[y_{21}' \quad y_{22}'] = a_{21}[y_{11} \quad y_{12}] + a_{22}[y_{21} \quad y_{22}].$$

(c) Use properties of determinants to deduce from **(a)** and **(a)** that

$$\begin{vmatrix} y_{11}' & y_{12}' \\ y_{21} & y_{22} \end{vmatrix} = a_{11}W \quad \text{and} \quad \begin{vmatrix} y_{11} & y_{12} \\ y_{21}' & y_{22}' \end{vmatrix} = a_{22}W.$$

(d) Conclude from **(c)** that

$$W' = (a_{11} + a_{22})W,$$

and use this to show that if $a < t_0 < b$ then

$$W(t) = W(t_0)\exp\left(\int_{t_0}^{t} [a_{11}(s) + a_{22}(s)]\, ds\right) \quad a < t < b.$$

5. Suppose the $n \times n$ matrix $A = A(t)$ is continuous on (a, b). Let

$$Y = \begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{bmatrix},$$

where the columns of Y are solutions of $\mathbf{y}' = A(t)\mathbf{y}$. Let

$$r_i = [y_{i1}\, y_{i2}\, \ldots\, y_{in}]$$

be the ith row of Y, and let W be the determinant of Y.

(a) Deduce from the definition of determinant that

$$W' = W_1 + W_2 + \cdots + W_n,$$

where, for $1 \leqslant m \leqslant n$, the ith row of W_m is r_i if $i \neq m$, and r'_m if $i = m$.

(b) Use the equation $Y' = AY$ and the definition of matrix multiplication to show that

$$r'_m = a_{m1}r_1 + a_{m2}r_2 + \cdots + a_{mn}r_n.$$

(c) Use properties of determinants to deduce from **(b)** that

$$\det(W_m) = a_{mm}W.$$

(d) Conclude from **(a)** and **(c)** that

$$W' = (a_{11} + a_{22} + \cdots + a_{nn})W,$$

and use this to show that if $a < t_0 < b$ then

$$W(t) = W(t_0)\exp\left(\int_{t_0}^{t} \big[a_{11}(s) + a_{22}(s) + \cdots + a_{nn}(s)\big]\, ds\right), \quad a < t < b.$$

6. Suppose the $n \times n$ matrix A is continuous on (a, b) and t_0 is a point in (a, b). Let Y be a fundamental matrix for $\mathbf{y}' = A(t)\mathbf{y}$ on (a, b).

(a) Show that $Y(t_0)$ is invertible.

(b) Show that if $\mathbf{k}$ is an arbitrary n-vector then the solution of the initial value problem

$$\mathbf{y}' = A(t)\mathbf{y}, \quad \mathbf{y}(t_0) = \mathbf{k}$$

is

$$\mathbf{y} = Y(t)Y^{-1}(t_0)\mathbf{k}.$$

7. Let

$$A = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}, \quad \mathbf{y}_1 = \begin{bmatrix} e^{6t} \\ e^{6t} \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} e^{-2t} \\ -e^{-2t} \end{bmatrix}, \quad \mathbf{k} = \begin{bmatrix} -3 \\ 9 \end{bmatrix}.$$

(a) Verify that $\{\mathbf{y}_1, \mathbf{y}_2\}$ is a fundamental set of solutions for $\mathbf{y}' = A\mathbf{y}$.

(b) Solve the initial value problem

$$\mathbf{y}' = A\mathbf{y}, \quad \mathbf{y}(0) = \mathbf{k}. \tag{A}$$

(c) Use the result of Exercise 6**(b)** to find a formula for the solution of (A) for an arbitrary initial vector $\mathbf{k}$.

8. Repeat Exercise 7 with

$$A = \begin{bmatrix} -2 & -2 \\ -5 & 1 \end{bmatrix}, \quad \mathbf{y}_1 = \begin{bmatrix} e^{-4t} \\ e^{-4t} \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} -2e^{3t} \\ 5e^{3t} \end{bmatrix}, \quad \mathbf{k} = \begin{bmatrix} 10 \\ -4 \end{bmatrix}.$$

9. Repeat Exercise 7 with

$$A = \begin{bmatrix} -4 & -10 \\ 3 & 7 \end{bmatrix}, \quad \mathbf{y}_1 = \begin{bmatrix} -5e^{2t} \\ 3e^{2t} \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} 2e^{t} \\ -e^{t} \end{bmatrix}, \quad \mathbf{k} = \begin{bmatrix} -19 \\ 11 \end{bmatrix}.$$

10. Repeat Exercise 7 with

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad \mathbf{y}_1 = \begin{bmatrix} e^{3t} \\ e^{3t} \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} e^{t} \\ -e^{t} \end{bmatrix}, \quad \mathbf{k} = \begin{bmatrix} 2 \\ 8 \end{bmatrix}.$$

11. Let

$$A = \begin{bmatrix} 3 & -1 & -1 \\ -2 & 3 & 2 \\ 4 & -1 & -2 \end{bmatrix},$$

$$\mathbf{y}_1 = \begin{bmatrix} e^{2t} \\ 0 \\ e^{2t} \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} e^{3t} \\ -e^{3t} \\ e^{3t} \end{bmatrix}, \quad \mathbf{y}_3 = \begin{bmatrix} e^{-t} \\ -3e^{-t} \\ 7e^{-t} \end{bmatrix}, \quad \mathbf{k} = \begin{bmatrix} 2 \\ -7 \\ 20 \end{bmatrix}.$$

(a) Verify that $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$ is a fundamental set of solutions for $\mathbf{y}' = A\mathbf{y}$.

(b) Solve the initial value problem

$$\mathbf{y}' = A\mathbf{y}, \quad \mathbf{y}(0) = \mathbf{k}. \tag{A}$$

(c) Use the result of Exercise 6**(b)** to find a formula for the solution of (A) for an arbitrary initial vector $\mathbf{k}$.

12. Repeat Exercise 11 with

$$A = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix},$$

$$\mathbf{y}_1 = \begin{bmatrix} -e^{-2t} \\ 0 \\ e^{-2t} \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} -e^{-2t} \\ e^{-2t} \\ 0 \end{bmatrix}, \quad \mathbf{y}_3 = \begin{bmatrix} e^{4t} \\ e^{4t} \\ e^{4t} \end{bmatrix}, \quad \mathbf{k} = \begin{bmatrix} 0 \\ -9 \\ 12 \end{bmatrix}.$$

13. Repeat Exercise 11 with

$$A = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 1 & 6 \\ 0 & 0 & -2 \end{bmatrix},$$

$$\mathbf{y}_1 = \begin{bmatrix} e^{t} \\ e^{t} \\ 0 \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} e^{-t} \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{y}_3 = \begin{bmatrix} e^{-2t} \\ -2e^{-2t} \\ e^{-2t} \end{bmatrix}, \quad \mathbf{k} = \begin{bmatrix} 5 \\ 5 \\ -1 \end{bmatrix}.$$

14. Suppose Y and Z are fundamental matrices for the $n \times n$ system $\mathbf{y}' = A(t)\mathbf{y}$. Then some of the four matrices YZ^{-1}, $Y^{-1}Z$, $Z^{-1}Y$, ZY^{-1} are necessarily constant. Identify them and prove that they are constant.

15. Suppose the columns of an $n \times n$ matrix Y are solutions of the $n \times n$ system $\mathbf{y}' = A\mathbf{y}$ and C is an $n \times n$ constant matrix.

(a) Show that the matrix $Z = YC$ satisfies the differential equation $Z' = AZ$.

(b) Show that Z is a fundamental matrix for $\mathbf{y}' = A(t)\mathbf{y}$ if and only if C is invertible and Y is a fundamental matrix for $\mathbf{y}' = A(t)\mathbf{y}$.

16. Suppose the $n \times n$ matrix $A = A(t)$ is continuous on (a, b) and t_0 is in (a, b). For $i = 1, 2, \dots, n$, let $\mathbf{y}_i$ be the solution of the initial value problem $\mathbf{y}_i' = A(t)\mathbf{y}_i$, $\mathbf{y}_i(t_0) = \mathbf{e}_i$, where

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \cdots \quad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix};$$

that is, the jth component of $\mathbf{e}_i$ is 1 if $j = i$, or 0 if $j \neq i$.

(a) Show that$\{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n\}$ is a fundamental set of solutions of $\mathbf{y}' = A(t)\mathbf{y}$ on (a, b).

(b) Conclude from **(a)** and Exercise 15 that $\mathbf{y}' = A(t)\mathbf{y}$ has infinitely many fundamental sets of solutions on (a, b).

17. Show that Y is a fundamental matrix for the system $\mathbf{y}' = A(t)\mathbf{y}$ if and only if Y^{-1} is a fundamental matrix for $\mathbf{y}' = -A^T(t)\mathbf{y}$, where A^T denotes the transpose of A. HINT: *See Exercise* 11.

18. Let Z be the fundamental matrix for the constant coefficient system $\mathbf{y}' = A\mathbf{y}$ such that $Z(0) = I$.

(a) Show that $Z(t)Z(s) = Z(t+s)$ for all s and t. HINT: *For fixed s let $\Gamma_1(t) = Z(t)Z(s)$ and $\Gamma_2(t) = Z(t+s)$. Show that Γ_1 and Γ_2 are both solutions of the matrix initial value problem $\Gamma' = A\Gamma, \quad \Gamma(0) = Z(s)$. Then conclude from Theorem* 6.2.1 *that $\Gamma_1 = \Gamma_2$.*

(b) Show that $(Z(t))^{-1} = Z(-t)$.

(c) The matrix Z defined above is sometimes denoted by e^{tA}. Discuss the motivation for this notation.

6.4 CONSTANT COEFFICIENT HOMOGENEOUS SYSTEMS I

We'll now begin our study of the homogeneous system

$$\mathbf{y}' = A\mathbf{y}, \tag{6.4.1}$$

where A is an $n \times n$ constant matrix. Since A is continuous on $(-\infty, \infty)$, Theorem 6.2.1 implies that all solutions of (6.4.1) are defined on $(-\infty, \infty)$. Therefore, when we speak of solutions of $\mathbf{y}' = A\mathbf{y}$, we'll mean solutions on $(-\infty, \infty)$.

In this section we assume that all the eigenvalues of A are real and that A has a set of n linearly independent eigenvectors. In the next two sections we consider the cases where some of the eigenvalues of A are complex, or where A does not have n linearly independent eigenvectors.

In Example 6.3.2 we showed that the vector functions

$$\mathbf{y}_1 = \begin{bmatrix} -e^{2t} \\ 2e^{2t} \end{bmatrix} \quad \text{and} \quad \mathbf{y}_2 = \begin{bmatrix} -e^{-t} \\ e^{-t} \end{bmatrix}$$

form a fundamental set of solutions of the system

$$\mathbf{y}' = \begin{bmatrix} -4 & -3 \\ 6 & 5 \end{bmatrix} \mathbf{y}, \tag{6.4.2}$$

but we did not show how we obtained $\mathbf{y}_1$ and $\mathbf{y}_2$ in the first place. To see how these solutions can be obtained we write (6.4.2) as

$$\begin{aligned} y_1' &= -4y_1 - 3y_2 \\ y_2' &= 6y_1 + 5y_2 \end{aligned} \tag{6.4.3}$$

and look for solutions of the form

$$y_1 = x_1 e^{\lambda t} \quad \text{and} \quad y_2 = x_2 e^{\lambda t}, \tag{6.4.4}$$

where x_1, x_2, and λ are constants to be determined. Differentiating (6.4.4) yields

$$y_1' = \lambda x_1 e^{\lambda t} \quad \text{and} \quad y_2' = \lambda x_2 e^{\lambda t}.$$

Substituting this and (6.4.4) into (6.4.3) and canceling the common factor $e^{\lambda t}$ yields

$$\begin{aligned} -4x_1 - 3x_2 &= \lambda x_1 \\ 6x_1 + 5x_2 &= \lambda x_2. \end{aligned}$$

For a given λ, this is a homogeneous algebraic system, since it can be rewritten as

$$\begin{aligned} (-4-\lambda)x_1 - 3x_2 &= 0 \\ 6x_1 + (5-\lambda)x_2 &= 0. \end{aligned} \tag{6.4.5}$$

The trivial solution $x_1 = x_2 = 0$ of this system isn't useful, since it corresponds to the trivial solution $y_1 \equiv y_2 \equiv 0$ of (6.4.3), which can't be part of a fundamental set of solutions of (6.4.2). Therefore we consider only those values of λ for which (6.4.5) has nontrivial solutions. These are the values of λ for which the determinant of (6.4.5) is zero; that is,

$$\begin{aligned} \left| \begin{array}{cc} -4-\lambda & -3 \\ 6 & 5-\lambda \end{array} \right| &= (-4-\lambda)(5-\lambda)+18 \\ &= \lambda^2 - \lambda - 2 \\ &= (\lambda-2)(\lambda+1) = 0, \end{aligned}$$

which has the solutions $\lambda_1 = 2$ and $\lambda_2 = -1$.

Taking $\lambda = 2$ in (6.4.5) yields

$$\begin{aligned} -6x_1 - 3x_2 &= 0 \\ 6x_1 + 3x_2 &= 0, \end{aligned}$$

which implies that $x_1 = -x_2/2$, where x_2 can be chosen arbitrarily. Choosing $x_2 = 2$ yields the solution $y_1 = -e^{2t}$, $y_2 = 2e^{2t}$ of (6.4.3). We can write this solution in vector form as

$$\mathbf{y}_1 = \left[\begin{array}{r} -1 \\ 2 \end{array} \right] e^{2t}. \tag{6.4.6}$$

Taking $\lambda = -1$ in (6.4.5) yields the system

$$\begin{aligned} -3x_1 - 3x_2 &= 0 \\ 6x_1 + 6x_2 &= 0, \end{aligned}$$

so $x_1 = -x_2$. Taking $x_2 = 1$ here yields the solution $y_1 = -e^{-t}$, $y_2 = e^{-t}$ of (6.4.3). We can write this solution in vector form as

$$\mathbf{y}_2 = \left[\begin{array}{r} -1 \\ 1 \end{array} \right] e^{-t}. \tag{6.4.7}$$

In (6.4.6) and (6.4.7) the constant coefficients in the arguments of the exponential functions are the eigenvalues of the coefficient matrix in (6.4.2), and the vector coefficients of the exponential functions are associated eigenvectors. This illustrates the next theorem.

Theorem 6.4.1 *Suppose the $n \times n$ constant matrix A has n real eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ (which need not be distinct) with associated linearly independent eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$. Then the functions*

$$\mathbf{y}_1 = \mathbf{x}_1 e^{\lambda_1 t},\ \mathbf{y}_2 = \mathbf{x}_2 e^{\lambda_2 t},\ \dots,\ \mathbf{y}_n = \mathbf{x}_n e^{\lambda_n t}$$

form a fundamental set of solutions of $\mathbf{y}' = A\mathbf{y}$; that is, the general solution of this system is

$$\mathbf{y} = c_1\mathbf{x}_1 e^{\lambda_1 t} + c_2\mathbf{x}_2 e^{\lambda_2 t} + \cdots + c_n\mathbf{x}_n e^{\lambda_n t}.$$

Proof Differentiating $\mathbf{y}_i = \mathbf{x}_i e^{\lambda_i t}$ and recalling that $A\mathbf{x}_i = \lambda_i \mathbf{x}_i$ yields

$$\mathbf{y}_i' = \lambda_i \mathbf{x}_i e^{\lambda_i t} = A\mathbf{x}_i e^{\lambda_i t} = A\mathbf{y}_i.$$

This shows that $\mathbf{y}_i$ is a solution of $\mathbf{y}' = A\mathbf{y}$.

The Wronskian of $\{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n\}$ is

$$\left|\begin{array}{cccc} x_{11}e^{\lambda_1 t} & x_{12}e^{\lambda_2 t} & \cdots & x_{1n}e^{\lambda_n t} \\ x_{21}e^{\lambda_1 t} & x_{22}e^{\lambda_2 t} & \cdots & x_{2n}e^{\lambda_n t} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1}e^{\lambda_1 t} & x_{n2}e^{\lambda_2 t} & \cdots & x_{nn}e^{\lambda x_n t} \end{array}\right| = e^{\lambda_1 t}e^{\lambda_2 t}\cdots e^{\lambda_n t}\left|\begin{array}{cccc} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{array}\right|.$$

Since the columns of the determinant on the right are $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$, which are assumed to be linearly independent, the determinant is nonzero. Therefore Theorem 6.3.3 implies that $\{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n\}$ is a fundamental set of solutions of $\mathbf{y}' = A\mathbf{y}$.

Example 6.4.1

(a) Find the general solution of

$$\mathbf{y}' = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}\mathbf{y}. \tag{6.4.8}$$

(b) Solve the initial value problem

$$\mathbf{y}' = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}\mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 5 \\ -1 \end{bmatrix}. \tag{6.4.9}$$

SOLUTION(a) The characteristic polynomial of the coefficient matrix A in (6.4.8) is

$$\begin{aligned} \left|\begin{array}{cc} 2-\lambda & 4 \\ 4 & 2-\lambda \end{array}\right| &= (\lambda-2)^2 - 16 \\ &= (\lambda-2-4)(\lambda-2+4) \\ &= (\lambda-6)(\lambda+2). \end{aligned}$$

Hence, $\lambda_1 = 6$ and $\lambda_2 = -2$ are eigenvalues of A. To obtain the eigenvectors, we must solve the system

$$\begin{bmatrix} 2-\lambda & 4 \\ 4 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{6.4.10}$$

with $\lambda = 6$ and $\lambda = -2$. Setting $\lambda = 6$ in (6.4.10) yields

$$\begin{bmatrix} -4 & 4 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which implies that $x_1 = x_2$. Taking $x_2 = 1$ yields the eigenvector

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

so

$$\mathbf{y}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{6t}$$

is a solution of (6.4.8). Setting $\lambda = -2$ in (6.4.10) yields

$$\begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which implies that $x_1 = -x_2$. Taking $x_2 = 1$ yields the eigenvector

$$\mathbf{x}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

so

$$\mathbf{y}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}$$

is a solution of (6.4.8). From Theorem 6.4.1, the general solution of (6.4.8) is

$$\mathbf{y} = c_1\mathbf{y}_1 + c_2\mathbf{y}_2 = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{6t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}. \tag{6.4.11}$$

SOLUTION(**b**) To satisfy the initial condition in (6.4.9), we must choose c_1 and c_2 in (6.4.11) so that

$$c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}.$$

This is equivalent to the system

$$\begin{aligned} c_1 - c_2 &= 5 \\ c_1 + c_2 &= -1, \end{aligned}$$

so $c_1 = 2, c_2 = -3$. Therefore the solution of (6.4.9) is

$$\mathbf{y} = 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{6t} - 3\begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t},$$

or, in terms of components,

$$y_1 = 2e^{6t} + 3e^{-2t}, \quad y_2 = 2e^{6t} - 3e^{-2t}.$$

Example 6.4.2

(a) Find the general solution of

$$\mathbf{y}' = \begin{bmatrix} 3 & -1 & -1 \\ -2 & 3 & 2 \\ 4 & -1 & -2 \end{bmatrix} \mathbf{y}. \tag{6.4.12}$$

(b) Solve the initial value problem

$$\mathbf{y}' = \begin{bmatrix} 3 & -1 & -1 \\ -2 & 3 & 2 \\ 4 & -1 & -2 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 2 \\ -1 \\ 8 \end{bmatrix}. \tag{6.4.13}$$

SOLUTION(a) The characteristic polynomial of the coefficient matrix A in (6.4.12) is

$$\begin{vmatrix} 3-\lambda & -1 & -1 \\ -2 & 3-\lambda & 2 \\ 4 & -1 & -2-\lambda \end{vmatrix} = -(\lambda-2)(\lambda-3)(\lambda+1).$$

Hence, the eigenvalues of A are $\lambda_1 = 2$, $\lambda_2 = 3$, and $\lambda_3 = -1$. To find the eigenvectors, we must solve the system

$$\begin{bmatrix} 3-\lambda & -1 & -1 \\ -2 & 3-\lambda & 2 \\ 4 & -1 & -2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{6.4.14}$$

with $\lambda = 2, 3, -1$. With $\lambda = 2$, the augmented matrix of (6.4.14) is

$$\left[\begin{array}{rrrcr} 1 & -1 & -1 & \vdots & 0 \\ -2 & 1 & 2 & \vdots & 0 \\ 4 & -1 & -4 & \vdots & 0 \end{array}\right],$$

which is row equivalent to

$$\left[\begin{array}{rrrcr} 1 & 0 & -1 & \vdots & 0 \\ 0 & 1 & 0 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{array}\right].$$

Hence, $x_1 = x_3$ and $x_2 = 0$. Taking $x_3 = 1$ yields

$$\mathbf{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{2t}$$

as a solution of (6.4.12). With $\lambda = 3$, the augmented matrix of (6.4.14) is

$$\left[\begin{array}{rrrcr} 0 & -1 & -1 & \vdots & 0 \\ -2 & 0 & 2 & \vdots & 0 \\ 4 & -1 & -5 & \vdots & 0 \end{array}\right],$$

which is row equivalent to

$$\left[\begin{array}{rrrcr} 1 & 0 & -1 & \vdots & 0 \\ 0 & 1 & 1 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{array}\right].$$

Hence, $x_1 = x_3$ and $x_2 = -x_3$. Taking $x_3 = 1$ yields

$$\mathbf{y}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} e^{3t}$$

as a solution of (6.4.12). With $\lambda = -1$, the augmented matrix of (6.4.14) is

$$\left[\begin{array}{rrrcr} 4 & -1 & -1 & \vdots & 0 \\ -2 & 4 & 2 & \vdots & 0 \\ 4 & -1 & -1 & \vdots & 0 \end{array}\right],$$

which is row equivalent to

$$\left[\begin{array}{rrrcr} 1 & 0 & -\frac{1}{7} & \vdots & 0 \\ 0 & 1 & \frac{3}{7} & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{array}\right].$$

Hence, $x_1 = x_3/7$ and $x_2 = -3x_3/7$. Taking $x_3 = 7$ yields

$$\mathbf{y}_3 = \begin{bmatrix} 1 \\ -3 \\ 7 \end{bmatrix} e^{-t}$$

as a solution of (6.4.12). By Theorem 6.4.1, the general solution of (6.4.12) is

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} e^{3t} + c_3 \begin{bmatrix} 1 \\ -3 \\ 7 \end{bmatrix} e^{-t},$$

which can also be written as

$$\mathbf{y} = \begin{bmatrix} e^{2t} & e^{3t} & e^{-t} \\ 0 & -e^{3t} & -3e^{-t} \\ e^{2t} & e^{3t} & 7e^{-t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}. \tag{6.4.15}$$

SOLUTION(b) To satisfy the initial condition in (6.4.13) we must choose c_1, c_2, c_3 in (6.4.15) so that

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -3 \\ 1 & 1 & 7 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 8 \end{bmatrix}.$$

Solving this system yields $c_1 = 3$, $c_2 = -2$, $c_3 = 1$. Hence, the solution of (6.4.13) is

$$\begin{aligned} \mathbf{y} &= \begin{bmatrix} e^{2t} & e^{3t} & e^{-t} \\ 0 & -e^{3t} & -3e^{-t} \\ e^{2t} & e^{3t} & 7e^{-t} \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} \\ &= 3\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{2t} - 2\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} e^{3t} + \begin{bmatrix} 1 \\ -3 \\ 7 \end{bmatrix} e^{-t}. \end{aligned}$$

Example 6.4.3 Find the general solution of

$$\mathbf{y}' = \begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix} \mathbf{y}. \tag{6.4.16}$$

Solution The characteristic polynomial of the coefficient matrix A in (6.4.16) is

$$\begin{vmatrix} -3-\lambda & 2 & 2 \\ 2 & -3-\lambda & 2 \\ 2 & 2 & -3-\lambda \end{vmatrix} = -(\lambda-1)(\lambda+5)^2.$$

Hence, $\lambda_1 = 1$ is an eigenvalue of multiplicity 1, while $\lambda_2 = -5$ is an eigenvalue of multiplicity 2. Eigenvectors associated with $\lambda_1 = 1$ are solutions of the system with augmented matrix

$$\left[\begin{array}{rrr|r} -4 & 2 & 2 & 0 \\ 2 & -4 & 2 & 0 \\ 2 & 2 & -4 & 0 \end{array}\right],$$

which is row equivalent to

$$\left[\begin{array}{rrr|r} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Hence, $x_1 = x_2 = x_3$, and we choose $x_3 = 1$ to obtain the solution

$$\mathbf{y}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^t \tag{6.4.17}$$

of (6.4.16). Eigenvectors associated with $\lambda_2 = -5$ are solutions of the system with augmented matrix

$$\begin{bmatrix} 2 & 2 & 2 & \vdots & 0 \\ 2 & 2 & 2 & \vdots & 0 \\ 2 & 2 & 2 & \vdots & 0 \end{bmatrix}.$$

Hence, the components of these eigenvectors need only satisfy the single condition

$$x_1 + x_2 + x_3 = 0.$$

Since there's only one equation here, we can choose x_2 and x_3 arbitrarily. We obtain one eigenvector by choosing $x_2 = 0$ and $x_3 = 1$, and another by choosing $x_2 = 1$ and $x_3 = 0$. In both cases $x_1 = -1$. Therefore

$$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

are linearly independent eigenvectors associated with $\lambda_2 = -5$, and the corresponding solutions of (6.4.16) are

$$\mathbf{y}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{-5t} \quad \text{and} \quad \mathbf{y}_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^{-5t}.$$

Because of this and (6.4.17), Theorem 6.4.1 implies that the general solution of (6.4.16) is

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{-5t} + c_3 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^{-5t}.$$

Geometric Properties of Solutions when $n = 2$

We'll now consider the geometric properties of solutions of a 2×2 constant coefficient system

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}. \tag{6.4.18}$$

It is convenient to think of a "y_1-y_2 plane," where a point is identified by rectangular coordinates (y_1, y_2). If $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ is a non-constant solution of (6.4.18), then the point

$(y_1(t), y_2(t))$ moves along a curve C in the y_1-y_2 plane as t varies from $-\infty$ to ∞. We call C the *trajectory* of $\mathbf{y}$. (We also say that C is a trajectory of the system (6.4.18).) I's important to note that C is the trajectory of infinitely many solutions of (6.4.18), since if τ is any real number, then $\mathbf{y}(t-\tau)$ is a solution of (6.4.18) (Exercise 28(b)), and $(y_1(t-\tau), y_2(t-\tau))$ also moves along C as t varies from $-\infty$ to ∞. Moreover, Exercise 28**(c)** implies that distinct trajectories of (6.4.18) can't intersect, and that two solutions $\mathbf{y}_1$ and $\mathbf{y}_2$ of (6.4.18) have the same trajectory if and only if $\mathbf{y}_2(t) = \mathbf{y}_1(t-\tau)$ for some τ.

From Exercise 28**(a)**, a trajectory of a nontrivial solution of (6.4.18) can't contain $(0, 0)$, which we define to be the trajectory of the trivial solution $\mathbf{y} \equiv 0$. More generally, if $\mathbf{y} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \neq \mathbf{0}$ is a constant solution of (6.4.18) (which could occur if zero is an eigenvalue of the matrix of (6.4.18)), we define the trajectory of $\mathbf{y}$ to be the single point (k_1, k_2).

To be specific, this is the question: What do the trajectories look like, and how are they traversed? In this section we'll answer this question, assuming that the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

of (6.4.18) has real eigenvalues λ_1 and λ_2 with associated linearly independent eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$. Then the general solution of (6.4.18) is

$$\mathbf{y} = c_1\mathbf{x}_1 e^{\lambda_1 t} + c_2\mathbf{x}_2 e^{\lambda_2 t}. \qquad (6.4.19)$$

We'll consider other situations in the next two sections.

We leave it to you (Exercise 35) to classify the trajectories of (6.4.18) if zero is an eigenvalue of A. We'll confine our attention here to the case where both eigenvalues are nonzero. In this case the simplest situation is where $\lambda_1 = \lambda_2 \neq 0$, so (6.4.19) becomes

$$\mathbf{y} = (c_1\mathbf{x}_1 + c_2\mathbf{x}_2)e^{\lambda_1 t}.$$

Since $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent, an arbitrary vector $\mathbf{x}$ can be written as $\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2$. Therefore the general solution of (6.4.18) can be written as $\mathbf{y} = \mathbf{x}e^{\lambda_1 t}$ where $\mathbf{x}$ is an arbitrary 2-vector, and the trajectories of nontrivial solutions of (6.4.18) are half-lines through (but not including) the origin. The direction of motion is away from the origin if $\lambda_1 > 0$ (Figure 6.1), toward it if $\lambda_1 < 0$ (Figure 6.2). (In these and the next figures an arrow through a point indicates the direction of motion along the trajectory through the point.)

Figure 6.1 Trajectories of a 2×2 system with a repeated positive eigenvalue

Figure 6.2 Trajectories of a 2×2 system with a repeated negative eigenvalue

Now suppose $\lambda_2 > \lambda_1$, and let L_1 and L_2 denote lines through the origin parallel to $\mathbf{x}_1$ and $\mathbf{x}_2$, respectively. By a half-line of L_1 (or L_2), we mean either of the rays obtained by removing the origin from L_1 (or L_2).

Letting $c_2 = 0$ in (6.4.19) yields $\mathbf{y} = c_1\mathbf{x}_1e^{\lambda_1 t}$. If $c_1 \neq 0$, the trajectory defined by this solution is a half-line of L_1. The direction of motion is away from the origin if $\lambda_1 > 0$, toward the origin if $\lambda_1 < 0$. Similarly, the trajectory of $\mathbf{y} = c_2\mathbf{x}_2e^{\lambda_2 t}$ with $c_2 \neq 0$ is a half-line of L_2.

Henceforth, we assume that c_1 and c_2 in (6.4.19) are both nonzero. In this case, the trajectory of (6.4.19) can't intersect L_1 or L_2, since every point on these lines is on the trajectory of a solution for which either $c_1 = 0$ or $c_2 = 0$. (Remember: distinct trajectories can't intersect!). Therefore the trajectory of (6.4.19) must lie entirely in one of the four open sectors bounded by L_1 and L_2, but do not any point on L_1 or L_2. Since the initial point $(y_1(0), y_2(0))$ defined by

$$\mathbf{y}(0) = c_1\mathbf{x}_1 + c_2\mathbf{x}_2$$

is on the trajectory, we can determine which sector contains the trajectory from the signs of c_1 and c_2, as shown in Figure 6.3.

The direction of $\mathbf{y}(t)$ in (6.4.19) is the same as that of

$$e^{-\lambda_2 t}\mathbf{y}(t) = c_1\mathbf{x}_1e^{-(\lambda_2-\lambda_1)t} + c_2\mathbf{x}_2 \tag{6.4.20}$$

and of

$$e^{-\lambda_1 t}\mathbf{y}(t) = c_1\mathbf{x}_1 + c_2\mathbf{x}_2e^{(\lambda_2-\lambda_1)t}. \tag{6.4.21}$$

Since the right side of (6.4.20) approaches $c_2\mathbf{x}_2$ as $t \to \infty$, the trajectory is asymptotically parallel to L_2 as $t \to \infty$. Since the right side of (6.4.21) approaches $c_1\mathbf{x}_1$ as $t \to -\infty$, the trajectory is asymptotically parallel to L_1 as $t \to -\infty$.

The shape and direction of traversal of the trajectory of (6.4.19) depend upon whether λ_1 and λ_2 are both positive, both negative, or of opposite signs. We'll now analyze these three cases.

Henceforth $\|\mathbf{u}\|$ denote the length of the vector $\mathbf{u}$.

Figure 6.3 Four open sectors bounded by L_1 and L_2

Figure 6.4 Two positive eigenvalues; motion away from origin

Case 1: $\lambda_2 > \lambda_1 > 0$

Figure 6.4 shows some typical trajectories. In this case, $\lim_{t\to-\infty} \|\mathbf{y}(t)\| = 0$, so the trajectory is not only asymptotically parallel to L_1 as $t \to -\infty$, but is actually asymptotically tangent to L_1 at the origin. On the other hand, $\lim_{t\to\infty} \|\mathbf{y}(t)\| = \infty$ and

$$\lim_{t\to\infty} \left\|\mathbf{y}(t) - c_2\mathbf{x}_2e^{\lambda_2 t}\right\| = \lim_{t\to\infty} \|c_1\mathbf{x}_1e^{\lambda_1 t}\| = \infty,$$

so, although the trajectory is asymptotically parallel to L_2 as $t \to \infty$, it's not asymptotically tangent to L_2. The direction of motion along each trajectory is away from the origin.

Case 2: $0 > \lambda_2 > \lambda_1$

Figure 6.5 shows some typical trajectories. In this case, $\lim_{t\to\infty}\|\mathbf{y}(t)\| = 0$, so the trajectory is asymptotically tangent to L_2 at the origin as $t \to \infty$. On the other hand, $\lim_{t\to-\infty}\|\mathbf{y}(t)\| = \infty$ and

$$\lim_{t\to-\infty}\left\|\mathbf{y}(t) - c_1\mathbf{x}_1 e^{\lambda_1 t}\right\| = \lim_{t\to-\infty}\|c_2\mathbf{x}_2 e^{\lambda_2 t}\| = \infty,$$

so, although the trajectory is asymptotically parallel to L_1 as $t \to -\infty$, it's not asymptotically tangent to it. The direction of motion along each trajectory is toward the origin.

Figure 6.5 Two negative eigenvalues; motion toward the origin

Figure 6.6 Eigenvalues of different signs

Case 3: $\lambda_2 > 0 > \lambda_1$

Figure 6.6 shows some typical trajectories. In this case,

$$\lim_{t\to\infty}\|\mathbf{y}(t)\| = \infty \quad \text{and} \quad \lim_{t\to\infty}\left\|\mathbf{y}(t) - c_2\mathbf{x}_2 e^{\lambda_2 t}\right\| = \lim_{t\to\infty}\|c_1\mathbf{x}_1 e^{\lambda_1 t}\| = 0,$$

so the trajectory is asymptotically tangent to L_2 as $t \to \infty$. Similarly,

$$\lim_{t\to-\infty}\|\mathbf{y}(t)\| = \infty \quad \text{and} \quad \lim_{t\to-\infty}\left\|\mathbf{y}(t) - c_1\mathbf{x}_1 e^{\lambda_1 t}\right\| = \lim_{t\to-\infty}\|c_2\mathbf{x}_2 e^{\lambda_2 t}\| = 0,$$

so the trajectory is asymptotically tangent to L_1 as $t \to -\infty$. The direction of motion is toward the origin on L_1 and away from the origin on L_2. The direction of motion along any other trajectory is away from L_1, toward L_2.

6.4 Exercises

In Exercises 1–15 find the general solution.

1. $\mathbf{y}' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\mathbf{y}$

2. $\mathbf{y}' = \frac{1}{4}\begin{bmatrix} -5 & 3 \\ 3 & -5 \end{bmatrix}\mathbf{y}$

3. $\mathbf{y}' = \frac{1}{5}\begin{bmatrix} -4 & 3 \\ -2 & -11 \end{bmatrix}\mathbf{y}$

4. $\mathbf{y}' = \begin{bmatrix} -1 & -4 \\ -1 & -1 \end{bmatrix}\mathbf{y}$

5. $\mathbf{y}' = \begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix}\mathbf{y}$

6. $\mathbf{y}' = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}\mathbf{y}$

7. $\mathbf{y}' = \begin{bmatrix} -6 & -3 \\ 1 & -2 \end{bmatrix}\mathbf{y}$

8. $\mathbf{y}' = \begin{bmatrix} 1 & -1 & -2 \\ 1 & -2 & -3 \\ -4 & 1 & -1 \end{bmatrix}\mathbf{y}$

9. $\mathbf{y}' = \begin{bmatrix} -6 & -4 & -8 \\ -4 & 0 & -4 \\ -8 & -4 & -6 \end{bmatrix} \mathbf{y}$

10. $\mathbf{y}' = \begin{bmatrix} 3 & 5 & 8 \\ 1 & -1 & -2 \\ -1 & -1 & -1 \end{bmatrix} \mathbf{y}$

11. $\mathbf{y}' = \begin{bmatrix} 1 & -1 & 2 \\ 12 & -4 & 10 \\ -6 & 1 & -7 \end{bmatrix} \mathbf{y}$

12. $\mathbf{y}' = \begin{bmatrix} 4 & -1 & -4 \\ 4 & -3 & -2 \\ 1 & -1 & -1 \end{bmatrix} \mathbf{y}$

13. $\mathbf{y}' = \begin{bmatrix} -2 & 2 & -6 \\ 2 & 6 & 2 \\ -2 & -2 & 2 \end{bmatrix} \mathbf{y}$

14. $\mathbf{y}' = \begin{bmatrix} 3 & 2 & -2 \\ -2 & 7 & -2 \\ -10 & 10 & -5 \end{bmatrix} \mathbf{y}$

15. $\mathbf{y}' = \begin{bmatrix} 3 & 1 & -1 \\ 3 & 5 & 1 \\ -6 & 2 & 4 \end{bmatrix} \mathbf{y}$

In Exercises 16–27 *solve the initial value problem.*

16. $\mathbf{y}' = \begin{bmatrix} -7 & 4 \\ -6 & 7 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$

17. $\mathbf{y}' = \frac{1}{6}\begin{bmatrix} 7 & 2 \\ -2 & 2 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$

18. $\mathbf{y}' = \begin{bmatrix} 21 & -12 \\ 24 & -15 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$

19. $\mathbf{y}' = \begin{bmatrix} -7 & 4 \\ -6 & 7 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} -1 \\ 7 \end{bmatrix}$

20. $\mathbf{y}' = \frac{1}{6}\begin{bmatrix} 1 & 2 & 0 \\ 4 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 4 \\ 7 \\ 1 \end{bmatrix}$

21. $\mathbf{y}' = \frac{1}{3}\begin{bmatrix} 2 & -2 & 3 \\ -4 & 4 & 3 \\ 2 & 1 & 0 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}$

22. $\mathbf{y}' = \begin{bmatrix} 6 & -3 & -8 \\ 2 & 1 & -2 \\ 3 & -3 & -5 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}$

23. $\mathbf{y}' = \frac{1}{3}\begin{bmatrix} 2 & 4 & -7 \\ 1 & 5 & -5 \\ -4 & 4 & -1 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$

24. $\mathbf{y}' = \begin{bmatrix} 3 & 0 & 1 \\ 11 & -2 & 7 \\ 1 & 0 & 3 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 2 \\ 7 \\ 6 \end{bmatrix}$

25. $\mathbf{y}' = \begin{bmatrix} -2 & -5 & -1 \\ -4 & -1 & 1 \\ 4 & 5 & 3 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 8 \\ -10 \\ -4 \end{bmatrix}$

26. $\mathbf{y}' = \begin{bmatrix} 3 & -1 & 0 \\ 4 & -2 & 0 \\ 4 & -4 & 2 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 7 \\ 10 \\ 2 \end{bmatrix}$

27. $\mathbf{y}' = \begin{bmatrix} -2 & 2 & 6 \\ 2 & 6 & 2 \\ -2 & -2 & 2 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 6 \\ -10 \\ 7 \end{bmatrix}$

28. Let A be an $n \times n$ constant matrix. Then Theorem 6.2.1 implies that the solutions of

$$\mathbf{y}' = A\mathbf{y} \tag{A}$$

are all defined on $(-\infty, \infty)$.

(a) Use Theorem 6.2.1 to show that the only solution of (A) that can ever equal the zero vector is $\mathbf{y} \equiv \mathbf{0}$.

(b) Suppose $\mathbf{y}_1$ is a solution of (A) and $\mathbf{y}_2$ is defined by $\mathbf{y}_2(t) = \mathbf{y}_1(t - \tau)$, where τ is an arbitrary real number. Show that $\mathbf{y}_2$ is also a solution of (A).

(c) Suppose $\mathbf{y}_1$ and $\mathbf{y}_2$ are solutions of (A) and there are real numbers t_1 and t_2 such that $\mathbf{y}_1(t_1) = \mathbf{y}_2(t_2)$. Show that $\mathbf{y}_2(t) = \mathbf{y}_1(t - \tau)$ for all t, where $\tau = t_2 - t_1$. HINT: *Show that* $\mathbf{y}_1(t - \tau)$ *and* $\mathbf{y}_2(t)$ *are solutions of the same initial value problem for* (A), *and apply the uniqueness assertion of Theorem* 6.2.1.

In Exercises 29- 34 *describe and graph trajectories of the given system.*

29. C/G $\mathbf{y}' = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \mathbf{y}$

30. C/G $\mathbf{y}' = \begin{bmatrix} -4 & 3 \\ -2 & -11 \end{bmatrix} \mathbf{y}$

31. C/G $\mathbf{y}' = \begin{bmatrix} 9 & -3 \\ -1 & 11 \end{bmatrix} \mathbf{y}$

32. C/G $\mathbf{y}' = \begin{bmatrix} -1 & -10 \\ -5 & 4 \end{bmatrix} \mathbf{y}$

33. C/G $\mathbf{y}' = \begin{bmatrix} 5 & -4 \\ 1 & 10 \end{bmatrix} \mathbf{y}$

34. C/G $\mathbf{y}' = \begin{bmatrix} -7 & 1 \\ 3 & -5 \end{bmatrix} \mathbf{y}$

35. Suppose the eigenvalues of the 2×2 matrix A are $\lambda = 0$ and $\mu \neq 0$, with corresponding eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$. Let L_1 be the line through the origin parallel to $\mathbf{x}_1$.

(a) Show that every point on L_1 is the trajectory of a constant solution of $\mathbf{y}' = A\mathbf{y}$.

(b) Show that the trajectories of nonconstant solutions of $\mathbf{y}' = A\mathbf{y}$ are half-lines parallel to $\mathbf{x}_2$ and on either side of L_1, and that the direction of motion along these trajectories is away from L_1 if $\mu > 0$, or toward L_1 if $\mu < 0$.

The matrices of the systems in Exercises 36-41 are singular. Describe and graph the trajectories of nonconstant solutions of the given systems.

36. C/G $\mathbf{y}' = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\mathbf{y}$

37. C/G $\mathbf{y}' = \begin{bmatrix} -1 & -3 \\ 2 & 6 \end{bmatrix}\mathbf{y}$

38. C/G $\mathbf{y}' = \begin{bmatrix} 1 & -3 \\ -1 & 3 \end{bmatrix}\mathbf{y}$

39. C/G $\mathbf{y}' = \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix}\mathbf{y}$

40. C/G $\mathbf{y}' = \begin{bmatrix} -4 & -4 \\ 1 & 1 \end{bmatrix}\mathbf{y}$

41. C/G $\mathbf{y}' = \begin{bmatrix} 3 & -1 \\ -3 & 1 \end{bmatrix}\mathbf{y}$

42. L Let $P = P(t)$ and $Q = Q(t)$ be the populations of two species at time t, and assume that each population would grow exponentially if the other didn't exist; that is, in the absence of competition,

$$P' = aP \quad \text{and} \quad Q' = bQ, \tag{A}$$

where a and b are positive constants. One way to model the effect of competition is to assume that the growth rate per individual of each population is reduced by an amount proportional to the other population, so (A) is replaced by

$$\begin{aligned} P' &= aP - \alpha Q \\ Q' &= -\beta P + bQ, \end{aligned}$$

where α and β are positive constants. (Since negative population doesn't make sense, this system holds only while P and Q are both positive.) Now suppose $P(0) = P_0 > 0$ and $Q(0) = Q_0 > 0$.

(a) For several choices of a, b, α, and β, verify experimentally (by graphing trajectories of (A) in the P-Q plane) that there's a constant $\rho > 0$ (depending upon a, b, α, and β) with the following properties:

(i) If $Q_0 > \rho P_0$, then P decreases monotonically to zero in finite time, during which Q remains positive.

(ii) If $Q_0 < \rho P_0$, then Q decreases monotonically to zero in finite time, during which P remains positive.

(b) Conclude from **(a)** that exactly one of the species becomes extinct in finite time if $Q_0 \neq \rho P_0$. Determine experimentally what happens if $Q_0 = \rho P_0$.

(c) Confirm your experimental results and determine γ by expressing the eigenvalues and associated eigenvectors of

$$A = \begin{bmatrix} a & -\alpha \\ -\beta & b \end{bmatrix}$$

in terms of a, b, α, and β, and applying the geometric arguments developed at the end of this section.

6.5 CONSTANT COEFFICIENT HOMOGENEOUS SYSTEMS II

We saw in Section 10.4 that if an $n \times n$ constant matrix A has n real eigenvalues λ_1, λ_2, $\dots$, λ_n (which need not be distinct) with associated linearly independent eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$, then the general solution of $\mathbf{y}' = A\mathbf{y}$ is

$$\mathbf{y} = c_1\mathbf{x}_1e^{\lambda_1 t} + c_2\mathbf{x}_2e^{\lambda_2 t} + \cdots + c_n\mathbf{x}_n e^{\lambda_n t}.$$

In this section we consider the case where A has n real eigenvalues, but does not have n linearly independent eigenvectors. It is shown in linear algebra that this occurs if and only if A has at least one eigenvalue of multiplicity $r > 1$ such that the associated eigenspace has dimension less than r. In this case A is said to be *defective*. Since it's beyond the scope of this book to give a complete analysis of systems with defective coefficient matrices, we will restrict our attention to some commonly occurring special cases.

Example 6.5.1 Show that the system

$$\mathbf{y}' = \begin{bmatrix} 11 & -25 \\ 4 & -9 \end{bmatrix}\mathbf{y} \tag{6.5.1}$$

does not have a fundamental set of solutions of the form $\{\mathbf{x}_1e^{\lambda_1 t}, \mathbf{x}_2e^{\lambda_2 t}\}$, where λ_1 and λ_2 are eigenvalues of the coefficient matrix A of (6.5.1) and $\mathbf{x}_1$, and $\mathbf{x}_2$ are associated linearly independent eigenvectors.

Solution The characteristic polynomial of A is

$$\begin{aligned} \begin{vmatrix} 11-\lambda & -25 \\ 4 & -9-\lambda \end{vmatrix} &= (\lambda-11)(\lambda+9)+100 \\ &= \lambda^2-2\lambda+1 = (\lambda-1)^2. \end{aligned}$$

Hence, $\lambda = 1$ is the only eigenvalue of A. The augmented matrix of the system $(A-I)\mathbf{x} = \mathbf{0}$ is

$$\left[\begin{array}{rrcr} 10 & -25 & \vdots & 0 \\ 4 & -10 & \vdots & 0 \end{array}\right],$$

which is row equivalent to

$$\left[\begin{array}{rrcr} 1 & -\frac{5}{2} & \vdots & 0 \\ 0 & 0 & \vdots & 0 \end{array}\right].$$

Hence, $x_1 = 5x_2/2$ where x_2 is arbitrary. Therefore all eigenvectors of A are scalar multiples of $\mathbf{x}_1 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$, so A does not have a set of two linearly independent eigenvectors.

■

From Example 6.5.1, we know that all scalar multiples of $\mathbf{y}_1 = \begin{bmatrix} 5 \\ 2 \end{bmatrix} e^t$ are solutions of (6.5.1); however, to find the general solution we must find a second solution $\mathbf{y}_2$ such that $\{\mathbf{y}_1, \mathbf{y}_2\}$ is linearly independent. Based on your recollection of the procedure for solving a constant coefficient scalar equation

$$ay'' + by' + cy = 0$$

in the case where the characteristic polynomial has a repeated root, you might expect to obtain a second solution of (6.5.1) by multiplying the first solution by t. However, this yields $\mathbf{y}_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix} te^t$, which doesn't work, since

$$\mathbf{y}_2' = \begin{bmatrix} 5 \\ 2 \end{bmatrix} (te^t + e^t), \quad \text{while} \quad \begin{bmatrix} 11 & -25 \\ 4 & -9 \end{bmatrix} \mathbf{y}_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix} te^t.$$

The next theorem shows what to do in this situation.

Theorem 6.5.1 *Suppose the $n \times n$ matrix A has an eigenvalue λ_1 of multiplicity $\geqslant 2$ and the associated eigenspace has dimension 1; that is, all λ_1-eigenvectors of A are scalar multiples of an eigenvector $\mathbf{x}$. Then there are infinitely many vectors $\mathbf{u}$ such that*

$$(A - \lambda_1 I)\mathbf{u} = \mathbf{x}. \tag{6.5.2}$$

Moreover, if $\mathbf{u}$ is any such vector then

$$\mathbf{y}_1 = \mathbf{x}e^{\lambda_1 t} \quad \text{and} \quad \mathbf{y}_2 = \mathbf{u}e^{\lambda_1 t} + \mathbf{x}te^{\lambda_1 t} \tag{6.5.3}$$

are linearly independent solutions of $\mathbf{y}' = A\mathbf{y}$.

A complete proof of this theorem is beyond the scope of this book. The difficulty is in proving that there's a vector $\mathbf{u}$ satisfying (6.5.2), since $\det(A - \lambda_1 I) = 0$. We'll take this without proof and verify the other assertions of the theorem.

We already know that $\mathbf{y}_1$ in (6.5.3) is a solution of $\mathbf{y}' = A\mathbf{y}$. To see that $\mathbf{y}_2$ is also a solution, we compute

$$\begin{aligned} \mathbf{y}_2' - A\mathbf{y}_2 &= \lambda_1 \mathbf{u}e^{\lambda_1 t} + \mathbf{x}e^{\lambda_1 t} + \lambda_1 \mathbf{x}te^{\lambda_1 t} - A\mathbf{u}e^{\lambda_1 t} - A\mathbf{x}te^{\lambda_1 t} \\ &= (\lambda_1 \mathbf{u} + \mathbf{x} - A\mathbf{u})e^{\lambda_1 t} + (\lambda_1 \mathbf{x} - A\mathbf{x})te^{\lambda_1 t}. \end{aligned}$$

Since $A\mathbf{x} = \lambda_1 \mathbf{x}$, this can be written as

$$\mathbf{y}_2' - A\mathbf{y}_2 = -\left((A - \lambda_1 I)\mathbf{u} - \mathbf{x}\right) e^{\lambda_1 t},$$

and now (6.5.2) implies that $\mathbf{y}_2' = A\mathbf{y}_2$.

To see that $\mathbf{y}_1$ and $\mathbf{y}_2$ are linearly independent, suppose c_1 and c_2 are constants such that

$$c_1\mathbf{y}_1 + c_2\mathbf{y}_2 = c_1\mathbf{x}e^{\lambda_1 t} + c_2(\mathbf{u}e^{\lambda_1 t} + \mathbf{x}te^{\lambda_1 t}) = \mathbf{0}. \tag{6.5.4}$$

We must show that $c_1 = c_2 = 0$. Multiplying (6.5.4) by $e^{-\lambda_1 t}$ shows that

$$c_1\mathbf{x} + c_2(\mathbf{u} + \mathbf{x}t) = \mathbf{0}. \tag{6.5.5}$$

By differentiating this with respect to t, we see that $c_2\mathbf{x} = \mathbf{0}$, which implies $c_2 = 0$, because $\mathbf{x} \neq \mathbf{0}$. Substituting $c_2 = 0$ into (6.5.5) yields $c_1\mathbf{x} = \mathbf{0}$, which implies that $c_1 = 0$, again because $\mathbf{x} \neq \mathbf{0}$

Example 6.5.2 Use Theorem 6.5.1 to find the general solution of the system

$$\mathbf{y}' = \begin{bmatrix} 11 & -25 \\ 4 & -9 \end{bmatrix} \mathbf{y} \tag{6.5.6}$$

considered in Example 6.5.1.

Solution In Example 6.5.1 we saw that $\lambda_1 = 1$ is an eigenvalue of multiplicity 2 of the coefficient matrix A in (6.5.6), and that all of the eigenvectors of A are multiples of

$$\mathbf{x} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}.$$

Therefore

$$\mathbf{y}_1 = \begin{bmatrix} 5 \\ 2 \end{bmatrix} e^t$$

is a solution of (6.5.6). From Theorem 6.5.1, a second solution is given by $\mathbf{y}_2 = \mathbf{u}e^t + \mathbf{x}te^t$, where $(A - I)\mathbf{u} = \mathbf{x}$. The augmented matrix of this system is

$$\left[\begin{array}{rrcr} 10 & -25 & \vdots & 5 \\ 4 & -10 & \vdots & 2 \end{array}\right],$$

which is row equivalent to

$$\left[\begin{array}{rrcr} 1 & -\frac{5}{2} & \vdots & \frac{1}{2} \\ 0 & 0 & \vdots & 0 \end{array}\right].$$

Therefore the components of $\mathbf{u}$ must satisfy

$$u_1 - \frac{5}{2}u_2 = \frac{1}{2},$$

where u_2 is arbitrary. We choose $u_2 = 0$, so that $u_1 = 1/2$ and

$$\mathbf{u} = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}.$$

Thus,

$$\mathbf{y}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \frac{e^t}{2} + \begin{bmatrix} 5 \\ 2 \end{bmatrix} te^t.$$

Since $\mathbf{y}_1$ and $\mathbf{y}_2$ are linearly independent by Theorem 6.5.1, they form a fundamental set of solutions of (6.5.6). Therefore the general solution of (6.5.6) is

$$\mathbf{y} = c_1 \begin{bmatrix} 5 \\ 2 \end{bmatrix} e^t + c_2 \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \frac{e^t}{2} + \begin{bmatrix} 5 \\ 2 \end{bmatrix} te^t \right). \blacksquare$$

Note that choosing the arbitrary constant u_2 to be nonzero is equivalent to adding a scalar multiple of $\mathbf{y}_1$ to the second solution $\mathbf{y}_2$ (Exercise 33).

Example 6.5.3 Find the general solution of

$$\mathbf{y}' = \begin{bmatrix} 3 & 4 & -10 \\ 2 & 1 & -2 \\ 2 & 2 & -5 \end{bmatrix} \mathbf{y}. \tag{6.5.7}$$

Solution The characteristic polynomial of the coefficient matrix A in (6.5.7) is

$$\begin{vmatrix} 3-\lambda & 4 & -10 \\ 2 & 1-\lambda & -2 \\ 2 & 2 & -5-\lambda \end{vmatrix} = -(\lambda-1)(\lambda+1)^2.$$

Hence, the eigenvalues are $\lambda_1 = 1$ with multiplicity 1 and $\lambda_2 = -1$ with multiplicity 2.

Eigenvectors associated with $\lambda_1 = 1$ must satisfy $(A - I)\mathbf{x} = \mathbf{0}$. The augmented matrix of this system is

$$\left[\begin{array}{ccc|c} 2 & 4 & -10 & 0 \\ 2 & 0 & -2 & 0 \\ 2 & 2 & -6 & 0 \end{array}\right],$$

which is row equivalent to

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Hence, $x_1 = x_3$ and $x_2 = 2x_3$, where x_3 is arbitrary. Choosing $x_3 = 1$ yields the eigenvector

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

Therefore

$$\mathbf{y}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} e^t$$

is a solution of (6.5.7).

Eigenvectors associated with $\lambda_2 = -1$ satisfy $(A + I)\mathbf{x} = \mathbf{0}$. The augmented matrix of this system is

$$\left[\begin{array}{rrrcr} 4 & 4 & -10 & \vdots & 0 \\ 2 & 2 & -2 & \vdots & 0 \\ 2 & 2 & -4 & \vdots & 0 \end{array}\right],$$

which is row equivalent to

$$\left[\begin{array}{rrrcr} 1 & 1 & 0 & \vdots & 0 \\ 0 & 0 & 1 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{array}\right].$$

Hence, $x_3 = 0$ and $x_1 = -x_2$, where x_2 is arbitrary. Choosing $x_2 = 1$ yields the eigenvector

$$\mathbf{x}_2 = \left[\begin{array}{r} -1 \\ 1 \\ 0 \end{array}\right],$$

so

$$\mathbf{y}_2 = \left[\begin{array}{r} -1 \\ 1 \\ 0 \end{array}\right] e^{-t}$$

is a solution of (6.5.7).

Since all the eigenvectors of A associated with $\lambda_2 = -1$ are multiples of $\mathbf{x}_2$, we must now use Theorem 6.5.1 to find a third solution of (6.5.7) in the form

$$\mathbf{y}_3 = \mathbf{u}e^{-t} + \left[\begin{array}{r} -1 \\ 1 \\ 0 \end{array}\right] te^{-t}, \tag{6.5.8}$$

where $\mathbf{u}$ is a solution of $(A + I)\mathbf{u} = \mathbf{x}_2$. The augmented matrix of this system is

$$\left[\begin{array}{rrrcr} 4 & 4 & -10 & \vdots & -1 \\ 2 & 2 & -2 & \vdots & 1 \\ 2 & 2 & -4 & \vdots & 0 \end{array}\right],$$

which is row equivalent to

$$\left[\begin{array}{rrrcr} 1 & 1 & 0 & \vdots & 1 \\ 0 & 0 & 1 & \vdots & \frac{1}{2} \\ 0 & 0 & 0 & \vdots & 0 \end{array}\right].$$

Hence, $u_3 = 1/2$ and $u_1 = 1 - u_2$, where u_2 is arbitrary. Choosing $u_2 = 0$ yields

$$\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ \frac{1}{2} \end{bmatrix},$$

and substituting this into (6.5.8) yields the solution

$$\mathbf{y}_3 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \frac{e^{-t}}{2} + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} te^{-t}$$

of (6.5.7).

Since the Wronskian of $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$ at $t = 0$ is

$$\begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & \frac{1}{2} \end{vmatrix} = \frac{1}{2},$$

$\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$ is a fundamental set of solutions of (6.5.7). Therefore the general solution of (6.5.7) is

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^{-t} + c_3 \left(\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \frac{e^{-t}}{2} + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} te^{-t} \right).$$

Theorem 6.5.2 *Suppose the $n \times n$ matrix A has an eigenvalue λ_1 of multiplicity $\geqslant 3$ and the associated eigenspace is one–dimensional; that is, all eigenvectors associated with λ_1 are scalar multiples of the eigenvector $\mathbf{x}$. Then there are infinitely many vectors $\mathbf{u}$ such that*

$$(A - \lambda_1 I)\mathbf{u} = \mathbf{x}, \tag{6.5.9}$$

and, if $\mathbf{u}$ is any such vector, there are infinitely many vectors $\mathbf{v}$ such that

$$(A - \lambda_1 I)\mathbf{v} = \mathbf{u}. \tag{6.5.10}$$

If $\mathbf{u}$ satisfies (6.5.9) and $\mathbf{v}$ satisfies (6.5.10), then

$$\begin{aligned} \mathbf{y}_1 &= \mathbf{x}e^{\lambda_1 t}, \\ \mathbf{y}_2 &= \mathbf{u}e^{\lambda_1 t} + \mathbf{x}te^{\lambda_1 t}, \text{ and} \\ \mathbf{y}_3 &= \mathbf{v}e^{\lambda_1 t} + \mathbf{u}te^{\lambda_1 t} + \mathbf{x}\frac{t^2 e^{\lambda_1 t}}{2} \end{aligned}$$

are linearly independent solutions of $\mathbf{y}' = A\mathbf{y}$.

Again, it's beyond the scope of this book to prove that there are vectors $\mathbf{u}$ and $\mathbf{v}$ that satisfy (6.5.9) and (6.5.10). Theorem 6.5.1 implies that $\mathbf{y}_1$ and $\mathbf{y}_2$ are solutions of $\mathbf{y}' = A\mathbf{y}$. We leave the rest of the proof to you (Exercise 34).

Example 6.5.4 Use Theorem 6.5.2 to find the general solution of

$$\mathbf{y}' = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & -1 \\ 0 & 2 & 2 \end{bmatrix} \mathbf{y}. \tag{6.5.11}$$

Solution The characteristic polynomial of the coefficient matrix A in (6.5.11) is

$$\begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 3-\lambda & -1 \\ 0 & 2 & 2-\lambda \end{vmatrix} = -(\lambda-2)^3.$$

Hence, $\lambda_1 = 2$ is an eigenvalue of multiplicity 3. The associated eigenvectors satisfy $(A-2I)\mathbf{x} = \mathbf{0}$. The augmented matrix of this system is

$$\left[\begin{array}{rrrcr} -1 & 1 & 1 & \vdots & 0 \\ 1 & 1 & -1 & \vdots & 0 \\ 0 & 2 & 0 & \vdots & 0 \end{array}\right],$$

which is row equivalent to

$$\left[\begin{array}{rrrcr} 1 & 0 & -1 & \vdots & 0 \\ 0 & 1 & 0 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{array}\right].$$

Hence, $x_1 = x_3$ and $x_2 = 0$, so the eigenvectors are all scalar multiples of

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Therefore

$$\mathbf{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{2t}$$

is a solution of (6.5.11).

We now find a second solution of (6.5.11) in the form

$$\mathbf{y}_2 = \mathbf{u}e^{2t} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} te^{2t},$$

where $\mathbf{u}$ satisfies $(A-2I)\mathbf{u}=\mathbf{x}_1$. The augmented matrix of this system is

$$\left[\begin{array}{rrrcr} -1 & 1 & 1 & \vdots & 1\\ 1 & 1 & -1 & \vdots & 0\\ 0 & 2 & 0 & \vdots & 1 \end{array}\right],$$

which is row equivalent to

$$\left[\begin{array}{rrrcr} 1 & 0 & -1 & \vdots & -\frac{1}{2}\\ 0 & 1 & 0 & \vdots & \frac{1}{2}\\ 0 & 0 & 0 & \vdots & 0 \end{array}\right].$$

Letting $u_3=0$ yields $u_1=-1/2$ and $u_2=1/2$; hence,

$$\mathbf{u}=\frac{1}{2}\left[\begin{array}{r} -1\\ 1\\ 0 \end{array}\right]$$

and

$$\mathbf{y}_2=\left[\begin{array}{r} -1\\ 1\\ 0 \end{array}\right]\frac{e^{2t}}{2}+\left[\begin{array}{r} 1\\ 0\\ 1 \end{array}\right]te^{2t}$$

is a solution of (6.5.11).

We now find a third solution of (6.5.11) in the form

$$\mathbf{y}_3=\mathbf{v}e^{2t}+\left[\begin{array}{r} -1\\ 1\\ 0 \end{array}\right]\frac{te^{2t}}{2}+\left[\begin{array}{r} 1\\ 0\\ 1 \end{array}\right]\frac{t^2e^{2t}}{2}$$

where $\mathbf{v}$ satisfies $(A-2I)\mathbf{v}=\mathbf{u}$. The augmented matrix of this system is

$$\left[\begin{array}{rrrcr} -1 & 1 & 1 & \vdots & -\frac{1}{2}\\ 1 & 1 & -1 & \vdots & \frac{1}{2}\\ 0 & 2 & 0 & \vdots & 0 \end{array}\right],$$

which is row equivalent to

$$\left[\begin{array}{rrrcr} 1 & 0 & -1 & \vdots & \frac{1}{2}\\ 0 & 1 & 0 & \vdots & 0\\ 0 & 0 & 0 & \vdots & 0 \end{array}\right].$$

Letting $v_3=0$ yields $v_1=1/2$ and $v_2=0$; hence,

$$\mathbf{v}=\frac{1}{2}\left[\begin{array}{r} 1\\ 0\\ 0 \end{array}\right].$$

Therefore

$$\mathbf{y}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \frac{e^{2t}}{2} + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \frac{te^{2t}}{2} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \frac{t^2e^{2t}}{2}$$

is a solution of (6.5.11). Since $\mathbf{y}_1$, $\mathbf{y}_2$, and $\mathbf{y}_3$ are linearly independent by Theorem 6.5.2, they form a fundamental set of solutions of (6.5.11). Therefore the general solution of (6.5.11) is

$$\begin{aligned} \mathbf{y} &= c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{2t} + c_2 \left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \frac{e^{2t}}{2} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} te^{2t} \right) \\ &\quad + c_3 \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \frac{e^{2t}}{2} + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \frac{te^{2t}}{2} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \frac{t^2e^{2t}}{2} \right). \end{aligned}$$

Theorem 6.5.3 *Suppose the $n \times n$ matrix A has an eigenvalue λ_1 of multiplicity $\geqslant 3$ and the associated eigenspace is two–dimensional; that is, all eigenvectors of A associated with λ_1 are linear combinations of two linearly independent eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$. Then there are constants α and β (not both zero) such that if*

$$\mathbf{x}_3 = \alpha \mathbf{x}_1 + \beta \mathbf{x}_2, \tag{6.5.12}$$

then there are infinitely many vectors $\mathbf{u}$ such that

$$(A - \lambda_1 I)\mathbf{u} = \mathbf{x}_3. \tag{6.5.13}$$

If $\mathbf{u}$ satisfies (6.5.13), then

$$\begin{aligned} \mathbf{y}_1 &= \mathbf{x}_1 e^{\lambda_1 t}, \\ \mathbf{y}_2 &= \mathbf{x}_2 e^{\lambda_1 t}, \text{ and} \\ \mathbf{y}_3 &= \mathbf{u} e^{\lambda_1 t} + \mathbf{x}_3 t e^{\lambda_1 t}, \end{aligned} \tag{6.5.14}$$

are linearly independent solutions of $\mathbf{y}' = A\mathbf{y}$.

We omit the proof of this theorem.

Example 6.5.5 Use Theorem 6.5.3 to find the general solution of

$$\mathbf{y}' = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & 1 \\ -1 & 0 & 2 \end{bmatrix} \mathbf{y}. \tag{6.5.15}$$

Solution The characteristic polynomial of the coefficient matrix A in (6.5.15) is

$$\begin{vmatrix} -\lambda & 0 & 1 \\ -1 & 1-\lambda & 1 \\ -1 & 0 & 2-\lambda \end{vmatrix} = -(\lambda-1)^3.$$

Hence, $\lambda_1 = 1$ is an eigenvalue of multiplicity 3. The associated eigenvectors satisfy $(A - I)\mathbf{x} = \mathbf{0}$. The augmented matrix of this system is

$$\left[\begin{array}{rrr|r} -1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 \end{array}\right],$$

which is row equivalent to

$$\left[\begin{array}{rrr|r} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Hence, $x_1 = x_3$ and x_2 is arbitrary, so the eigenvectors are of the form

$$\mathbf{x}_1 = \begin{bmatrix} x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Therefore the vectors

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tag{6.5.16}$$

form a basis for the eigenspace, and

$$\mathbf{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^t \quad \text{and} \quad \mathbf{y}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^t$$

are linearly independent solutions of (6.5.15).

To find a third linearly independent solution of (6.5.15), we must find constants α and β (not both zero) such that the system

$$(A - I)\mathbf{u} = \alpha\mathbf{x}_1 + \beta\mathbf{x}_2 \tag{6.5.17}$$

has a solution $\mathbf{u}$. The augmented matrix of this system is

$$\left[\begin{array}{rrrcc} -1 & 0 & 1 & \vdots & \alpha \\ -1 & 0 & 1 & \vdots & \beta \\ -1 & 0 & 1 & \vdots & \alpha \end{array}\right],$$

which is row equivalent to

$$\left[\begin{array}{rrrcr} 1 & 0 & -1 & \vdots & -\alpha \\ 0 & 0 & 0 & \vdots & \beta-\alpha \\ 0 & 0 & 0 & \vdots & 0 \end{array}\right]. \tag{6.5.18}$$

Therefore (6.5.17) has a solution if and only if $\beta = \alpha$, where α is arbitrary. If $\alpha = \beta = 1$ then (6.5.12) and (6.5.16) yield

$$\mathbf{x}_3 = \mathbf{x}_1 + \mathbf{x}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

and the augmented matrix (6.5.18) becomes

$$\left[\begin{array}{rrrcr} 1 & 0 & -1 & \vdots & -1 \\ 0 & 0 & 0 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{array}\right].$$

This implies that $u_1 = -1 + u_3$, while u_2 and u_3 are arbitrary. Choosing $u_2 = u_3 = 0$ yields

$$\mathbf{u} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}.$$

Therefore (6.5.14) implies that

$$\mathbf{y}_3 = \mathbf{u}e^t + \mathbf{x}_3 te^t = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} e^t + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} te^t$$

is a solution of (6.5.15). Since $\mathbf{y}_1$, $\mathbf{y}_2$, and $\mathbf{y}_3$ are linearly independent by Theorem 6.5.3, they form a fundamental set of solutions for (6.5.15). Therefore the general solution of (6.5.15) is

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^t + c_3 \left(\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} e^t + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} te^t \right). \blacksquare$$

Geometric Properties of Solutions when $n = 2$

We'll now consider the geometric properties of solutions of a 2×2 constant coefficient system

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \tag{6.5.19}$$

under the assumptions of this section; that is, when the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

has a repeated eigenvalue λ_1 and the associated eigenspace is one-dimensional. In this case we know from Theorem 6.5.1 that the general solution of (6.5.19) is

$$\mathbf{y} = c_1\mathbf{x}e^{\lambda_1 t} + c_2(\mathbf{u}e^{\lambda_1 t} + \mathbf{x}te^{\lambda_1 t}), \tag{6.5.20}$$

where $\mathbf{x}$ is an eigenvector of A and $\mathbf{u}$ is any one of the infinitely many solutions of

$$(A - \lambda_1 I)\mathbf{u} = \mathbf{x}. \tag{6.5.21}$$

We assume that $\lambda_1 \neq 0$.

Figure 6.1 Positive and negative half-planes

Let L denote the line through the origin parallel to $\mathbf{x}$. By a *half-line* of L we mean either of the rays obtained by removing the origin from L. Eqn. (6.5.20) is a parametric equation of the half-line of L in the direction of $\mathbf{x}$ if $c_1 > 0$, or of the half-line of L in the direction of $-\mathbf{x}$ if $c_1 < 0$. The origin is the trajectory of the trivial solution $\mathbf{y} \equiv \mathbf{0}$.

Henceforth, we assume that $c_2 \neq 0$. In this case, the trajectory of (6.5.20) can't intersect L, since every point of L is on a trajectory obtained by setting $c_2 = 0$. Therefore the trajectory of (6.5.20) must lie entirely in one of the open half-planes bounded by L, but does not contain any point on L. Since the initial point $(y_1(0), y_2(0))$ defined by $\mathbf{y}(0) = c_1\mathbf{x}_1 + c_2\mathbf{u}$ is on the trajectory, we can determine which half-plane contains the trajectory from the sign of c_2, as shown in Figure 340. For convenience we'll call the half-plane where $c_2 > 0$ the *positive half-plane*. Similarly, the-half plane where $c_2 < 0$ is the *negative half-plane*. You should convince yourself (Exercise 35) that even though there are infinitely many vectors $\mathbf{u}$ that satisfy (6.5.21), they all define the same positive and negative half-planes. In the figures simply regard $\mathbf{u}$ as an arrow pointing to the positive half-plane, since wen't attempted to give $\mathbf{u}$ its proper length or direction in comparison with $\mathbf{x}$. For our purposes here, only the relative orientation of $\mathbf{x}$ and $\mathbf{u}$ is important; that is, whether the positive half-plane is to the right of an observer facing the direction of $\mathbf{x}$ (as in Figures 6.2 and 6.5), or to the left of the observer (as in Figures 6.3 and 6.4).

Multiplying (6.5.20) by $e^{-\lambda_1 t}$ yields

$$e^{-\lambda_1 t}\mathbf{y}(t) = c_1\mathbf{x} + c_2\mathbf{u} + c_2 t\mathbf{x}.$$

Since the last term on the right is dominant when $|t|$ is large, this provides the following information on the direction of $\mathbf{y}(t)$:

(a) Along trajectories in the positive half-plane ($c_2 > 0$), the direction of $\mathbf{y}(t)$ approaches the direction of $\mathbf{x}$ as $t \to \infty$ and the direction of $-\mathbf{x}$ as $t \to -\infty$.

(b) Along trajectories in the negative half-plane ($c_2 < 0$), the direction of $\mathbf{y}(t)$ approaches the direction of $-\mathbf{x}$ as $t \to \infty$ and the direction of $\mathbf{x}$ as $t \to -\infty$.

Since

$$\lim_{t\to\infty} \|\mathbf{y}(t)\| = \infty \quad \text{and} \quad \lim_{t\to-\infty} \mathbf{y}(t) = \mathbf{0} \quad \text{if} \quad \lambda_1 > 0,$$

or

$$\lim_{t-\to\infty} \|\mathbf{y}(t)\| = \infty \quad \text{and} \quad \lim_{t\to\infty} \mathbf{y}(t) = \mathbf{0} \quad \text{if} \quad \lambda_1 < 0,$$

there are four possible patterns for the trajectories of (6.5.19), depending upon the signs of c_2 and λ_1. Figures 6.2-6.5 illustrate these patterns, and reveal the following principle:

If λ_1 and c_2 have the same sign then the direction of the traectory approaches the direction of $-\mathbf{x}$ as $\|\mathbf{y}\| \to 0$ and the direction of $\mathbf{x}$ as $\|\mathbf{y}\| \to \infty$. If λ_1 and c_2 have opposite signs then the direction of the trajectory approaches the direction of $\mathbf{x}$ as $\|\mathbf{y}\| \to 0$ and the direction of $-\mathbf{x}$ as $\|\mathbf{y}\| \to \infty$.

Figure 6.2 Positive eigenvalue; motion away from the origin

Figure 6.3 Positive eigenvalue; motion away from the origin

Figure 6.4 Negative eigenvalue; motion toward the origin

Figure 6.5 Negative eigenvalue; motion toward the origin

6.5 Exercises

In Exercises 1–12 find the general solution.

1. $\mathbf{y}' = \begin{bmatrix} 3 & 4 \\ -1 & 7 \end{bmatrix} \mathbf{y}$

2. $\mathbf{y}' = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} \mathbf{y}$

3. $\mathbf{y}' = \begin{bmatrix} -7 & 4 \\ -1 & -11 \end{bmatrix} \mathbf{y}$

4. $\mathbf{y}' = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix} \mathbf{y}$

5. $\mathbf{y}' = \begin{bmatrix} 4 & 12 \\ -3 & -8 \end{bmatrix} \mathbf{y}$

6. $\mathbf{y}' = \begin{bmatrix} -10 & 9 \\ -4 & 2 \end{bmatrix} \mathbf{y}$

7. $\mathbf{y}' = \begin{bmatrix} -13 & 16 \\ -9 & 11 \end{bmatrix} \mathbf{y}$

8. $\mathbf{y}' = \begin{bmatrix} 0 & 2 & 1 \\ -4 & 6 & 1 \\ 0 & 4 & 2 \end{bmatrix} \mathbf{y}$

9. $\mathbf{y}' = \frac{1}{3} \begin{bmatrix} 1 & 1 & -3 \\ -4 & -4 & 3 \\ -2 & 1 & 0 \end{bmatrix} \mathbf{y}$

10. $\mathbf{y}' = \begin{bmatrix} -1 & 1 & -1 \\ -2 & 0 & 2 \\ -1 & 3 & -1 \end{bmatrix} \mathbf{y}$

11. $\mathbf{y}' = \begin{bmatrix} 4 & -2 & -2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \mathbf{y}$

12. $\mathbf{y}' = \begin{bmatrix} 6 & -5 & 3 \\ 2 & -1 & 3 \\ 2 & 1 & 1 \end{bmatrix} \mathbf{y}$

In Exercises 13–23 *solve the initial value problem.*

13. $\mathbf{y}' = \begin{bmatrix} -11 & 8 \\ -2 & -3 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$

14. $\mathbf{y}' = \begin{bmatrix} 15 & -9 \\ 16 & -9 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$

15. $\mathbf{y}' = \begin{bmatrix} -3 & -4 \\ 1 & -7 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

16. $\mathbf{y}' = \begin{bmatrix} -7 & 24 \\ -6 & 17 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

17. $\mathbf{y}' = \begin{bmatrix} -7 & 3 \\ -3 & -1 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

18. $\mathbf{y}' = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & -2 \\ -1 & -1 & -1 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 6 \\ 5 \\ -7 \end{bmatrix}$

19. $\mathbf{y}' = \begin{bmatrix} -2 & 2 & 1 \\ -2 & 2 & 1 \\ -3 & 3 & 2 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} -6 \\ -2 \\ 0 \end{bmatrix}$

20. $\mathbf{y}' = \begin{bmatrix} -7 & -4 & 4 \\ -1 & 0 & 1 \\ -9 & -5 & 6 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(\mathbf{0}) = \begin{bmatrix} -6 \\ 9 \\ -1 \end{bmatrix}$

21. $\mathbf{y}' = \begin{bmatrix} -1 & -4 & -1 \\ 3 & 6 & 1 \\ -3 & -2 & 3 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(\mathbf{0}) = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$

22. $\mathbf{y}' = \begin{bmatrix} 4 & -8 & -4 \\ -3 & -1 & -3 \\ 1 & -1 & 9 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} -4 \\ 1 \\ -3 \end{bmatrix}$

23. $\mathbf{y}' = \begin{bmatrix} -5 & -1 & 11 \\ -7 & 1 & 13 \\ -4 & 0 & 8 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$

The coefficient matrices in Exercises 24–32 *have eigenvalues of multiplicity* 3. *Find the general solution.*

24. $\mathbf{y}' = \begin{bmatrix} 5 & -1 & 1 \\ -1 & 9 & -3 \\ -2 & 2 & 4 \end{bmatrix} \mathbf{y}$

25. $\mathbf{y}' = \begin{bmatrix} 1 & 10 & -12 \\ 2 & 2 & 3 \\ 2 & -1 & 6 \end{bmatrix} \mathbf{y}$

26. $\mathbf{y}' = \begin{bmatrix} -6 & -4 & -4 \\ 2 & -1 & 1 \\ 2 & 3 & 1 \end{bmatrix} \mathbf{y}$

27. $\mathbf{y}' = \begin{bmatrix} 0 & 2 & -2 \\ -1 & 5 & -3 \\ 1 & 1 & 1 \end{bmatrix} \mathbf{y}$

28. $\mathbf{y}' = \begin{bmatrix} -2 & -12 & 10 \\ 2 & -24 & 11 \\ 2 & -24 & 8 \end{bmatrix} \mathbf{y}$

29. $\mathbf{y}' = \begin{bmatrix} -1 & -12 & 8 \\ 1 & -9 & 4 \\ 1 & -6 & 1 \end{bmatrix} \mathbf{y}$

30. $\mathbf{y}' = \begin{bmatrix} -4 & 0 & -1 \\ -1 & -3 & -1 \\ 1 & 0 & -2 \end{bmatrix} \mathbf{y}$

31. $\mathbf{y}' = \begin{bmatrix} -3 & -3 & 4 \\ 4 & 5 & -8 \\ 2 & 3 & -5 \end{bmatrix} \mathbf{y}$

32. $\mathbf{y}' = \begin{bmatrix} -3 & -1 & 0 \\ 1 & -1 & 0 \\ -1 & -1 & -2 \end{bmatrix} \mathbf{y}$

33. Under the assumptions of Theorem 6.5.1, suppose $\mathbf{u}$ and $\hat{\mathbf{u}}$ are vectors such that

$$(A - \lambda_1 I)\mathbf{u} = \mathbf{x} \quad \text{and} \quad (A - \lambda_1 I)\hat{\mathbf{u}} = \mathbf{x},$$

and let

$$\mathbf{y}_2 = \mathbf{u}e^{\lambda_1 t} + \mathbf{x}te^{\lambda_1 t} \quad \text{and} \quad \hat{\mathbf{y}}_2 = \hat{\mathbf{u}}e^{\lambda_1 t} + \mathbf{x}te^{\lambda_1 t}.$$

Show that $\mathbf{y}_2 - \hat{\mathbf{y}}_2$ is a scalar multiple of $\mathbf{y}_1 = \mathbf{x}e^{\lambda_1 t}$.

34. Under the assumptions of Theorem 6.5.2, let

$$\begin{aligned} \mathbf{y}_1 &= \mathbf{x}e^{\lambda_1 t}, \\ \mathbf{y}_2 &= \mathbf{u}e^{\lambda_1 t} + \mathbf{x}te^{\lambda_1 t}, \text{ and} \\ \mathbf{y}_3 &= \mathbf{v}e^{\lambda_1 t} + \mathbf{u}te^{\lambda_1 t} + \mathbf{x}\frac{t^2e^{\lambda_1 t}}{2}. \end{aligned}$$

Complete the proof of Theorem 6.5.2 by showing that $\mathbf{y}_3$ is a solution of $\mathbf{y}' = A\mathbf{y}$ and that $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$ is linearly independent.

35. Suppose the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

has a repeated eigenvalue λ_1 and the associated eigenspace is one-dimensional. Let $\mathbf{x}$ be a λ_1-eigenvector of A. Show that if $(A - \lambda_1 I)\mathbf{u}_1 = \mathbf{x}$ and $(A - \lambda_1 I)\mathbf{u}_2 = \mathbf{x}$, then $\mathbf{u}_2 - \mathbf{u}_1$ is parallel to $\mathbf{x}$. Conclude from this that all vectors $\mathbf{u}$ such that $(A - \lambda_1 I)\mathbf{u} = \mathbf{x}$ define the same positive and negative half-planes with respect to the line L through the origin parallel to $\mathbf{x}$.

In Exercises 36- 45 *plot trajectories of the given system.*

36. C/G $\mathbf{y}' = \begin{bmatrix} -3 & -1 \\ 4 & 1 \end{bmatrix}\mathbf{y}$

37. C/G $\mathbf{y}' = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}\mathbf{y}$

38. C/G $\mathbf{y}' = \begin{bmatrix} -1 & -3 \\ 3 & 5 \end{bmatrix}\mathbf{y}$

39. C/G $\mathbf{y}' = \begin{bmatrix} -5 & 3 \\ -3 & 1 \end{bmatrix}\mathbf{y}$

40. C/G $\mathbf{y}' = \begin{bmatrix} -2 & -3 \\ 3 & 4 \end{bmatrix}\mathbf{y}$

41. C/G $\mathbf{y}' = \begin{bmatrix} -4 & -3 \\ 3 & 2 \end{bmatrix}\mathbf{y}$

42. C/G $\mathbf{y}' = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix}\mathbf{y}$

43. C/G $\mathbf{y}' = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}\mathbf{y}$

44. C/G $\mathbf{y}' = \begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix}\mathbf{y}$

45. C/G $\mathbf{y}' = \begin{bmatrix} 0 & -4 \\ 1 & -4 \end{bmatrix}\mathbf{y}$

6.6 CONSTANT COEFFICIENT HOMOGENEOUS SYSTEMS III

We now consider the system $\mathbf{y}' = A\mathbf{y}$, where A has a complex eigenvalue $\lambda = \alpha + i\beta$ with $\beta \neq 0$. We continue to assume that A has real entries, so the characteristic polynomial of A has real coefficients. This implies that $\overline{\lambda} = \alpha - i\beta$ is also an eigenvalue of A.

An eigenvector $\mathbf{x}$ of A associated with $\lambda = \alpha + i\beta$ will have complex entries, so we'll write

$$\mathbf{x} = \mathbf{u} + i\mathbf{v}$$

where $\mathbf{u}$ and $\mathbf{v}$ have real entries; that is, $\mathbf{u}$ and $\mathbf{v}$ are the real and imaginary parts of $\mathbf{x}$. Since $A\mathbf{x} = \lambda\mathbf{x}$,

$$A(\mathbf{u} + i\mathbf{v}) = (\alpha + i\beta)(\mathbf{u} + i\mathbf{v}). \tag{6.6.1}$$

Taking complex conjugates here and recalling that A has real entries yields

$$A(\mathbf{u} - i\mathbf{v}) = (\alpha - i\beta)(\mathbf{u} - i\mathbf{v}),$$

which shows that $\mathbf{x} = \mathbf{u} - i\mathbf{v}$ is an eigenvector associated with $\overline{\lambda} = \alpha - i\beta$. The complex conjugate eigenvalues λ and $\overline{\lambda}$ can be separately associated with linearly independent solutions $\mathbf{y}' = A\mathbf{y}$; however, we won't pursue this approach, since solutions obtained in this way turn out to be complex–valued. Instead, we'll obtain solutions of $\mathbf{y}' = A\mathbf{y}$ in the form

$$\mathbf{y} = f_1\mathbf{u} + f_2\mathbf{v} \tag{6.6.2}$$

where f_1 and f_2 are real–valued scalar functions. The next theorem shows how to do this.

Theorem 6.6.1 *Let A be an $n \times n$ matrix with real entries. Let $\lambda = \alpha + i\beta$ $(\beta \neq 0)$ be a complex eigenvalue of A and let $\mathbf{x} = \mathbf{u} + i\mathbf{v}$ be an associated eigenvector, where $\mathbf{u}$ and $\mathbf{v}$ have real components. Then $\mathbf{u}$ and $\mathbf{v}$ are both nonzero and*

$$\mathbf{y}_1 = e^{\alpha t}(\mathbf{u}\cos\beta t - \mathbf{v}\sin\beta t) \quad \text{and} \quad \mathbf{y}_2 = e^{\alpha t}(\mathbf{u}\sin\beta t + \mathbf{v}\cos\beta t),$$

which are the real and imaginary parts of

$$e^{\alpha t}(\cos\beta t + i\sin\beta t)(\mathbf{u} + i\mathbf{v}), \tag{6.6.3}$$

are linearly independent solutions of $\mathbf{y}' = A\mathbf{y}$.

Proof A function of the form (6.6.2) is a solution of $\mathbf{y}' = A\mathbf{y}$ if and only if

$$f_1'\mathbf{u} + f_2'\mathbf{v} = f_1A\mathbf{u} + f_2A\mathbf{v}. \tag{6.6.4}$$

Carrying out the multiplication indicated on the right side of (6.6.1) and collecting the real and imaginary parts of the result yields

$$A(\mathbf{u} + i\mathbf{v}) = (\alpha\mathbf{u} - \beta\mathbf{v}) + i(\alpha\mathbf{v} + \beta\mathbf{u}).$$

Equating real and imaginary parts on the two sides of this equation yields

$$\begin{aligned} A\mathbf{u} &= \alpha\mathbf{u} - \beta\mathbf{v} \\ A\mathbf{v} &= \alpha\mathbf{v} + \beta\mathbf{u}. \end{aligned}$$

We leave it to you (Exercise 25) to show from this that $\mathbf{u}$ and $\mathbf{v}$ are both nonzero. Substituting from these equations into (6.6.4) yields

$$\begin{aligned} f_1'\mathbf{u} + f_2'\mathbf{v} &= f_1(\alpha\mathbf{u} - \beta\mathbf{v}) + f_2(\alpha\mathbf{v} + \beta\mathbf{u}) \\ &= (\alpha f_1 + \beta f_2)\mathbf{u} + (-\beta f_1 + \alpha f_2)\mathbf{v}. \end{aligned}$$

This is true if

$$\begin{aligned} f_1' &= \alpha f_1 + \beta f_2 \\ f_2' &= -\beta f_1 + \alpha f_2, \end{aligned} \quad \text{or, equivalently,} \quad \begin{aligned} f_1' - \alpha f_1 &= \beta f_2 \\ f_2' - \alpha f_2 &= -\beta f_1. \end{aligned}$$

If we let $f_1 = g_1 e^{\alpha t}$ and $f_2 = g_2 e^{\alpha t}$, where g_1 and g_2 are to be determined, then the last two equations become

$$\begin{aligned} g_1' &= \beta g_2 \\ g_2' &= -\beta g_1, \end{aligned}$$

which implies that

$$g_1'' = \beta g_2' = -\beta^2 g_1,$$

so

$$g_1'' + \beta^2 g_1 = 0.$$

The general solution of this equation is

$$g_1 = c_1 \cos\beta t + c_2 \sin\beta t.$$

Moreover, since $g_2 = g_1'/\beta$,

$$g_2 = -c_1 \sin\beta t + c_2 \cos\beta t.$$

Multiplying g_1 and g_2 by $e^{\alpha t}$ shows that

$$\begin{aligned} f_1 &= e^{\alpha t}(c_1 \cos\beta t + c_2 \sin\beta t), \\ f_2 &= e^{\alpha t}(-c_1 \sin\beta t + c_2 \cos\beta t). \end{aligned}$$

Substituting these into (6.6.2) shows that

$$\begin{aligned} \mathbf{y} &= e^{\alpha t}\left[(c_1 \cos\beta t + c_2 \sin\beta t)\mathbf{u} + (-c_1 \sin\beta t + c_2 \cos\beta t)\mathbf{v}\right] \\ &= c_1 e^{\alpha t}(\mathbf{u}\cos\beta t - \mathbf{v}\sin\beta t) + c_2 e^{\alpha t}(\mathbf{u}\sin\beta t + \mathbf{v}\cos\beta t) \end{aligned} \tag{6.6.5}$$

is a solution of $\mathbf{y}' = A\mathbf{y}$ for any choice of the constants c_1 and c_2. In particular, by first taking $c_1 = 1$ and $c_2 = 0$ and then taking $c_1 = 0$ and $c_2 = 1$, we see that $\mathbf{y}_1$ and $\mathbf{y}_2$ are solutions of $\mathbf{y}' = A\mathbf{y}$. We leave it to you to verify that they are, respectively, the real and imaginary parts of (6.6.3) (Exercise 26), and that they are linearly independent (Exercise 27).

Example 6.6.1 Find the general solution of

$$\mathbf{y}' = \begin{bmatrix} 4 & -5 \\ 5 & -2 \end{bmatrix} \mathbf{y}. \tag{6.6.6}$$

Solution The characteristic polynomial of the coefficient matrix A in (6.6.6) is

$$\left|\begin{array}{cc} 4-\lambda & -5 \\ 5 & -2-\lambda \end{array}\right| = (\lambda-1)^2+16.$$

Hence, $\lambda = 1+4i$ is an eigenvalue of A. The associated eigenvectors satisfy $(A-(1+4i)\,I)\,\mathbf{x} = \mathbf{0}$. The augmented matrix of this system is

$$\left[\begin{array}{ccccc} 3-4i & -5 & \vdots & 0 \\ 5 & -3-4i & \vdots & 0 \end{array}\right],$$

which is row equivalent to

$$\left[\begin{array}{cccc} 1 & -\frac{3+4i}{5} & \vdots & 0 \\ 0 & 0 & \vdots & 0 \end{array}\right].$$

Therefore $x_1 = (3+4i)x_2/5$. Taking $x_2 = 5$ yields $x_1 = 3+4i$, so

$$\mathbf{x} = \left[\begin{array}{c} 3+4i \\ 5 \end{array}\right]$$

is an eigenvector. The real and imaginary parts of

$$e^t(\cos 4t + i\sin 4t)\left[\begin{array}{c} 3+4i \\ 5 \end{array}\right]$$

are

$$\mathbf{y}_1 = e^t\left[\begin{array}{c} 3\cos 4t - 4\sin 4t \\ 5\cos 4t \end{array}\right] \quad \text{and} \quad \mathbf{y}_2 = e^t\left[\begin{array}{c} 3\sin 4t + 4\cos 4t \\ 5\sin 4t \end{array}\right],$$

which are linearly independent solutions of (6.6.6). The general solution of (6.6.6) is

$$\mathbf{y} = c_1e^t\left[\begin{array}{c} 3\cos 4t - 4\sin 4t \\ 5\cos 4t \end{array}\right] + c_2e^t\left[\begin{array}{c} 3\sin 4t + 4\cos 4t \\ 5\sin 4t \end{array}\right].$$

Example 6.6.2 Find the general solution of

$$\mathbf{y}' = \left[\begin{array}{cc} -14 & 39 \\ -6 & 16 \end{array}\right]\mathbf{y}. \tag{6.6.7}$$

Solution The characteristic polynomial of the coefficient matrix A in (6.6.7) is

$$\left|\begin{array}{cc} -14-\lambda & 39 \\ -6 & 16-\lambda \end{array}\right| = (\lambda-1)^2+9.$$

Hence, $\lambda = 1{+}3i$ is an eigenvalue of A. The associated eigenvectors satisfy $(A - (1+3i)I)\,\mathbf{x} = \mathbf{0}$. The augmented augmented matrix of this system is

$$\left[\begin{array}{cccc} -15-3i & 39 & \vdots & 0 \\ -6 & 15-3i & \vdots & 0 \end{array}\right],$$

which is row equivalent to

$$\left[\begin{array}{cccc} 1 & \frac{-5+i}{2} & \vdots & 0 \\ 0 & 0 & \vdots & 0 \end{array}\right].$$

Therefore $x_1 = (5-i)/2$. Taking $x_2 = 2$ yields $x_1 = 5 - i$, so

$$\mathbf{x} = \left[\begin{array}{c} 5-i \\ 2 \end{array}\right]$$

is an eigenvector. The real and imaginary parts of

$$e^t(\cos 3t + i \sin 3t)\left[\begin{array}{c} 5-i \\ 2 \end{array}\right]$$

are

$$\mathbf{y}_1 = e^t\left[\begin{array}{c} \sin 3t + 5\cos 3t \\ 2\cos 3t \end{array}\right] \quad \text{and} \quad \mathbf{y}_2 = e^t\left[\begin{array}{c} -\cos 3t + 5\sin 3t \\ 2\sin 3t \end{array}\right],$$

which are linearly independent solutions of (6.6.7). The general solution of (6.6.7) is

$$\mathbf{y} = c_1e^t\left[\begin{array}{c} \sin 3t + 5\cos 3t \\ 2\cos 3t \end{array}\right] + c_2e^t\left[\begin{array}{c} -\cos 3t + 5\sin 3t \\ 2\sin 3t \end{array}\right].$$

Example 6.6.3 Find the general solution of

$$\mathbf{y}' = \left[\begin{array}{ccc} -5 & 5 & 4 \\ -8 & 7 & 6 \\ 1 & 0 & 0 \end{array}\right]\mathbf{y}. \tag{6.6.8}$$

Solution The characteristic polynomial of the coefficient matrix A in (6.6.8) is

$$\left|\begin{array}{ccc} -5-\lambda & 5 & 4 \\ -8 & 7-\lambda & 6 \\ 1 & 0 & -\lambda \end{array}\right| = -(\lambda-2)(\lambda^2+1).$$

Hence, the eigenvalues of A are $\lambda_1 = 2$, $\lambda_2 = i$, and $\lambda_3 = -i$. The augmented matrix of $(A - 2I)\mathbf{x} = \mathbf{0}$ is

$$\left[\begin{array}{ccccc} -7 & 5 & 4 & \vdots & 0 \\ -8 & 5 & 6 & \vdots & 0 \\ 1 & 0 & -2 & \vdots & 0 \end{array}\right],$$

which is row equivalent to

$$\left[\begin{array}{rrrcr} 1 & 0 & -2 & \vdots & 0 \\ 0 & 1 & -2 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{array}\right].$$

Therefore $x_1 = x_2 = 2x_3$. Taking $x_3 = 1$ yields

$$\mathbf{x}_1 = \left[\begin{array}{c} 2 \\ 2 \\ 1 \end{array}\right],$$

so

$$\mathbf{y}_1 = \left[\begin{array}{c} 2 \\ 2 \\ 1 \end{array}\right] e^{2t}$$

is a solution of (6.6.8).

The augmented matrix of $(A - iI)\mathbf{x} = \mathbf{0}$ is

$$\left[\begin{array}{cccccc} -5-i & 5 & 4 & \vdots & 0 \\ -8 & 7-i & 6 & \vdots & 0 \\ 1 & 0 & -i & \vdots & 0 \end{array}\right],$$

which is row equivalent to

$$\left[\begin{array}{ccccc} 1 & 0 & -i & \vdots & 0 \\ 0 & 1 & 1-i & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{array}\right].$$

Therefore $x_1 = ix_3$ and $x_2 = -(1-i)x_3$. Taking $x_3 = 1$ yields the eigenvector

$$\mathbf{x}_2 = \left[\begin{array}{c} i \\ -1+i \\ 1 \end{array}\right].$$

The real and imaginary parts of

$$(\cos t + i \sin t) \left[\begin{array}{c} i \\ -1+i \\ 1 \end{array}\right]$$

are

$$\mathbf{y}_2 = \left[\begin{array}{c} -\sin t \\ -\cos t - \sin t \\ \cos t \end{array}\right] \quad \text{and} \quad \mathbf{y}_3 = \left[\begin{array}{c} \cos t \\ \cos t - \sin t \\ \sin t \end{array}\right],$$

which are solutions of (6.6.8). Since the Wronskian of $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$ at $t = 0$ is

$$\begin{vmatrix} 2 & 0 & 1 \\ 2 & -1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 1,$$

$\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$ is a fundamental set of solutions of (6.6.8). The general solution of (6.6.8) is

$$\mathbf{y} = c_1 \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} -\sin t \\ -\cos t - \sin t \\ \cos t \end{bmatrix} + c_3 \begin{bmatrix} \cos t \\ \cos t - \sin t \\ \sin t \end{bmatrix}.$$

Example 6.6.4 Find the general solution of

$$\mathbf{y}' = \begin{bmatrix} 1 & -1 & -2 \\ 1 & 3 & 2 \\ 1 & -1 & 2 \end{bmatrix} \mathbf{y}. \tag{6.6.9}$$

Solution The characteristic polynomial of the coefficient matrix A in (6.6.9) is

$$\begin{vmatrix} 1-\lambda & -1 & -2 \\ 1 & 3-\lambda & 2 \\ 1 & -1 & 2-\lambda \end{vmatrix} = -(\lambda - 2)\left((\lambda-2)^2 + 4\right).$$

Hence, the eigenvalues of A are $\lambda_1 = 2$, $\lambda_2 = 2 + 2i$, and $\lambda_3 = 2 - 2i$. The augmented matrix of $(A - 2I)\mathbf{x} = \mathbf{0}$ is

$$\begin{bmatrix} -1 & -1 & -2 & \vdots & 0 \\ 1 & 1 & 2 & \vdots & 0 \\ 1 & -1 & 0 & \vdots & 0 \end{bmatrix},$$

which is row equivalent to

$$\begin{bmatrix} 1 & 0 & 1 & \vdots & 0 \\ 0 & 1 & 1 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}.$$

Therefore $x_1 = x_2 = -x_3$. Taking $x_3 = 1$ yields

$$\mathbf{x}_1 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix},$$

so

$$\mathbf{y}_1 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} e^{2t}$$

is a solution of (6.6.9).

The augmented matrix of $(A - (2+2i)I)\,\mathbf{x} = \mathbf{0}$ is

$$\left[\begin{array}{rrrcr} -1-2i & -1 & -2 & \vdots & 0 \\ 1 & 1-2i & 2 & \vdots & 0 \\ 1 & -1 & -2i & \vdots & 0 \end{array}\right],$$

which is row equivalent to

$$\left[\begin{array}{rrrcr} 1 & 0 & -i & \vdots & 0 \\ 0 & 1 & i & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{array}\right].$$

Therefore $x_1 = ix_3$ and $x_2 = -ix_3$. Taking $x_3 = 1$ yields the eigenvector

$$\mathbf{x}_2 = \left[\begin{array}{r} i \\ -i \\ 1 \end{array}\right]$$

The real and imaginary parts of

$$e^{2t}(\cos 2t + i \sin 2t)\left[\begin{array}{r} i \\ -i \\ 1 \end{array}\right]$$

are

$$\mathbf{y}_2 = e^{2t}\left[\begin{array}{r} -\sin 2t \\ \sin 2t \\ \cos 2t \end{array}\right] \quad \text{and} \quad \mathbf{y}_2 = e^{2t}\left[\begin{array}{r} \cos 2t \\ -\cos 2t \\ \sin 2t \end{array}\right],$$

which are solutions of (6.6.9). Since the Wronskian of $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$ at $t = 0$ is

$$\left|\begin{array}{rrr} -1 & 0 & 1 \\ -1 & 0 & -1 \\ 1 & 1 & 0 \end{array}\right| = -2,$$

$\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$ is a fundamental set of solutions of (6.6.9). The general solution of (6.6.9) is

$$\mathbf{y} = c_1\left[\begin{array}{r} -1 \\ -1 \\ 1 \end{array}\right]e^{2t} + c_2e^{2t}\left[\begin{array}{r} -\sin 2t \\ \sin 2t \\ \cos 2t \end{array}\right] + c_3e^{2t}\left[\begin{array}{r} \cos 2t \\ -\cos 2t \\ \sin 2t \end{array}\right].$$

Geometric Properties of Solutions when $n = 2$

We'll now consider the geometric properties of solutions of a 2×2 constant coefficient system

$$\left[\begin{array}{c} y_1' \\ y_2' \end{array}\right] = \left[\begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array}\right]\left[\begin{array}{c} y_1 \\ y_2 \end{array}\right] \tag{6.6.10}$$

under the assumptions of this section; that is, when the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

has a complex eigenvalue $\lambda = \alpha + i\beta$ ($\beta \neq 0$) and $\mathbf{x} = \mathbf{u} + i\mathbf{v}$ is an associated eigenvector, where $\mathbf{u}$ and $\mathbf{v}$ have real components. To describe the trajectories accurately it's necessary to introduce a new rectangular coordinate system in the y_1-y_2 plane. This raises a point that hasn't come up before: It is always possible to choose $\mathbf{x}$ so that $(\mathbf{u}, \mathbf{v}) = 0$. A special effort is required to do this, since not every eigenvector has this property. However, if we know an eigenvector that doesn't, we can multiply it by a suitable complex constant to obtain one that does. To see this, note that if $\mathbf{x}$ is a λ-eigenvector of A and k is an arbitrary real number, then

$$\mathbf{x}_1 = (1 + ik)\mathbf{x} = (1 + ik)(\mathbf{u} + i\mathbf{v}) = (\mathbf{u} - k\mathbf{v}) + i(\mathbf{v} + k\mathbf{u})$$

is also a λ-eigenvector of A, since

$$A\mathbf{x}_1 = A((1 + ik)\mathbf{x}) = (1 + ik)A\mathbf{x} = (1 + ik)\lambda\mathbf{x} = \lambda((1 + ik)\mathbf{x}) = \lambda\mathbf{x}_1.$$

The real and imaginary parts of $\mathbf{x}_1$ are

$$\mathbf{u}_1 = \mathbf{u} - k\mathbf{v} \quad \text{and} \quad \mathbf{v}_1 = \mathbf{v} + k\mathbf{u}, \tag{6.6.11}$$

so

$$(\mathbf{u}_1, \mathbf{v}_1) = (\mathbf{u} - k\mathbf{v}, \mathbf{v} + k\mathbf{u}) = -\left[(\mathbf{u}, \mathbf{v})k^2 + (\|\mathbf{v}\|^2 - \|\mathbf{u}\|^2)k - (\mathbf{u}, \mathbf{v})\right].$$

Therefore $(\mathbf{u}_1, \mathbf{v}_1) = 0$ if

$$(\mathbf{u}, \mathbf{v})k^2 + (\|\mathbf{v}\|^2 - \|\mathbf{u}\|^2)k - (\mathbf{u}, \mathbf{v}) = 0. \tag{6.6.12}$$

If $(\mathbf{u}, \mathbf{v}) \neq 0$ we can use the quadratic formula to find two real values of k such that $(\mathbf{u}_1, \mathbf{v}_1) = 0$ (Exercise 28).

Example 6.6.5 In Example 6.6.1 we found the eigenvector

$$\mathbf{x} = \begin{bmatrix} 3 + 4i \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix} + i\begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

for the matrix of the system (6.6.6). Here $\mathbf{u} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$ are not orthogonal, since $(\mathbf{u}, \mathbf{v}) = 12$. Since $\|\mathbf{v}\|^2 - \|\mathbf{u}\|^2 = -18$, (6.6.12) is equivalent to

$$2k^2 - 3k - 2 = 0.$$

The zeros of this equation are $k_1 = 2$ and $k_2 = -1/2$. Letting $k = 2$ in (6.6.11) yields

$$\mathbf{u}_1 = \mathbf{u} - 2\mathbf{v} = \begin{bmatrix} -5 \\ 5 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_1 = \mathbf{v} + 2\mathbf{u} = \begin{bmatrix} 10 \\ 10 \end{bmatrix},$$

and $(\mathbf{u}_1, \mathbf{v}_1) = 0$. Letting $k = -1/2$ in (6.6.11) yields

$$\mathbf{u}_1 = \mathbf{u} + \frac{\mathbf{v}}{2} = \begin{bmatrix} 5 \\ 5 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_1 = \mathbf{v} - \frac{\mathbf{u}}{2} = \frac{1}{2}\begin{bmatrix} -5 \\ 5 \end{bmatrix},$$

and again $(\mathbf{u}_1, \mathbf{v}_1) = 0$. ■

(The numbers don't always work out as nicely as in this example. You'll need a calculator or computer to do Exercises 29-40.)

Henceforth, we'll assume that $(\mathbf{u}, \mathbf{v}) = 0$. Let $\mathbf{U}$ and $\mathbf{V}$ be unit vectors in the directions of $\mathbf{u}$ and $\mathbf{v}$, respectively; that is, $\mathbf{U} = \mathbf{u}/\|\mathbf{u}\|$ and $\mathbf{V} = \mathbf{v}/\|\mathbf{v}\|$. The new rectangular coordinate system will have the same origin as the y_1-y_2 system. The coordinates of a point in this system will be denoted by (z_1, z_2), where z_1 and z_2 are the displacements in the directions of $\mathbf{U}$ and $\mathbf{V}$, respectively.

From (6.6.5), the solutions of (6.6.10) are given by

$$\mathbf{y} = e^{\alpha t}\left[(c_1 \cos\beta t + c_2 \sin\beta t)\mathbf{u} + (-c_1 \sin\beta t + c_2 \cos\beta t)\mathbf{v}\right]. \qquad (6.6.13)$$

For convenience, let's call the curve traversed by $e^{-\alpha t}\mathbf{y}(t)$ a *shadow trajectory* of (6.6.10). Multiplying (6.6.13) by $e^{-\alpha t}$ yields

$$e^{-\alpha t}\mathbf{y}(t) = z_1(t)\mathbf{U} + z_2(t)\mathbf{V},$$

where

$$\begin{aligned} z_1(t) &= \|\mathbf{u}\|(c_1 \cos\beta t + c_2 \sin\beta t) \\ z_2(t) &= \|\mathbf{v}\|(-c_1 \sin\beta t + c_2 \cos\beta t). \end{aligned}$$

Therefore

$$\frac{(z_1(t))^2}{\|\mathbf{u}\|^2} + \frac{(z_2(t))^2}{\|\mathbf{v}\|^2} = c_1^2 + c_2^2$$

(verify!), which means that the shadow trajectories of (6.6.10) are ellipses centered at the origin, with axes of symmetry parallel to $\mathbf{U}$ and $\mathbf{V}$. Since

$$z_1' = \frac{\beta\|\mathbf{u}\|}{\|\mathbf{v}\|} z_2 \quad \text{and} \quad z_2' = -\frac{\beta\|\mathbf{v}\|}{\|\mathbf{u}\|} z_1,$$

the vector from the origin to a point on the shadow ellipse rotates in the same direction that $\mathbf{V}$ would have to be rotated by $\pi/2$ radians to bring it into coincidence with $\mathbf{U}$ (Figures 6.1 and 6.2).

Figure 6.1 Shadow trajectories traversed clockwise

Figure 6.2 Shadow trajectories traversed counterclockwise

If $\alpha = 0$, then any trajectory of (6.6.10) is a shadow trajectory of (6.6.10); therefore, if λ is purely imaginary, then the trajectories of (6.6.10) are ellipses traversed periodically as indicated in Figures 6.1 and 6.2.

If $\alpha > 0$, then

$$\lim_{t\to\infty} \|\mathbf{y}(t)\| = \infty \quad \text{and} \quad \lim_{t\to-\infty} \mathbf{y}(t) = 0,$$

so the trajectory spirals away from the origin as t varies from $-\infty$ to ∞. The direction of the spiral depends upon the relative orientation of $\mathbf{U}$ and $\mathbf{V}$, as shown in Figures 6.3 and 6.4.

If $\alpha < 0$, then

$$\lim_{t\to-\infty} \|\mathbf{y}(t)\| = \infty \quad \text{and} \quad \lim_{t\to\infty} \mathbf{y}(t) = 0,$$

so the trajectory spirals toward the origin as t varies from $-\infty$ to ∞. Again, the direction of the spiral depends upon the relative orientation of U and V, as shown in Figures 6.5

Figure 6.3 $\alpha > 0$; shadow trajectory spiraling outward

Figure 6.4 $\alpha > 0$; shadow trajectory spiraling outward

Figure 6.5 $\alpha < 0$; shadow trajectory spiraling inward

Figure 6.6 $\alpha < 0$; shadow trajectory spiraling inward

6.6 Exercises

In Exercises 1–16 find the general solution.

1. $\mathbf{y}' = \begin{bmatrix} -1 & 2 \\ -5 & 5 \end{bmatrix} \mathbf{y}$

2. $\mathbf{y}' = \begin{bmatrix} -11 & 4 \\ -26 & 9 \end{bmatrix} \mathbf{y}$

3. $\mathbf{y}' = \begin{bmatrix} 1 & 2 \\ -4 & 5 \end{bmatrix} \mathbf{y}$

4. $\mathbf{y}' = \begin{bmatrix} 5 & -6 \\ 3 & -1 \end{bmatrix} \mathbf{y}$

5. $\mathbf{y}' = \begin{bmatrix} 3 & -3 & 1 \\ 0 & 2 & 2 \\ 5 & 1 & 1 \end{bmatrix} \mathbf{y}$

6. $\mathbf{y}' = \begin{bmatrix} -3 & 3 & 1 \\ 1 & -5 & -3 \\ -3 & 7 & 3 \end{bmatrix} \mathbf{y}$

7. $\mathbf{y}' = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \mathbf{y}$

8. $\mathbf{y}' = \begin{bmatrix} -3 & 1 & -3 \\ 4 & -1 & 2 \\ 4 & -2 & 3 \end{bmatrix} \mathbf{y}$

9. $\mathbf{y}' = \begin{bmatrix} 5 & -4 \\ 10 & 1 \end{bmatrix} \mathbf{y}$

10. $\mathbf{y}' = \frac{1}{3} \begin{bmatrix} 7 & -5 \\ 2 & 5 \end{bmatrix} \mathbf{y}$

11. $\mathbf{y}' = \begin{bmatrix} 3 & 2 \\ -5 & 1 \end{bmatrix} \mathbf{y}$

12. $\mathbf{y}' = \begin{bmatrix} 34 & 52 \\ -20 & -30 \end{bmatrix} \mathbf{y}$

13. $\mathbf{y}' = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & -1 \\ -1 & -2 & -1 \end{bmatrix} \mathbf{y}$

14. $\mathbf{y}' = \begin{bmatrix} 3 & -4 & -2 \\ -5 & 7 & -8 \\ -10 & 13 & -8 \end{bmatrix} \mathbf{y}$

15. $\mathbf{y}' = \begin{bmatrix} 6 & 0 & -3 \\ -3 & 3 & 3 \\ 1 & -2 & 6 \end{bmatrix} \mathbf{y}'$

16. $\mathbf{y}' = \begin{bmatrix} 1 & 2 & -2 \\ 0 & 2 & -1 \\ 1 & 0 & 0 \end{bmatrix} \mathbf{y}'$

In Exercises 17–24 *solve the initial value problem.*

17. $\mathbf{y}' = \begin{bmatrix} 4 & -6 \\ 3 & -2 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$

18. $\mathbf{y}' = \begin{bmatrix} 7 & 15 \\ -3 & 1 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$

19. $\mathbf{y}' = \begin{bmatrix} 7 & -15 \\ 3 & -5 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 17 \\ 7 \end{bmatrix}$

20. $\mathbf{y}' = \frac{1}{6}\begin{bmatrix} 4 & -2 \\ 5 & 2 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

21. $\mathbf{y}' = \begin{bmatrix} 5 & 2 & -1 \\ -3 & 2 & 2 \\ 1 & 3 & 2 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 4 \\ 0 \\ 6 \end{bmatrix}$

22. $\mathbf{y}' = \begin{bmatrix} 4 & 4 & 0 \\ 8 & 10 & -20 \\ 2 & 3 & -2 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 8 \\ 6 \\ 5 \end{bmatrix}$

23. $\mathbf{y}' = \begin{bmatrix} 1 & 15 & -15 \\ -6 & 18 & -22 \\ -3 & 11 & -15 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 15 \\ 17 \\ 10 \end{bmatrix}$

24. $\mathbf{y}' = \begin{bmatrix} 4 & -4 & 4 \\ -10 & 3 & 15 \\ 2 & -3 & 1 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 16 \\ 14 \\ 6 \end{bmatrix}$

25. Suppose an $n \times n$ matrix A with real entries has a complex eigenvalue $\lambda = \alpha + i\beta$ $(\beta \neq 0)$ with associated eigenvector $\mathbf{x} = \mathbf{u} + i\mathbf{v}$, where $\mathbf{u}$ and $\mathbf{v}$ have real components. Show that $\mathbf{u}$ and $\mathbf{v}$ are both nonzero.

26. Verify that

$$\mathbf{y}_1 = e^{\alpha t}(\mathbf{u}\cos\beta t - \mathbf{v}\sin\beta t) \quad \text{and} \quad \mathbf{y}_2 = e^{\alpha t}(\mathbf{u}\sin\beta t + \mathbf{v}\cos\beta t),$$

are the real and imaginary parts of

$$e^{\alpha t}(\cos\beta t + i\sin\beta t)(\mathbf{u} + i\mathbf{v}).$$

27. Show that if the vectors $\mathbf{u}$ and $\mathbf{v}$ are not both $\mathbf{0}$ and $\beta \neq 0$ then the vector functions

$$\mathbf{y}_1 = e^{\alpha t}(\mathbf{u}\cos\beta t - \mathbf{v}\sin\beta t) \quad \text{and} \quad \mathbf{y}_2 = e^{\alpha t}(\mathbf{u}\sin\beta t + \mathbf{v}\cos\beta t)$$

are linearly independent on every interval. HINT: *There are two cases to consider:* (i) $\{\mathbf{u}, \mathbf{v}\}$ *linearly independent, and* (ii) $\{\mathbf{u}, \mathbf{v}\}$ *linearly dependent. In either case, exploit the the linear independence of* $\{\cos\beta t, \sin\beta t\}$ *on every interval.*

28. Suppose $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ are not orthogonal; that is, $(\mathbf{u}, \mathbf{v}) \neq 0$.

(a) Show that the quadratic equation

$$(\mathbf{u}, \mathbf{v})k^2 + (\|\mathbf{v}\|^2 - \|\mathbf{u}\|^2)k - (\mathbf{u}, \mathbf{v}) = 0$$

has a positive root k_1 and a negative root $k_2 = -1/k_1$.

(b) Let $\mathbf{u}_1^{(1)} = \mathbf{u} - k_1\mathbf{v}$, $\mathbf{v}_1^{(1)} = \mathbf{v} + k_1\mathbf{u}$, $\mathbf{u}_1^{(2)} = \mathbf{u} - k_2\mathbf{v}$, and $\mathbf{v}_1^{(2)} = \mathbf{v} + k_2\mathbf{u}$, so that $(\mathbf{u}_1^{(1)}, \mathbf{v}_1^{(1)}) = (\mathbf{u}_1^{(2)}, \mathbf{v}_1^{(2)}) = 0$, from the discussion given above. Show that

$$\mathbf{u}_1^{(2)} = \frac{\mathbf{v}_1^{(1)}}{k_1} \quad \text{and} \quad \mathbf{v}_1^{(2)} = -\frac{\mathbf{u}_1^{(1)}}{k_1}.$$

(c) Let $\mathbf{U}_1$, $\mathbf{V}_1$, $\mathbf{U}_2$, and $\mathbf{V}_2$ be unit vectors in the directions of $\mathbf{u}_1^{(1)}$, $\mathbf{v}_1^{(1)}$, $\mathbf{u}_1^{(2)}$, and $\mathbf{v}_1^{(2)}$, respectively. Conclude from **(a)** that $\mathbf{U}_2 = \mathbf{V}_1$ and $\mathbf{V}_2 = -\mathbf{U}_1$, and that therefore the counterclockwise angles from $\mathbf{U}_1$ to $\mathbf{V}_1$ and from $\mathbf{U}_2$ to $\mathbf{V}_2$ are both $\pi/2$ or both $-\pi/2$.

In Exercises 29-32 find vectors $\mathbf{U}$ *and* $\mathbf{V}$ *parallel to the axes of symmetry of the trajectories, and plot some typical trajectories.*

29. C/G $\mathbf{y}' = \begin{bmatrix} 3 & -5 \\ 5 & -3 \end{bmatrix}\mathbf{y}$ **30.** C/G $\mathbf{y}' = \begin{bmatrix} -15 & 10 \\ -25 & 15 \end{bmatrix}\mathbf{y}$

31. C/G $\mathbf{y}' = \begin{bmatrix} -4 & 8 \\ -4 & 4 \end{bmatrix}\mathbf{y}$ **32.** C/G $\mathbf{y}' = \begin{bmatrix} -3 & -15 \\ 3 & 3 \end{bmatrix}\mathbf{y}$

In Exercises 33-40 find vectors $\mathbf{U}$ *and* $\mathbf{V}$ *parallel to the axes of symmetry of the shadow trajectories, and plot a typical trajectory.*

33. C/G $\mathbf{y}' = \begin{bmatrix} -5 & 6 \\ -12 & 7 \end{bmatrix}\mathbf{y}$ **34.** C/G $\mathbf{y}' = \begin{bmatrix} 5 & -12 \\ 6 & -7 \end{bmatrix}\mathbf{y}$

35. C/G $\mathbf{y}' = \begin{bmatrix} 4 & -5 \\ 9 & -2 \end{bmatrix}\mathbf{y}$ **36.** C/G $\mathbf{y}' = \begin{bmatrix} -4 & 9 \\ -5 & 2 \end{bmatrix}\mathbf{y}$

37. C/G $\mathbf{y}' = \begin{bmatrix} -1 & 10 \\ -10 & -1 \end{bmatrix}\mathbf{y}$ **38.** C/G $\mathbf{y}' = \begin{bmatrix} -1 & -5 \\ 20 & -1 \end{bmatrix}\mathbf{y}$

39. C/G $\mathbf{y}' = \begin{bmatrix} -7 & 10 \\ -10 & 9 \end{bmatrix}\mathbf{y}$ **40.** C/G $\mathbf{y}' = \begin{bmatrix} -7 & 6 \\ -12 & 5 \end{bmatrix}\mathbf{y}$

6.7 VARIATION OF PARAMETERS FOR NONHOMOGENEOUS LINEAR SYSTEMS

We now consider the nonhomogeneous linear system

$$\mathbf{y}' = A(t)\mathbf{y} + \mathbf{f}(t),$$

where A is an $n \times n$ matrix function and $\mathbf{f}$ is an n-vector forcing function. Associated with this system is the *complementary system* $\mathbf{y}' = A(t)\mathbf{y}$.

The next theorem is analogous to Theorems **??** and **??**. It shows how to find the general solution of $\mathbf{y}' = A(t)\mathbf{y} + \mathbf{f}(t)$ if we know a particular solution of $\mathbf{y}' = A(t)\mathbf{y} + \mathbf{f}(t)$ and a fundamental set of solutions of the complementary system. We leave the proof as an exercise (Exercise 21).

Theorem 6.7.1 *Suppose the $n \times n$ matrix function A and the n-vector function $\mathbf{f}$ are continuous on (a, b). Let $\mathbf{y}_p$ be a particular solution of $\mathbf{y}' = A(t)\mathbf{y} + \mathbf{f}(t)$ on (a, b), and let $\{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n\}$ be a fundamental set of solutions of the complementary equation $\mathbf{y}' = A(t)\mathbf{y}$ on (a, b). Then $\mathbf{y}$ is a solution of $\mathbf{y}' = A(t)\mathbf{y} + \mathbf{f}(t)$ on (a, b) if and only if*

$$\mathbf{y} = \mathbf{y}_p + c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + \cdots + c_n\mathbf{y}_n,$$

where $c_1, c_2, \dots, c_n$ are constants.

Finding a Particular Solution of a Nonhomogeneous System

We now discuss an extension of the method of variation of parameters to linear nonhomogeneous systems. This method will produce a particular solution of a nonhomogenous system $\mathbf{y}' = A(t)\mathbf{y} + \mathbf{f}(t)$ provided that we know a fundamental matrix for the complementary system. To derive the method, suppose Y is a fundamental matrix for the complementary system; that is,

$$Y = \begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{bmatrix},$$

where

$$\mathbf{y}_1 = \begin{bmatrix} y_{11} \\ y_{21} \\ \vdots \\ y_{n1} \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} y_{12} \\ y_{22} \\ \vdots \\ y_{n2} \end{bmatrix}, \quad \dots, \quad \mathbf{y}_n = \begin{bmatrix} y_{1n} \\ y_{2n} \\ \vdots \\ y_{nn} \end{bmatrix}$$

is a fundamental set of solutions of the complementary system. In Section 10.3 we saw that $Y' = A(t)Y$. We seek a particular solution of

$$\mathbf{y}' = A(t)\mathbf{y} + \mathbf{f}(t) \tag{6.7.1}$$

of the form

$$\mathbf{y}_p = Y\mathbf{u}, \tag{6.7.2}$$

where $\mathbf{u}$ is to be determined. Differentiating (6.7.2) yields

$$\begin{aligned} \mathbf{y}_p' &= Y'\mathbf{u} + Y\mathbf{u}' \\ &= AY\mathbf{u} + Y\mathbf{u}' \text{ (since } Y' = AY) \\ &= A\mathbf{y}_p + Y\mathbf{u}' \text{ (since } Y\mathbf{u} = \mathbf{y}_p). \end{aligned}$$

Comparing this with (6.7.1) shows that $\mathbf{y}_p = Y\mathbf{u}$ is a solution of (6.7.1) if and only if

$$Y\mathbf{u}' = \mathbf{f}.$$

Thus, we can find a particular solution $\mathbf{y}_p$ by solving this equation for $\mathbf{u}'$, integrating to obtain $\mathbf{u}$, and computing $Y\mathbf{u}$. We can take all constants of integration to be zero, since any particular solution will suffice.

Exercise 22 sketches a proof that this method is analogous to the method of variation of parameters discussed in Sections 5.7 and 9.4 for scalar linear equations.

Example 6.7.1

(a) Find a particular solution of the system

$$\mathbf{y}' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\mathbf{y} + \begin{bmatrix} 2e^{4t} \\ e^{4t} \end{bmatrix}, \tag{6.7.3}$$

which we considered in Example 6.2.1.

(b) Find the general solution of (6.7.3).

SOLUTION(**a**) The complementary system is

$$\mathbf{y}' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\mathbf{y}. \tag{6.7.4}$$

The characteristic polynomial of the coefficient matrix is

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = (\lambda+1)(\lambda-3).$$

Using the method of Section 10.4, we find that

$$\mathbf{y}_1 = \begin{bmatrix} e^{3t} \\ e^{3t} \end{bmatrix} \quad \text{and} \quad \mathbf{y}_2 = \begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix}$$

are linearly independent solutions of (6.7.4). Therefore

$$Y = \begin{bmatrix} e^{3t} & e^{-t} \\ e^{3t} & -e^{-t} \end{bmatrix}$$

is a fundamental matrix for (6.7.4). We seek a particular solution $\mathbf{y}_p = Y\mathbf{u}$ of (6.7.3), where $Y\mathbf{u}' = \mathbf{f}$; that is,

$$\begin{bmatrix} e^{3t} & e^{-t} \\ e^{3t} & -e^{-t} \end{bmatrix}\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 2e^{4t} \\ e^{4t} \end{bmatrix}.$$

The determinant of Y is the Wronskian

$$\begin{vmatrix} e^{3t} & e^{-t} \\ e^{3t} & -e^{-t} \end{vmatrix} = -2e^{2t}.$$

By Cramer's rule,

$$\begin{aligned} u_1' &= -\frac{1}{2e^{2t}}\begin{vmatrix} 2e^{4t} & e^{-t} \\ e^{4t} & -e^{-t} \end{vmatrix} = \frac{3e^{3t}}{2e^{2t}} = \frac{3}{2}e^{t}, \\ u_2' &= -\frac{1}{2e^{2t}}\begin{vmatrix} e^{3t} & 2e^{4t} \\ e^{3t} & e^{4t} \end{vmatrix} = \frac{e^{7t}}{2e^{2t}} = \frac{1}{2}e^{5t}. \end{aligned}$$

Therefore

$$\mathbf{u}' = \frac{1}{2}\begin{bmatrix} 3e^{t} \\ e^{5t} \end{bmatrix}.$$

Integrating and taking the constants of integration to be zero yields

$$\mathbf{u} = \frac{1}{10}\begin{bmatrix} 15e^{t} \\ e^{5t} \end{bmatrix},$$

so

$$\mathbf{y}_p = Y\mathbf{u} = \frac{1}{10}\begin{bmatrix} e^{3t} & e^{-t} \\ e^{3t} & -e^{-t} \end{bmatrix}\begin{bmatrix} 15e^{t} \\ e^{5t} \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 8e^{4t} \\ 7e^{4t} \end{bmatrix}$$

is a particular solution of (6.7.3).

SOLUTION(**b**) From Theorem 6.7.1, the general solution of (6.7.3) is

$$\mathbf{y} = \mathbf{y}_p + c_1\mathbf{y}_1 + c_2\mathbf{y}_2 = \frac{1}{5}\begin{bmatrix} 8e^{4t} \\ 7e^{4t} \end{bmatrix} + c_1\begin{bmatrix} e^{3t} \\ e^{3t} \end{bmatrix} + c_2\begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix}, \tag{6.7.5}$$

which can also be written as

$$\mathbf{y} = \frac{1}{5}\begin{bmatrix} 8e^{4t} \\ 7e^{4t} \end{bmatrix} + \begin{bmatrix} e^{3t} & e^{-t} \\ e^{3t} & -e^{-t} \end{bmatrix}\mathbf{c},$$

where $\mathbf{c}$ is an arbitrary constant vector.

Writing (6.7.5) in terms of coordinates yields

$$\begin{aligned} y_1 &= \frac{8}{5}e^{4t} + c_1e^{3t} + c_2e^{-t} \\ y_2 &= \frac{7}{5}e^{4t} + c_1e^{3t} - c_2e^{-t}, \end{aligned}$$

so our result is consistent with Example 6.2.1. ■.

If A isn't a constant matrix, it's usually difficult to find a fundamental set of solutions for the system $\mathbf{y}' = A(t)\mathbf{y}$. It is beyond the scope of this text to discuss methods for doing this. Therefore, in the following examples and in the exercises involving systems with variable coefficient matrices we'll provide fundamental matrices for the complementary systems without explaining how they were obtained.

Example 6.7.2 Find a particular solution of

$$\mathbf{y}' = \begin{bmatrix} 2 & 2e^{-2t} \\ 2e^{2t} & 4 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \tag{6.7.6}$$

given that

$$Y = \begin{bmatrix} e^{4t} & -1 \\ e^{6t} & e^{2t} \end{bmatrix}$$

is a fundamental matrix for the complementary system.

Solution We seek a particular solution $\mathbf{y}_p = Y\mathbf{u}$ of (6.7.6) where $Y\mathbf{u}' = \mathbf{f}$; that is,

$$\begin{bmatrix} e^{4t} & -1 \\ e^{6t} & e^{2t} \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The determinant of Y is the Wronskian

$$\begin{vmatrix} e^{4t} & -1 \\ e^{6t} & e^{2t} \end{vmatrix} = 2e^{6t}.$$

By Cramer's rule,

$$\begin{aligned} u_1' &= \frac{1}{2e^{6t}} \begin{vmatrix} 1 & -1 \\ 1 & e^{2t} \end{vmatrix} &= \frac{e^{2t}+1}{2e^{6t}} &= \frac{e^{-4t}+e^{-6t}}{2} \\ u_2' &= \frac{1}{2e^{6t}} \begin{vmatrix} e^{4t} & 1 \\ e^{6t} & 1 \end{vmatrix} &= \frac{e^{4t}-e^{6t}}{2e^{6t}} &= \frac{e^{-2t}-1}{2}. \end{aligned}$$

Therefore

$$\mathbf{u}' = \frac{1}{2} \begin{bmatrix} e^{-4t}+e^{-6t} \\ e^{-2t}-1 \end{bmatrix}.$$

Integrating and taking the constants of integration to be zero yields

$$\mathbf{u} = -\frac{1}{24} \begin{bmatrix} 3e^{-4t}+2e^{-6t} \\ 6e^{-2t}+12t \end{bmatrix},$$

so

$$\mathbf{y}_p = Y\mathbf{u} = -\frac{1}{24} \begin{bmatrix} e^{4t} & -1 \\ e^{6t} & e^{2t} \end{bmatrix} \begin{bmatrix} 3e^{-4t}+2e^{-6t} \\ 6e^{-2t}+12t \end{bmatrix} = \frac{1}{24} \begin{bmatrix} 4e^{-2t}+12t-3 \\ -3e^{2t}(4t+1)-8 \end{bmatrix}$$

is a particular solution of (6.7.6).

Example 6.7.3 Find a particular solution of

$$\mathbf{y}' = -\frac{2}{t^2} \begin{bmatrix} t & -3t^2 \\ 1 & -2t \end{bmatrix} \mathbf{y} + t^2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \tag{6.7.7}$$

given that

$$Y = \begin{bmatrix} 2t & 3t^2 \\ 1 & 2t \end{bmatrix}$$

is a fundamental matrix for the complementary system on $(-\infty, 0)$ and $(0, \infty)$.

Solution We seek a particular solution $\mathbf{y}_p = Y\mathbf{u}$ of (6.7.7) where $Y\mathbf{u}' = \mathbf{f}$; that is,

$$\begin{bmatrix} 2t & 3t^2 \\ 1 & 2t \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} t^2 \\ t^2 \end{bmatrix}.$$

The determinant of Y is the Wronskian

$$\begin{vmatrix} 2t & 3t^2 \\ 1 & 2t \end{vmatrix} = t^2.$$

By Cramer's rule,

$$\begin{aligned} u_1' &= \frac{1}{t^2}\begin{vmatrix} t^2 & 3t^2 \\ t^2 & 2t \end{vmatrix} &=& \frac{2t^3-3t^4}{t^2} &=& 2t-3t^2, \\ u_2' &= \frac{1}{t^2}\begin{vmatrix} 2t & t^2 \\ 1 & t^2 \end{vmatrix} &=& \frac{2t^3-t^2}{t^2} &=& 2t-1. \end{aligned}$$

Therefore

$$\mathbf{u}' = \begin{bmatrix} 2t-3t^2 \\ 2t-1 \end{bmatrix}.$$

Integrating and taking the constants of integration to be zero yields

$$\mathbf{u} = \begin{bmatrix} t^2-t^3 \\ t^2-t \end{bmatrix},$$

so

$$\mathbf{y}_p = Y\mathbf{u} = \begin{bmatrix} 2t & 3t^2 \\ 1 & 2t \end{bmatrix} \begin{bmatrix} t^2-t^3 \\ t^2-t \end{bmatrix} = \begin{bmatrix} t^3(t-1) \\ t^2(t-1) \end{bmatrix}$$

is a particular solution of (6.7.7).

Example 6.7.4

(a) Find a particular solution of

$$\mathbf{y}' = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} \mathbf{y} + \begin{bmatrix} e^t \\ 0 \\ e^{-t} \end{bmatrix}. \tag{6.7.8}$$

(b) Find the general solution of (6.7.8).

SOLUTION(**a**) The complementary system for (6.7.8) is

$$\mathbf{y}' = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} \mathbf{y}. \tag{6.7.9}$$

The characteristic polynomial of the coefficient matrix is

$$\begin{vmatrix} 2-\lambda & -1 & -1 \\ 1 & -\lambda & -1 \\ 1 & -1 & -\lambda \end{vmatrix} = -\lambda(\lambda-1)^2.$$

Using the method of Section 10.4, we find that

$$\mathbf{y}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} e^t \\ e^t \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{y}_3 = \begin{bmatrix} e^t \\ 0 \\ e^t \end{bmatrix}$$

are linearly independent solutions of (6.7.9). Therefore

$$Y = \begin{bmatrix} 1 & e^t & e^t \\ 1 & e^t & 0 \\ 1 & 0 & e^t \end{bmatrix}$$

is a fundamental matrix for (6.7.9). We seek a particular solution $\mathbf{y}_p = Y\mathbf{u}$ of (6.7.8), where $Y\mathbf{u}' = \mathbf{f}$; that is,

$$\begin{bmatrix} 1 & e^t & e^t \\ 1 & e^t & 0 \\ 1 & 0 & e^t \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \\ u_3' \end{bmatrix} = \begin{bmatrix} e^t \\ 0 \\ e^{-t} \end{bmatrix}.$$

The determinant of Y is the Wronskian

$$\begin{vmatrix} 1 & e^t & e^t \\ 1 & e^t & 0 \\ 1 & 0 & e^t \end{vmatrix} = -e^{2t}.$$

Thus, by Cramer's rule,

$$\begin{aligned} u_1' &= -\frac{1}{e^{2t}} \begin{vmatrix} e^t & e^t & e^t \\ 0 & e^t & 0 \\ e^{-t} & 0 & e^t \end{vmatrix} &= -\frac{e^{3t}-e^t}{e^{2t}} &= e^{-t}-e^t \\ u_2' &= -\frac{1}{e^{2t}} \begin{vmatrix} 1 & e^t & e^t \\ 1 & 0 & 0 \\ 1 & e^{-t} & e^t \end{vmatrix} &= -\frac{1-e^{2t}}{e^{2t}} &= 1-e^{-2t} \\ u_3' &= -\frac{1}{e^{2t}} \begin{vmatrix} 1 & e^t & e^t \\ 1 & e^t & 0 \\ 1 & 0 & e^{-t} \end{vmatrix} &= \frac{e^{2t}}{e^{2t}} &= 1. \end{aligned}$$

Therefore

$$\mathbf{u}' = \begin{bmatrix} e^{-t}-e^t \\ 1-e^{-2t} \\ 1 \end{bmatrix}.$$

Integrating and taking the constants of integration to be zero yields

$$\mathbf{u} = \begin{bmatrix} -e^{t} - e^{-t} \\ \dfrac{e^{-2t}}{2} + t \\ t \end{bmatrix},$$

so

$$\mathbf{y}_p = Y\mathbf{u} = \begin{bmatrix} 1 & e^{t} & e^{t} \\ 1 & e^{t} & 0 \\ 1 & 0 & e^{t} \end{bmatrix} \begin{bmatrix} -e^{t} - e^{-t} \\ \dfrac{e^{-2t}}{2} + t \\ t \end{bmatrix} = \begin{bmatrix} e^{t}(2t-1) - \dfrac{e^{-t}}{2} \\ e^{t}(t-1) - \dfrac{e^{-t}}{2} \\ e^{t}(t-1) - e^{-t} \end{bmatrix}$$

is a particular solution of (6.7.8).

SOLUTION(a) From Theorem 6.7.1 the general solution of (6.7.8) is

$$\mathbf{y} = \mathbf{y}_p + c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + c_3\mathbf{y}_3 = \begin{bmatrix} e^{t}(2t-1) - \dfrac{e^{-t}}{2} \\ e^{t}(t-1) - \dfrac{e^{-t}}{2} \\ e^{t}(t-1) - e^{-t} \end{bmatrix} + c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} e^{t} \\ e^{t} \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} e^{t} \\ 0 \\ e^{t} \end{bmatrix},$$

which can be written as

$$\mathbf{y} = \mathbf{y}_p + Y\mathbf{c} = \begin{bmatrix} e^{t}(2t-1) - \dfrac{e^{-t}}{2} \\ e^{t}(t-1) - \dfrac{e^{-t}}{2} \\ e^{t}(t-1) - e^{-t} \end{bmatrix} + \begin{bmatrix} 1 & e^{t} & e^{t} \\ 1 & e^{t} & 0 \\ 1 & 0 & e^{t} \end{bmatrix} \mathbf{c}$$

where $\mathbf{c}$ is an arbitrary constant vector.

Example 6.7.5 Find a particular solution of

$$\mathbf{y}' = \frac{1}{2} \begin{bmatrix} 3 & e^{-t} & -e^{2t} \\ 0 & 6 & 0 \\ -e^{-2t} & e^{-3t} & -1 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 1 \\ e^{t} \\ e^{-t} \end{bmatrix}, \qquad (6.7.10)$$

given that

$$Y = \begin{bmatrix} e^{t} & 0 & e^{2t} \\ 0 & e^{3t} & e^{3t} \\ e^{-t} & 1 & 0 \end{bmatrix}$$

is a fundamental matrix for the complementary system.

Solution We seek a particular solution of (6.7.10) in the form $\mathbf{y}_p = Y\mathbf{u}$, where $Y\mathbf{u}' = \mathbf{f}$; that is,

$$\begin{bmatrix} e^t & 0 & e^{2t} \\ 0 & e^{3t} & e^{3t} \\ e^{-t} & 1 & 0 \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \\ u_3' \end{bmatrix} = \begin{bmatrix} 1 \\ e^t \\ e^{-t} \end{bmatrix}.$$

The determinant of Y is the Wronskian

$$\begin{vmatrix} e^t & 0 & e^{2t} \\ 0 & e^{3t} & e^{3t} \\ e^{-t} & 1 & 0 \end{vmatrix} = -2e^{4t}.$$

By Cramer's rule,

$$\begin{aligned}
u_1' &= -\frac{1}{2e^{4t}} \begin{vmatrix} 1 & 0 & e^{2t} \\ e^t & e^{3t} & e^{3t} \\ e^{-t} & 1 & 0 \end{vmatrix} = \frac{e^{4t}}{2e^{4t}} = \frac{1}{2} \\
u_2' &= -\frac{1}{2e^{4t}} \begin{vmatrix} e^t & 1 & e^{2t} \\ 0 & e^t & e^{3t} \\ e^{-t} & e^{-t} & 0 \end{vmatrix} = \frac{e^{3t}}{2e^{4t}} = \frac{1}{2}e^{-t} \\
u_3' &= -\frac{1}{2e^{4t}} \begin{vmatrix} e^t & 0 & 1 \\ 0 & e^{3t} & e^t \\ e^{-t} & 1 & e^{-t} \end{vmatrix} = -\frac{e^{3t} - 2e^{2t}}{2e^{4t}} = \frac{2e^{-2t} - e^{-t}}{2}.
\end{aligned}$$

Therefore

$$\mathbf{u}' = \frac{1}{2} \begin{bmatrix} 1 \\ e^{-t} \\ 2e^{-2t} - e^{-t} \end{bmatrix}.$$

Integrating and taking the constants of integration to be zero yields

$$\mathbf{u} = \frac{1}{2} \begin{bmatrix} t \\ -e^{-t} \\ e^{-t} - e^{-2t} \end{bmatrix},$$

so

$$\mathbf{y}_p = Y\mathbf{u} = \frac{1}{2} \begin{bmatrix} e^t & 0 & e^{2t} \\ 0 & e^{3t} & e^{3t} \\ e^{-t} & 1 & 0 \end{bmatrix} \begin{bmatrix} t \\ -e^{-t} \\ e^{-t} - e^{-2t} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} e^t(t+1) - 1 \\ -e^t \\ e^{-t}(t-1) \end{bmatrix}$$

is a particular solution of (6.7.10).

6.7 Exercises

In Exercises 1–10 find a particular solution.

1. $\mathbf{y}' = \begin{bmatrix} -1 & -4 \\ -1 & -1 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 21e^{4t} \\ 8e^{-3t} \end{bmatrix}$

2. $\mathbf{y}' = \frac{1}{5}\begin{bmatrix} -4 & 3 \\ -2 & -11 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 50e^{3t} \\ 10e^{-3t} \end{bmatrix}$

3. $\mathbf{y}' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 1 \\ t \end{bmatrix}$

4. $\mathbf{y}' = \begin{bmatrix} -4 & -3 \\ 6 & 5 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 2 \\ -2e^{t} \end{bmatrix}$

5. $\mathbf{y}' = \begin{bmatrix} -6 & -3 \\ 1 & -2 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 4e^{-3t} \\ 4e^{-5t} \end{bmatrix}$

6. $\mathbf{y}' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 1 \\ t \end{bmatrix}$

7. $\mathbf{y}' = \begin{bmatrix} 3 & 1 & -1 \\ 3 & 5 & 1 \\ -6 & 2 & 4 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix}$

8. $\mathbf{y}' = \begin{bmatrix} 3 & -1 & -1 \\ -2 & 3 & 2 \\ 4 & -1 & -2 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 1 \\ e^{t} \\ e^{t} \end{bmatrix}$

9. $\mathbf{y}' = \begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix} \mathbf{y} + \begin{bmatrix} e^{t} \\ e^{-5t} \\ e^{t} \end{bmatrix}$

10. $\mathbf{y}' = \frac{1}{3}\begin{bmatrix} 1 & 1 & -3 \\ -4 & -4 & 3 \\ -2 & 1 & 0 \end{bmatrix} \mathbf{y} + \begin{bmatrix} e^{t} \\ e^{t} \\ e^{t} \end{bmatrix}$

In Exercises 11–20 *find a particular solution, given that* Y *is a fundamental matrix for the complementary system.*

11. $\mathbf{y}' = \frac{1}{t}\begin{bmatrix} 1 & t \\ -t & 1 \end{bmatrix} \mathbf{y} + t\begin{bmatrix} \cos t \\ \sin t \end{bmatrix}; \quad Y = t\begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$

12. $\mathbf{y}' = \frac{1}{t}\begin{bmatrix} 1 & t \\ t & 1 \end{bmatrix} \mathbf{y} + \begin{bmatrix} t \\ t^2 \end{bmatrix}; \quad Y = t\begin{bmatrix} e^{t} & e^{-t} \\ e^{t} & -e^{-t} \end{bmatrix}$

13. $\mathbf{y}' = \frac{1}{t^2-1}\begin{bmatrix} t & -1 \\ -1 & t \end{bmatrix} \mathbf{y} + t\begin{bmatrix} 1 \\ -1 \end{bmatrix}; \quad Y = \begin{bmatrix} t & 1 \\ 1 & t \end{bmatrix}$

14. $\mathbf{y}' = \frac{1}{3}\begin{bmatrix} 1 & -2e^{-t} \\ 2e^{t} & -1 \end{bmatrix} \mathbf{y} + \begin{bmatrix} e^{2t} \\ e^{-2t} \end{bmatrix}; \quad Y = \begin{bmatrix} 2 & e^{-t} \\ e^{t} & 2 \end{bmatrix}$

15. $\mathbf{y}' = \frac{1}{2t^4}\begin{bmatrix} 3t^3 & t^6 \\ 1 & -3t^3 \end{bmatrix} \mathbf{y} + \frac{1}{t}\begin{bmatrix} t^2 \\ 1 \end{bmatrix}; \quad Y = \frac{1}{t^2}\begin{bmatrix} t^3 & t^4 \\ -1 & t \end{bmatrix}$

16. $\mathbf{y}' = \begin{bmatrix} \frac{1}{t-1} & -\frac{e^{-t}}{t-1} \\ \frac{e^{t}}{t+1} & \frac{1}{t+1} \end{bmatrix} \mathbf{y} + \begin{bmatrix} t^2-1 \\ t^2-1 \end{bmatrix}; \quad Y = \begin{bmatrix} t & e^{-t} \\ e^{t} & t \end{bmatrix}$

17. $\mathbf{y}' = \frac{1}{t}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ -2 & 2 & 2 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad Y = \begin{bmatrix} t^2 & t^3 & 1 \\ t^2 & 2t^3 & -1 \\ 0 & 2t^3 & 2 \end{bmatrix}$

18. $\mathbf{y}' = \begin{bmatrix} 3 & e^t & e^{2t} \\ e^{-t} & 2 & e^t \\ e^{-2t} & e^{-t} & 1 \end{bmatrix} \mathbf{y} + \begin{bmatrix} e^{3t} \\ 0 \\ 0 \end{bmatrix}; \quad Y = \begin{bmatrix} e^{5t} & e^{2t} & 0 \\ e^{4t} & 0 & e^t \\ e^{3t} & -1 & -1 \end{bmatrix}$

19. $\mathbf{y}' = \frac{1}{t} \begin{bmatrix} 1 & t & 0 \\ 0 & 1 & t \\ 0 & -t & 1 \end{bmatrix} \mathbf{y} + \begin{bmatrix} t \\ t \\ t \end{bmatrix}; \quad Y = t \begin{bmatrix} 1 & \cos t & \sin t \\ 0 & -\sin t & \cos t \\ 0 & -\cos t & -\sin t \end{bmatrix}$

20. $\mathbf{y}' = -\frac{1}{t} \begin{bmatrix} e^{-t} & -t & 1-e^{-t} \\ e^{-t} & 1 & -t-e^{-t} \\ e^{-t} & -t & 1-e^{-t} \end{bmatrix} \mathbf{y} + \frac{1}{t} \begin{bmatrix} e^t \\ 0 \\ e^t \end{bmatrix}; \quad Y = \frac{1}{t} \begin{bmatrix} e^t & e^{-t} & t \\ e^t & -e^{-t} & e^{-t} \\ e^t & e^{-t} & 0 \end{bmatrix}$

21. Prove Theorem 6.7.1.

22. **(a)** Convert the scalar equation

$$P_0(t)y^{(n)} + P_1(t)y^{(n-1)} + \cdots + P_n(t)y = F(t) \tag{A}$$

into an equivalent $n \times n$ system

$$\mathbf{y}' = A(t)\mathbf{y} + \mathbf{f}(t). \tag{B}$$

(b) Suppose (A) is normal on an interval (a, b) and $\{y_1, y_2, \dots, y_n\}$ is a fundamental set of solutions of

$$P_0(t)y^{(n)} + P_1(t)y^{(n-1)} + \cdots + P_n(t)y = 0 \tag{C}$$

on (a, b). Find a corresponding fundamental matrix Y for

$$\mathbf{y}' = A(t)\mathbf{y} \tag{D}$$

on (a, b) such that

$$y = c_1y_1 + c_2y_2 + \cdots + c_ny_n$$

is a solution of (C) if and only if $\mathbf{y} = Y\mathbf{c}$ with

$$\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

is a solution of (D).

(c) Let $y_p = u_1y_1 + u_1y_2 + \cdots + u_ny_n$ be a particular solution of (A), obtained by the method of variation of parameters for scalar equations as given in Section 9.4, and define

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}.$$

Show that $\mathbf{y}_p = Y\mathbf{u}$ is a solution of (B).

(d) Let $\mathbf{y}_p = Y\mathbf{u}$ be a particular solution of (B), obtained by the method of variation of parameters for systems as given in this section. Show that $y_p = u_1y_1 + u_1y_2 + \cdots + u_ny_n$ is a solution of (A).

23. Suppose the $n \times n$ matrix function A and the n–vector function $\mathbf{f}$ are continuous on (a, b). Let t_0 be in (a, b), let $\mathbf{k}$ be an arbitrary constant vector, and let Y be a fundamental matrix for the homogeneous system $\mathbf{y}' = A(t)\mathbf{y}$. Use variation of parameters to show that the solution of the initial value problem

$$\mathbf{y}' = A(t)\mathbf{y} + \mathbf{f}(t), \quad \mathbf{y}(t_0) = \mathbf{k}$$

is

$$\mathbf{y}(t) = Y(t)\left(Y^{-1}(t_0)\mathbf{k} + \int_{t_0}^{t} Y^{-1}(s)\mathbf{f}(s)\, ds\right).$$

APPENDIX A

ANSWERS

Section 1.2 Answers, pp. 20–21

1.2.1 (p. 20) **(a)** 3 **(b)** 2 **(c)** 1 **(d)** 2

1.2.3 (p. 20) **(a)** $y = -\frac{x^2}{2} + c$ **(b)** $y = x\cos x - \sin x + c$

(c) $y = \frac{x^2}{2}\ln x - \frac{x^2}{4} + c$ **(d)** $y = -x\cos x + 2\sin x + c_1 + c_2 x$

(e) $y = (2x-4)e^x + c_1 + c_2 x$ **(f)** $y = \frac{x^3}{3} - \sin x + e^x + c_1 + c_2 x$

(g) $y = \sin x + c_1 + c_2 x + c_3 x^2$ **(h)** $y = -\frac{x^5}{60} + e^x + c_1 + c_2 x + c_3 x^2$

(i) $y = \frac{7}{64}e^{4x} + c_1 + c_2 x + c_3 x^2$

1.2.4 (p. 20) **(a)** $y = -(x-1)e^x$ **(b)** $y = 1 - \frac{1}{2}\cos x^2$ **(c)** $y = 3 - \ln(\sqrt{2}\cos x)$

(d) $y = -\frac{47}{15} - \frac{37}{5}(x-2) + \frac{x^5}{30}$ **(e)** $y = \frac{1}{4}xe^{2x} - \frac{1}{4}e^{2x} + \frac{29}{4}$

(f) $y = x\sin x + 2\cos x - 3x - 1$ **(g)** $y = (x^2 - 6x + 12)e^x + \frac{x^2}{2} - 8x - 11$

(h) $y = \frac{x^3}{3} + \frac{\cos 2x}{6} + \frac{7}{4}x^2 - 6x + \frac{7}{8}$ **(i)** $y = \frac{x^4}{12} + \frac{x^3}{6} + \frac{1}{2}(x-2)^2 - \frac{26}{3}(x-2) - \frac{5}{3}$

1.2.7 (p. 20) **(a)** 576 ft **(b)** 10 s **1.2.8 (p. 20)** **(b)** $y = 0$ **1.2.10 (p. 21)** **(a)** $(-2c-2, \infty)$

$(-\infty, \infty)$

Section 2.1 Answers, pp. 48–??

2.1.1 (p. 48) $y = e^{-ax}$ **2.1.2 (p. 48)** $y = ce^{-x^3}$ **2.1.3 (p. 48)** $y = ce^{-(\ln x)^2/2}$

2.1.4 (p. 48) $y = \frac{c}{x^3}$ **2.1.5 (p. 48)** $y = ce^{1/x}$ **2.1.6 (p. 48)** $y = \frac{e^{-(x-1)}}{x}$ **2.1.7 (p. 48)** $y = \frac{e}{x \ln x}$ **2.1.8 (p. 48)** $y = \frac{\pi}{x \sin x}$ **2.1.9 (p. 48)** $y = 2(1 + x^2)$ **2.1.10 (p. 48)** $y = 3x^{-k}$

2.1.11 (p. 48) $y = c(\cos kx)^{1/k}$ **2.1.12 (p. 48)** $y = \frac{1}{3} + ce^{-3x}$ **2.1.13 (p. 48)** $y = \frac{2}{x} + \frac{c}{x}e^x$

2.1.14 (p. 48) $y = e^{-x^2}\left(\frac{x^2}{2} + c\right)$ **2.1.15 (p. 48)** $y = -\frac{e^{-x} + c}{1 + x^2}$ **2.1.16 (p. 48)** $y = \frac{7 \ln |x|}{x} + \frac{3}{2}x + \frac{c}{x}$

2.1.17 (p. 48) $y = (x - 1)^{-4}(\ln |x - 1| - \cos x + c)$ **2.1.18 (p. 48)** $y = e^{-x^2}\left(\frac{x^3}{4} + \frac{c}{x}\right)$

2.1.19 (p. 48) $y = \frac{2 \ln |x|}{x^2} + \frac{1}{2} + \frac{c}{x^2}$ **2.1.20 (p. 48)** $y = (x+c) \cos x$ **2.1.21 (p. 48)** $y = \frac{c - \cos x}{(1 + x)^2}$

2.1.22 (p. 49) $y = -\frac{1}{2}\frac{(x - 2)^3}{(x - 1)} + c\frac{(x - 2)^5}{(x - 1)}$ **2.1.23 (p. 49)** $y = (x + c)e^{-\sin^2 x}$

2.1.24 (p. 49) $y = \frac{e^x}{x^2} - \frac{e^x}{x^3} + \frac{c}{x^2}$. $y = \frac{e^{3x} - e^{-7x}}{10}$ **2.1.26 (p. 49)** $\frac{2x + 1}{(1 + x^2)^2}$

2.1.27 (p. 49) $y = \frac{1}{x^2} \ln\left(\frac{1 + x^2}{2}\right)$ **2.1.29 (p. 49)** $y = \frac{2 \ln |x|}{x} + \frac{x}{2} - \frac{1}{2x}$ **2.1.28 (p. 49)** $y = \frac{1}{2}(\sin x + \csc x)$ **2.1.29 (p. 49)** $y = \frac{2 \ln |x|}{x} + \frac{x}{2} - \frac{1}{2x}$ **2.1.30 (p. 49)** $y = (x-1)^{-3}\left[\ln(1 - x) - \cos x\right]$

2.1.31 (p. 49) $y = 2x^2 + \frac{1}{x^2}$ $(0, \infty)$ **2.1.32 (p. 49)** $y = x^2(1-\ln x)$ **2.1.33 (p. 49)** $y = \frac{1}{2} + \frac{5}{2}e^{-x^2}$

2.1.34 (p. 49) $y = \frac{\ln |x - 1| + \tan x + 1}{(x - 1)^3}$ **2.1.35 (p. 49)** $y = \frac{\ln |x| + x^2 + 1}{(x + 2)^4}$

2.1.36 (p. 49) $y = (x^2 - 1)\left(\frac{1}{2} \ln |x^2 - 1| - 4\right)$

2.1.37 (p. 49) $y = -(x^2 - 5)\left(7 + \ln |x^2 - 5|\right)$ **2.1.38 (p. 49)** $y = e^{-x^2}\left(3 + \int_0^x t^2 e^{t^2}\, dt\right)$

2.1.39 (p. 49) $y = \frac{1}{x}\left(2 + \int_1^x \frac{\sin t}{t}\, dt\right)$ **2.1.40 (p. 49)** $y = e^{-x} \int_1^x \frac{\tan t}{t}\, dt$

2.1.41 (p. 49) $y = \frac{1}{1 + x^2}\left(1 + \int_0^x \frac{e^t}{1 + t^2}\, dt\right)$ **2.1.42 (p. 50)** $y = \frac{1}{x}\left(2e^{-(x-1)} + e^{-x} \int_1^x e^t e^{t^2}\, dt\right)$

2.1.43 (p. 50) $G = \frac{r}{\lambda} + \left(G_0 - \frac{r}{\lambda}\right) e^{-\lambda t}$ $\lim_{t\to\infty} G(t) = \frac{r}{\lambda}$ **2.1.45 (p. 50)** **(a)** $y = y_0 e^{-a(x-x_0)} + e^{-ax}\int_{x_0}^{x} e^{at}$

2.1.48 (p. ??) **(a)** $y = \tan^{-1}\left(\frac{1}{3} + ce^{3x}\right)$ **(b)** $y = \pm\left[\ln\left(\frac{1}{x} + \frac{c}{x^2}\right)\right]^{1/2}$

(c) $y = \exp\left(x^2 + \frac{c}{x^2}\right)$ **(d)** $y = -1 + \frac{x}{c + 3\ln|x|}$

Section 2.2 Answers, pp. 58–??

2.2.1 (p. 58) $y = 2 \pm \sqrt{2(x^3 + x^2 + x + c)}$

2.2.2 (p. 58) $\ln(|\sin y|) = \cos x + c;$ $y \equiv k\pi,$ $k =$ integer

2.2.3 (p. 58) $y = \frac{c}{x-c}$ $y \equiv -1$ **2.2.4 (p. 58)** $\frac{(\ln y)^2}{2} = -\frac{x^3}{3} + c$

2.2.5 (p. 58) $y^3 + 3\sin y + \ln|y| + \ln(1+x^2) + \tan^{-1} x = c; y \equiv 0$

2.2.6 (p. 59) $y = \pm\left(1 + \left(\frac{x}{1+cx}\right)^2\right)^{1/2}; y \equiv \pm 1$

2.2.7 (p. 59) $y = \tan\left(\frac{x^3}{3} + c\right)$ **2.2.8 (p. 59)** $y = \frac{c}{\sqrt{1+x^2}}$ **2.2.9 (p. 59)** $y = \frac{2 - ce^{(x-1)^2/2}}{1 - ce^{(x-1)^2/2}};$ $y \equiv 1$

2.2.10 (p. 59) $y = 1 + \left(3x^2 + 9x + c\right)^{1/3}$

2.2.11 (p. 59) $y = 2 + \sqrt{\frac{2}{3}x^3 + 3x^2 + 4x - \frac{11}{3}}$ **2.2.12 (p. 59)** $y = \frac{e^{-(x^2-4)/2}}{2 - e^{-(x^2-4)/2}}$

2.2.13 (p. 59) $y^3 + 2y^2 + x^2 + \sin x = 3$ **2.2.14 (p. 59)** $(y+1)(y-1)^{-3}(y-2)^2 = -256(x+1)^{-6}$

2.2.15 (p. 59) $y = -1 + 3e^{-x^2}$ **2.2.16 (p. 59)** $y = \frac{1}{\sqrt{2e^{-2x^2} - 1}}$ **2.2.17 (p. 59)** $y \equiv -1;$ $(-\infty, \infty)$

2.2.18 (p. 59) $y = \frac{4 - e^{-x^2}}{2 - e^{-x^2}};$ $(-\infty, \infty)$ **2.2.19 (p. 59)** $y = \frac{-1 + \sqrt{4x^2 - 15}}{2};$ $\left(\frac{\sqrt{15}}{2}, \infty\right)$

2.2.20 (p. 59) $y = \frac{2}{1 + e^{-2x}}$ $(-\infty, \infty)$ **2.2.21 (p. 59)** $y = -\sqrt{25 - x^2};$ $(-5, 5)$

2.2.22 (p. 59) $y \equiv 2,$ $(-\infty, \infty)$ **2.2.23 (p. 59)** $y = 3\left(\frac{x+1}{2x-4}\right)^{1/3};$ $(-\infty, 2)$

2.2.24 (p. 59) $y = \frac{x+c}{1-cx}$ **2.2.25 (p. 59)** $y = -x\cos c + \sqrt{1-x^2}\sin c;$ $y \equiv 1; y \equiv -1$

2.2.26 (p. 59) $y = -x + 3\pi/2$ **2.2.28 (p. 60)** $P = \dfrac{P_0}{\alpha P_0 + (1 - \alpha P_0)e^{-at}}$; $\lim_{t\to\infty} P(t) = 1/\alpha$

2.2.29 (p. 60) $I = \dfrac{SI_0}{I_0 + (S - I_0)e^{-rSt}}$

2.2.30 (p. 60) If $q = rS$ then $I = \dfrac{I_0}{1 + rI_0 t}$ and $\lim_{t\to\infty} I(t) = 0$. If $q \neq Rs$, then $I = \dfrac{\alpha I_0}{I_0 + (\alpha - I_0)e^{-r\alpha t}}$. If $q < rs$, then $\lim_{t\to\infty} I(t) = \alpha = S - \dfrac{q}{r}$

if $q > rS$, then $\lim_{t\to\infty} I(t) = 0$ **2.2.34 (p. 60)** $f = ap$, where a=constant

2.2.35 (p. ??) $y = e^{-x}\left(-1 \pm \sqrt{2x^2 + c}\right)$ **2.2.36 (p. ??)** $y = x^2\left(-1 + \sqrt{x^2 + c}\right)$

2.2.37 (p. ??) $y = e^x\left(-1 + (3xe^x + c)^{1/3}\right)$

2.2.38 (p. ??) $y = e^{2x}(1 \pm \sqrt{c - x^2})$ **2.2.39 (p. ??)** **(a)** $y_1 = 1/x$; $g(x) = h(x)$

(b) $y_1 = x$; $g(x) = h(x)/x^2$ **(c)** $y_1 = e^{-x}$; $g(x) = e^x h(x)$

(d) $y_1 = x^{-r}$; $g(x) = x^{r-1}h(x)$ **(e)** $y_1 = 1/v(x)$; $g(x) = v(x)h(x)$

Section 2.3 Answers, pp. 65–??

2.3.1 (p. 65) **(a)**, **(b)** $x_0 \neq k\pi$ (k = integer) **2.3.2 (p. 65)** **(a)**, **(b)** $(x_0, y_0) \neq (0, 0)$

2.3.3 (p. 66) **(a)**, **(b)** $x_0 y_0 \neq (2k+1)\frac{\pi}{2}$ (k= integer) **2.3.4 (p. 66)** **(a)**, **(b)** $x_0 y_0 > 0$ and $x_0 y_0 \neq 1$

2.3.5 (p. 66) **(a)** all (x_0, y_0) **(b)** (x_0, y_0) with $y_0 \neq 0$ **2.3.6 (p. 66)** **(a), (b)** all (x_0, y_0)

2.3.7 (p. 66) **(a), (b)** all (x_0, y_0) **2.3.8 (p. 66)** **(a), (b)** (x_0, y_0) such that $x_0 \neq 4y_0$

2.3.9 (p. 66) **(a)** all (x_0, y_0) **(b)** all $(x_0, y_0) \neq (0, 0)$ **2.3.10 (p. 66)** **(a)** all (x_0, y_0)

(b) all (x_0, y_0) with $y_0 \neq \pm 1$ **2.3.11 (p. 66)** **(a), (b)** all (x_0, y_0)

2.3.12 (p. 66) **(a)**, **(b)** all (x_0, y_0) such that $x_0 + y_0 > 0$

2.3.13 (p. 66) **(a)**, **(b)** all (x_0, y_0) with $x_0 \neq 1, \quad y_0 \neq (2k+1)\frac{\pi}{2}$ ($k = \text{integer}$)

2.3.16 (p. ??) $y = \left(\frac{3}{5}x + 1\right)^{5/3}$, $-\infty < x < \infty$, is a solution.

Also,

$$y = \begin{cases} 0, & -\infty < x \leqslant -\frac{5}{3} \\ \left(\frac{3}{5}x+1\right)^{5/3}, & -\frac{5}{3} < x < \infty \end{cases}$$

is a solution, For every $a \geqslant \frac{5}{3}$, the following function is also a solution:

$$y = \begin{cases} \left(\frac{3}{5}(x+a)\right)^{5/3}, & -\infty < x < -a, \\ 0, & -a \leqslant x \leqslant -\frac{5}{3} \\ \left(\frac{3}{5}x+1\right)^{5/3}, & -\frac{5}{3} < x < \infty. \end{cases}$$

2.3.17 (p. ??) **(a)** all (x_0, y_0) **(b)** all (x_0, y_0) with $y_0 \neq 1$

2.3.18 (p. ??) $y_1 \equiv 1$; $y_2 = 1 + |x|^3$; $y_3 = 1 - |x|^3$; $y_4 = 1 + x^3$; $y_5 = 1 - x^3$

$$y_6 = \begin{cases} 1+x^3, & x \geqslant 0, \\ 1, & x < 0 \end{cases}; \quad y_7 = \begin{cases} 1-x^3, & x \geqslant 0, \\ 1, & x < 0 \end{cases};$$

$$y_8 = \begin{cases} 1, & x \geqslant 0, \\ 1+x^3, & x < 0 \end{cases}; \quad y_9 = \begin{cases} 1, & x \geqslant 0, \\ 1-x^3, & x < 0 \end{cases}$$

2.3.19 (p. ??) $y = 1 + (x^2+4)^{3/2}, \quad -\infty < x < \infty$

2.3.20 (p. ??) **(a)** The solution is unique on $(0, \infty)$. It is given by

$$y = \begin{cases} 1, & 0 < x \leqslant \sqrt{5}, \\ 1-(x^2-5)^{3/2}, & \sqrt{5} < x < \infty \end{cases}$$

(b)

$$y = \begin{cases} 1, & -\infty < x \leqslant \sqrt{5}, \\ 1-(x^2-5)^{3/2}, & \sqrt{5} < x < \infty \end{cases}$$

is a solution of (A) on $(-\infty,\infty)$. If $\alpha \geqslant 0$, then

$$y = \begin{cases} 1 + (x^2-\alpha^2)^{3/2}, & -\infty < x < -\alpha, \\ 1, & -\alpha \leqslant x \leqslant \sqrt{5}, \\ 1-(x^2-5)^{3/2}, & \sqrt{5} < x < \infty, \end{cases}$$

and

$$y = \begin{cases} 1 - (x^2-\alpha^2)^{3/2}, & -\infty < x < -\alpha, \\ 1, & -\alpha \leqslant x \leqslant \sqrt{5}, \\ 1-(x^2-5)^{3/2}, & \sqrt{5} < x < \infty, \end{cases}$$

are also solutions of (A) on $(-\infty,\infty)$.

Section 2.4 Answers, pp. 72–??

2.4.1 (p. 72) $y = \dfrac{1}{1-ce^x}$ **2.4.2 (p. 72)** $y = x^{2/7}(c - \ln|x|)^{1/7}$ **2.4.3 (p. 72)** $y = e^{2/x}(c - 1/x)^2$

2.4.4 (p. 72) $y = \pm\dfrac{\sqrt{2x+c}}{1+x^2}$ **2.4.5 (p. 72)** $y = \pm(1 - x^2 + ce^{-x^2})^{-1/2}$

2.4.6 (p. 72) $y = \left[\dfrac{x}{3(1-x)+ce^{-x}}\right]^{1/3}$ **2.4.7 (p. 72)** $y = \dfrac{2\sqrt{2}}{\sqrt{1-4x}}$ **2.4.8 (p. 72)** $y = \left[1 - \dfrac{3}{2}e^{-(x^2-1)/4}\right]^{-2}$ **2.4.9 (p. 72)** $y = \dfrac{1}{x(11-3x)^{1/3}}$ **2.4.10 (p. 72)** $y = (2e^x - 1)^2$

2.4.11 (p. 72) $y = (2e^{12x} - 1 - 12x)^{1/3}$ **2.4.12 (p. 72)** $y = \left[\dfrac{5x}{2(1+4x^5)}\right]^{1/2}$

2.4.13 (p. 72) $y = (4e^{x/2} - x - 2)^2$

2.4.14 (p. 72) $P = \dfrac{P_0e^{at}}{1 + aP_0\int_0^t \alpha(\tau)e^{a\tau}\,d\tau}$; $\lim_{t\to\infty} P(t) = \begin{cases} \infty & \text{if } L = 0, \\ 0 & \text{if } L = \infty, \\ 1/aL & \text{if } 0 < L < \infty. \end{cases}$

2.4.15 (p. 72) $y = x(\ln|x| + c)$ **2.4.16 (p. 72)** $y = \dfrac{cx^2}{1-cx}$ $\quad y = -x$

2.4.17 (p. 72) $y = \pm x(4\ln|x| + c)^{1/4}$ **2.4.18 (p. 72)** $y = x\sin^{-1}(\ln|x| + c)$

2.4.19 (p. 73) $y = x\tan(\ln|x| + c)$ **2.4.20 (p. 73)** $y = \pm x\sqrt{cx^2 - 1}$

2.4.21 (p. 73) $y = \pm x\ln(\ln|x| + c)$ **2.4.22 (p. 73)** $y = -\dfrac{2x}{2\ln|x| + 1}$

2.4.23 (p. 73) $y = x(3\ln x + 27)^{1/3}$ **2.4.24 (p. 73)** $y = \frac{1}{x}\left(\frac{9-x^4}{2}\right)^{1/2}$ **2.4.25 (p. 73)** $y = -x$

2.4.26 (p. 73) $y = -\frac{x(4x-3)}{(2x-3)}$ **2.4.27 (p. 73)** $y = x\sqrt{4x^6-1}$ **2.4.28 (p. 73)** $\tan^{-1}\frac{y}{x} - \frac{1}{2}\ln(x^2+y^2) = c$

2.4.29 (p. 73) $(x+y)\ln|x| + y(1-\ln|y|) + cx = 0$ **2.4.30 (p. 73)** $(y+x)^3 = 3x^3(\ln|x| + c)$

2.4.31 (p. 73) $(y+x) = c(y-x)^3; \quad y = x; \quad y = -x$

2.4.32 (p. 73) $y^2(y-3x) = c; \quad y \equiv 0; \; y = 3x$

2.4.33 (p. 73) $(x-y)^3(x+y) = cy^2x^4; \quad y = 0; \; y = x; \; y = -x$ **2.4.34 (p. 73)** $\frac{y}{x} + \frac{y^3}{x^3} = \ln|x| + c$

2.4.40 (p. ??) Choose X_0 and Y_0 so that

$$\begin{aligned} aX_0 + bY_0 &= \alpha \\ cX_0 + dY_0 &= \beta. \end{aligned}$$

2.4.41 (p. ??) $(y+2x+1)^4(2y-6x-3) = c; \quad y = 3x + 3/2; \quad y = -2x - 1$

2.4.42 (p. ??) $(y+x-1)(y-x-5)^3 = c; \quad y = x + 5; \quad y = -x + 1$

2.4.43 (p. ??) $\ln|y-x-6| - \frac{2(x+2)}{y-x-6} = c; \quad y = x + 6$ **2.4.44 (p. ??)** $(y_1 = x^{1/3})$ $y = x^{1/3}(\ln|x| + c)^{1/3}$

2.4.45 (p. ??) $y_1 = x^3; y = \pm x^3\sqrt{cx^6-1}$ **2.4.46 (p. ??)** $y_1 = x^2; y = \frac{x^2(1+cx^4)}{1-cx^4}$ $y = -x^2$

2.4.47 (p. ??) $y_1 = e^x; y = -\frac{e^x(1-2ce^x)}{1-ce^x}; \quad y = -2e^x$

2.4.48 (p. ??) $y_1 = \tan x; y = \tan x \; \tan(\ln|\tan x| + c)$

2.4.49 (p. ??) $y_1 = \ln x; y = \frac{2\ln x \; \left(1 + c(\ln x)^4\right)}{1 - c(\ln x)^4}; \quad y = -2\ln x$

2.4.50 (p. ??) $y_1 = x^{1/2}$; $y = x^{1/2}(-2 \pm \sqrt{\ln|x| + c})$

2.4.51 (p. ??) $y_1 = e^{x^2}$; $y = e^{x^2}(-1 \pm \sqrt{2x^2 + c})$ **2.4.52 (p. ??)** $y = \dfrac{-3 + \sqrt{1 + 60x}}{2x}$

2.4.53 (p. ??) $y = \dfrac{-5 + \sqrt{1 + 48x}}{2x^2}$ **2.4.56 (p. ??)** $y = 1 + \dfrac{1}{x + 1 + ce^x}$

2.4.57 (p. ??) $y = e^x - \dfrac{1}{1 + ce^{-x}}$ **2.4.58 (p. ??)** $y = 1 - \dfrac{1}{x(1 - cx)}$ **2.4.59 (p. ??)** $y = x - \dfrac{2x}{x^2 + c}$

Section 2.5 Answers, pp. 80–??

2.5.1 (p. 80) $2x^3y^2 = c$ **2.5.2 (p. 80)** $3y\sin x + 2x^2e^x + 3y = c$ **2.5.3 (p. 80)** Not exact

2.5.4 (p. 80) $x^2 - 2xy^2 + 4y^3 = c$ **2.5.5 (p. 80)** $x + y = c$ **2.5.6 (p. 80)** Not exact

2.5.7 (p. 80) $2y^2\cos x + 3xy^3 - x^2 = c$ **2.5.8 (p. 81)** Not exact

2.5.9 (p. 81) $x^3 + x^2y + 4xy^2 + 9y^2 = c$ **2.5.10 (p. 81)** Not exact **2.5.11 (p. 81)** $\ln|xy| + x^2 + y^2 = c$

2.5.12 (p. 81) Not exact **2.5.13 (p. 81)** $x^2 + y^2 = c$ **2.5.14 (p. 81)** $x^2y^2e^x + 2y + 3x^2 = c$

2.5.15 (p. 81) $x^3e^{x^2+y} - 4y^3 + 2x^2 = c$ **2.5.16 (p. 81)** $x^4e^{xy} + 3xy = c$

2.5.17 (p. 81) $x^3\cos xy + 4y^2 + 2x^2 = c$ **2.5.18 (p. 81)** $y = \dfrac{x + \sqrt{2x^2 + 3x - 1}}{x^2}$

2.5.19 (p. 81) $y = \sin x - \sqrt{1 - \dfrac{\tan x}{2}}$ **2.5.20 (p. 81)** $y = \left(\dfrac{e^x - 1}{e^x + 1}\right)^{1/3}$

2.5.21 (p. 81) $y = 1 + 2\tan x$ **2.5.22 (p. 81)** $y = \dfrac{x^2 - x + 6}{(x + 2)(x - 3)}$

2.5.23 (p. 81) $\dfrac{7x^2}{2} + 4xy + \dfrac{3y^2}{2} = c$ **2.5.24 (p. 81)** $(x^4y^2 + 1)e^x + y^2 = c$

2.5.29 (p. ??) **(a)** $M(x, y) = 2xy + f(x)$ **(b)** $M(x, y) = 2(\sin x + x\cos x)(y\sin y + \cos y) + f(x)$

(c) $M(x, y) = ye^x - e^y\cos x + f(x)$

2.5.30 (p. ??) **(a)** $N(x, y) = \dfrac{x^4y}{2} + x^2 + 6xy + g(y)$ **(b)** $N(x, y) = \dfrac{x}{y} + 2y\sin x + g(y)$

(c) $N(x,y) = x(\sin y + y\cos y) + g(y)$

2.5.33 (p. ??) $B = C$ **2.5.34 (p. ??)** $B = 2D, \quad E = 2C$

2.5.37 (p. ??) **(a)** $2x^2 + x^4y^4 + y^2 = c$ **(b)** $x^3 + 3xy^2 = c$ **(c)** $x^3 + y^2 + 2xy = c$

2.5.38 (p. ??) $y = -1 - \dfrac{1}{x^2}$ **2.5.39 (p. ??)** $y = x^3\left(\dfrac{-3(x^2+1) + \sqrt{9x^4 + 34x^2 + 21}}{2}\right)$

2.5.40 (p. ??) $y = -e^{-x^2}\left(\dfrac{2x + \sqrt{9 - 5x^2}}{3}\right).$

2.5.44 (p. ??) **(a)** $G(x,y) = 2xy + c$ **(b)** $G(x,y) = e^x \sin y + c$

(c) $G(x,y) = 3x^2y - y^3 + c$ **(d)** $G(x,y) = -\sin x \sinh y + c$

(e) $G(x,y) = \cos x \sinh y + c$

Section 2.6 Answers, pp. 86–??

2.6.3 (p. 86) $\mu(x) = 1/x^2;\ y = cx$ and $\mu(y) = 1/y^2; x = cy$

2.6.4 (p. 86) $\mu(x) = x^{-3/2};\ x^{3/2}y = c$ **2.6.5 (p. 86)** $\mu(y) = 1/y^3;\ y^3e^{2x} = c$

2.6.6 (p. 86) $\mu(x) = e^{5x/2};\ e^{5x/2}(xy+1) = c$ **2.6.7 (p. 86)** $\mu(x) = e^x;\ e^x(xy+y+x) = c$

2.6.8 (p. 86) $\mu(x) = x;\ x^2y^2(9x+4y) = c$ **2.6.9 (p. 86)** $\mu(y) = y^2;\ y^3(3x^2y+2x+1) = c$

2.6.10 (p. 86) $\mu(y) = ye^y;\ e^y(xy^3+1) = c$ **2.6.11 (p. 87)** $\mu(y) = y^2; y^3(3x^4+8x^3y+y) =$

c

2.6.12 (p. 87) $\mu(x) = xe^x;\ x^2y(x+1)e^x = c$

2.6.13 (p. 87) $\mu(x) = (x^3-1)^{-4/3};\ xy(x^3-1)^{-1/3} = c$ and $x \equiv 1$

2.6.14 (p. 87) $\mu(y) = e^y;\ e^y(\sin x \cos y + y - 1) = c$ **2.6.15 (p. ??)** $\mu(y) = e^{-y^2};$
$xye^{-y^2}(x+y) = c$ **2.6.16 (p. ??)** $\dfrac{xy}{\sin y} = c$ and $y = k\pi$ (k = integer) **2.6.17 (p. ??)**

$\mu(x, y) = x^4y^3$; $x^5y^4 \ln x = c$ **2.6.18 (p. ??)** $\mu(x, y) = 1/xy$; $|x|^{\alpha}|y|^{\beta}e^{\gamma x}e^{\delta y} = c$ and $x \equiv 0, y \equiv 0$

2.6.19 (p. ??) $\mu(x, y) = x^{-2}y^{-3}$; $3x^2y^2 + y = 1 + cxy^2$ and $x \equiv 0, y \equiv 0$

2.6.20 (p. ??) $\mu(x, y) = x^{-2}y^{-1}$; $-\frac{2}{x} + y^3 + 3\ln|y| = c$ and $x \equiv 0, y \equiv 0$

2.6.21 (p. ??) $\mu(x, y) = e^{ax}e^{by}$; $e^{ax}e^{by}\cos xy = c$

2.6.22 (p. ??) $\mu(x, y) = x^{-4}y^{-3}$ (and others) $xy = c$ **2.6.23 (p. ??)** $\mu(x, y) = xe^y$; $x^2ye^y \sin x = c$

2.6.24 (p. ??) $\mu(x) = 1/x^2$; $\frac{x^3y^3}{3} - \frac{y}{x} = c$ **2.6.25 (p. ??)** $\mu(x) = x+1$; $y(x+1)^2(x+y) = c$

2.6.26 (p. ??) $\mu(x, y) = x^2y^2$; $x^3y^3(3x + 2y^2) = c$

2.6.27 (p. ??) $\mu(x, y) = x^{-2}y^{-2}$; $3x^2y = cxy + 2$ and $x \equiv 0, y \equiv 0$

Section 3.1 Answers, pp. 103–106

3.1.1 (p. 103) $y_1 = 1.450000000$, $y_2 = 2.085625000$, $y_3 = 3.079099746$

3.1.2 (p. 103) $y_1 = 1.200000000$, $y_2 = 1.440415946$, $y_3 = 1.729880994$

3.1.3 (p. 103) $y_1 = 1.900000000$, $y_2 = 1.781375000$, $y_3 = 1.646612970$

3.1.4 (p. 104) $y_1 = 2.962500000$, $y_2 = 2.922635828$, $y_3 = 2.880205639$

3.1.5 (p. 104) $y_1 = 2.513274123$, $y_2 = 1.814517822$, $y_3 = 1.216364496$

3.1.6 (p. 104)

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	Exact
1.0	48.298147362	51.492825643	53.076673685	54.647937102

3.1.7 (p. 104)

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	Exact
2.0	1.390242009	1.370996758	1.361921132	1.353193719

3.1.8 (p. 104)

x	$h = 0.05$	$h = 0.025$	$h = 0.0125$	Exact
1.50	7.886170437	8.852463793	9.548039907	10.500000000

3.1.9 (p. 105)

x	$h=0.1$	$h=0.05$	$h=0.025$	$h=0.1$	$h=0.05$	$h=0.025$
3.0	1.469458241	1.462514486	1.459217010	0.3210	0.1537	0.0753
	Approximate Solutions			Residuals		

3.1.10 (p. 105)

x	$h=0.1$	$h=0.05$	$h=0.025$	$h=0.1$	$h=0.05$	$h=0.025$
2.0	0.473456737	0.483227470	0.487986391	-0.3129	-0.1563	-0.0781
	Approximate Solutions			Residuals		

3.1.11 (p. 105)

x	$h=0.1$	$h=0.05$	$h=0.025$	"Exact"
1.0	0.691066797	0.676269516	0.668327471	0.659957689

3.1.12 (p. 105)

x	$h=0.1$	$h=0.05$	$h=0.025$	"Exact"
2.0	-0.772381768	-0.761510960	-0.756179726	-0.750912371

3.1.13 (p. 105)

Euler's method				
x	$h=0.1$	$h=0.05$	$h=0.025$	Exact
1.0	0.538871178	0.593002325	0.620131525	0.647231889

Euler semilinear method				
x	$h=0.1$	$h=0.05$	$h=0.025$	Exact
1.0	0.647231889	0.647231889	0.647231889	0.647231889

Applying variation of parameters to the given initial value problem yields

$y = ue^{-3x}$, where (A) $u' = 7, \quad u(0) = 6$. Since $u'' = 0$, Euler's method yields the exact solution of (A). Therefore the Euler semilinear method produces the exact solution of the given problem

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3.1.14 (p. 105)

Euler's method				
x	$h=0.1$	$h=0.05$	$h=0.025$	"Exact"
3.0	12.804226135	13.912944662	14.559623055	15.282004826

Euler semilinear method				
x	$h=0.1$	$h=0.05$	$h=0.025$	"Exact"
3.0	15.354122287	15.317257705	15.299429421	15.282004826

3.1.15 (p. 105)

Euler's method				
x	$h=0.2$	$h=0.1$	$h=0.05$	"Exact"
2.0	0.867565004	0.885719263	0.895024772	0.904276722

Euler semilinear method				
x	$h = 0.2$	$h = 0.1$	$h = 0.05$	"Exact"
2.0	0.569670789	0.720861858	0.808438261	0.904276722

3.1.16 (p. 105)

Euler's method				
x	$h = 0.2$	$h = 0.1$	$h = 0.05$	"Exact"
3.0	0.922094379	0.945604800	0.956752868	0.967523153

Euler semilinear method				
x	$h = 0.2$	$h = 0.1$	$h = 0.05$	"Exact"
3.0	0.993954754	0.980751307	0.974140320	0.967523153

3.1.17 (p. 105)

Euler's method				
x	$h = 0.0500$	$h = 0.0250$	$h = 0.0125$	"Exact"
1.50	0.319892131	0.330797109	0.337020123	0.343780513

Euler semilinear method				
x	$h = 0.0500$	$h = 0.0250$	$h = 0.0125$	"Exact"
1.50	0.305596953	0.323340268	0.333204519	0.343780513

3.1.18 (p. 105)

Euler's method				
x	$h = 0.2$	$h = 0.1$	$h = 0.05$	"Exact"
2.0	0.754572560	0.743869878	0.738303914	0.732638628

Euler semilinear method				
x	$h = 0.2$	$h = 0.1$	$h = 0.05$	"Exact"
2.0	0.722610454	0.727742966	0.730220211	0.732638628

3.1.19 (p. 105)

Euler's method				
x	$h = 0.0500$	$h = 0.0250$	$h = 0.0125$	"Exact"
1.50	2.175959970	2.210259554	2.227207500	2.244023982

Euler semilinear method				
x	$h = 0.0500$	$h = 0.0250$	$h = 0.0125$	"Exact"
1.50	2.117953342	2.179844585	2.211647904	2.244023982

3.1.20 (p. 106)

Euler's method				
x	$h = 0.1$	$h = 0.05$	$h = 0.025$	"Exact"
1.0	0.032105117	0.043997045	0.050159310	0.056415515

Euler semilinear method				
x	$h = 0.1$	$h = 0.05$	$h = 0.025$	"Exact"
1.0	0.056020154	0.056243980	0.056336491	0.056415515

3.1.21 (p. 106)

Euler's method				
x	$h = 0.1$	$h = 0.05$	$h = 0.025$	"Exact"
1.0	28.987816656	38.426957516	45.367269688	54.729594761

Euler semilinear method				
x	$h = 0.1$	$h = 0.05$	$h = 0.025$	"Exact"
1.0	54.709134946	54.724150485	54.728228015	54.729594761

3.1.22 (p. 106)

Euler's method				
x	$h = 0.1$	$h = 0.05$	$h = 0.025$	"Exact"
3.0	1.361427907	1.361320824	1.361332589	1.361383810

Euler semilinear method				
x	$h = 0.1$	$h = 0.05$	$h = 0.025$	"Exact"
3.0	1.291345518	1.326535737	1.344004102	1.361383810

Section 3.2 Answers, pp. 118–106

3.2.1 (p. 118) $y_1 = 1.542812500,\ y_2 = 2.421622101,\ y_3 = 4.208020541$
3.2.2 (p. 118) $y_1 = 1.220207973,\ y_2 = 1.489578775\ y_3 = 1.819337186$
3.2.3 (p. 118) $y_1 = 1.890687500,\ y_2 = 1.763784003,\ y_3 = 1.622698378$
3.2.4 (p. 118) $y_1 = 2.961317914\ \ y_2 = 2.920132727\ \ y_3 = 2.876213748.$
3.2.5 (p. 118) $y_1 = 2.478055238,\ y_2 = 1.844042564,\ y_3 = 1.313882333$

3.2.6 (p. 118)

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	Exact
1.0	56.134480009	55.003390448	54.734674836	54.647937102

3.2.7 (p. 118)

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	Exact
2.0	1.353501839	1.353288493	1.353219485	1.353193719

3.2.8 (p. 118)

x	$h = 0.05$	$h = 0.025$	$h = 0.0125$	Exact
1.50	10.141969585	10.396770409	10.472502111	10.500000000

3.2.9 (p. 118)

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	$h = 0.1$	$h = 0.05$	$h = 0.025$
3.0	1.455674816	1.455935127	1.456001289	-0.00818	-0.00207	-0.000518
	Approximate Solutions			Residuals		

3.2.10 (p. 118)

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	$h = 0.1$	$h = 0.05$	$h = 0.025$
2.0	0.492862999	0.492709931	0.492674855	0.00335	0.000777	0.000187
	Approximate Solutions			Residuals		

3.2.11 (p. 118)

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	"Exact"
1.0	0.660268159	0.660028505	0.659974464	0.659957689

3.2.12 (p. 118)

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	"Exact"
2.0	-0.749751364	-0.750637632	-0.750845571	-0.750912371

3.2.13 (p. 118) Applying variation of parameters to the given initial value problem

$y = ue^{-3x}$, where (A) $u' = 1 - 2x,\quad u(0) = 2$. Since $u''' = 0$, the improved Euler method yields the exact solution of (A). Therefore the improved Euler semilinear method produces the exact solution of the given problem.

Improved Euler method				
x	$h = 0.1$	$h = 0.05$	$h = 0.025$	Exact
1.0	0.105660401	0.100924399	0.099893685	0.099574137

Improved Euler semilinear method				
x	$h = 0.1$	$h = 0.05$	$h = 0.025$	Exact
1.0	0.099574137	0.099574137	0.099574137	0.099574137

3.2.14 (p. 118)

Improved Euler method				
x	$h = 0.1$	$h = 0.05$	$h = 0.025$	"Exact"
3.0	15.107600968	15.234856000	15.269755072	15.282004826

Improved Euler semilinear method				
x	$h = 0.1$	$h = 0.05$	$h = 0.025$	"Exact"
3.0	15.285231726	15.282812424	15.282206780	15.282004826

3.2.15 (p. 118)

Improved Euler method				
x	$h = 0.2$	$h = 0.1$	$h = 0.05$	"Exact"
2.0	0.924335375	0.907866081	0.905058201	0.904276722

Improved Euler semilinear method				
x	$h = 0.2$	$h = 0.1$	$h = 0.05$	"Exact"
2.0	0.969670789	0.920861858	0.908438261	0.904276722

3.2.16 (p. 118)

Improved Euler method				
x	$h = 0.2$	$h = 0.1$	$h = 0.05$	"Exact"
3.0	0.967473721	0.967510790	0.967520062	0.967523153

Improved Euler semilinear method				
x	$h = 0.2$	$h = 0.1$	$h = 0.05$	"Exact"
3.0	0.967473721	0.967510790	0.967520062	0.967523153

3.2.17 (p. 118)

Improved Euler method				
x	$h = 0.0500$	$h = 0.0250$	$h = 0.0125$	"Exact"
1.50	0.349176060	0.345171664	0.344131282	0.343780513

Improved Euler semilinear method				
x	$h = 0.0500$	$h = 0.0250$	$h = 0.0125$	"Exact"
1.50	0.349350206	0.345216894	0.344142832	0.343780513

3.2.18 (p. 118)

Improved Euler method				
x	$h = 0.2$	$h = 0.1$	$h = 0.05$	"Exact"
2.0	0.732679223	0.732721613	0.732667905	0.732638628

Improved Euler semilinear method				
x	$h = 0.2$	$h = 0.1$	$h = 0.05$	"Exact"
2.0	0.732166678	0.732521078	0.732609267	0.732638628

3.2.19 (p. 118)

Improved Euler method				
x	$h = 0.0500$	$h = 0.0250$	$h = 0.0125$	"Exact"
1.50	2.247880315	2.244975181	2.244260143	2.244023982

Improved Euler semilinear method				
x	$h = 0.0500$	$h = 0.0250$	$h = 0.0125$	"Exact"
1.50	2.248603585	2.245169707	2.244310465	2.244023982

3.2.20 (p. 118)

Improved Euler method				
x	$h = 0.1$	$h = 0.05$	$h = 0.025$	"Exact"
1.0	0.059071894	0.056999028	0.056553023	0.056415515

Improved Euler semilinear method				
x	$h = 0.1$	$h = 0.05$	$h = 0.025$	"Exact"
1.0	0.056295914	0.056385765	0.056408124	0.056415515

3.2.21 (p. 118)

Improved Euler method				
x	$h = 0.1$	$h = 0.05$	$h = 0.025$	"Exact"
1.0	50.534556346	53.483947013	54.391544440	54.729594761

Improved Euler semilinear method				
x	$h = 0.1$	$h = 0.05$	$h = 0.025$	"Exact"
1.0	54.709041434	54.724083572	54.728191366	54.729594761

3.2.22 (p. 118)

Improved Euler method				
x	$h = 0.1$	$h = 0.05$	$h = 0.025$	"Exact"
3.0	1.361395309	1.361379259	1.361382239	1.361383810

Improved Euler semilinear method				
x	$h = 0.1$	$h = 0.05$	$h = 0.025$	"Exact"
3.0	1.375699933	1.364730937	1.362193997	1.361383810

3.2.23 (p. 118)

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	Exact
2.0	1.349489056	1.352345900	1.352990822	1.353193719

3.2.24 (p. 118)

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	Exact
2.0	1.350890736	1.352667599	1.353067951	1.353193719

3.2.25 (p. 118)

x	$h = 0.05$	$h = 0.025$	$h = 0.0125$	Exact
1.50	10.133021311	10.391655098	10.470731411	10.500000000

3.2.26 (p. 118)

x	$h = 0.05$	$h = 0.025$	$h = 0.0125$	Exact
1.50	10.136329642	10.393419681	10.470731411	10.500000000

3.2.27 (p. 118)

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	"Exact"
1.0	0.660846835	0.660189749	0.660016904	0.659957689

3.2.28 (p. 118)

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	"Exact"
1.0	0.660658411	0.660136630	0.660002840	0.659957689

3.2.29 (p. 119)

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	"Exact"
2.0	-0.750626284	-0.750844513	-0.750895864	-0.751331499

3.2.30 (p. 119)

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	"Exact"
2.0	-0.750335016	-0.750775571	-0.750879100	-0.751331499

Section 3.3 Answers, pp. 126–127

3.3.1 (p. 126) $y_1 = 1.550598190,\ y_2 = 2.469649729$ **3.3.2 (p. 126)** $y_1 = 1.221551366,\ y_2 = 1.492920208$

3.3.3 (p. 126) $y_1 = 1.890339767,\ y_2 = 1.763094323$ **3.3.4 (p. 126)** $y_1 = 2.961316248\ y_2 = 2.920128958.$

3.3.5 (p. 126) $y_1 = 2.475605264,\ y_2 = 1.825992433$

3.3.6 (p. 126)

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	Exact
1.0	54.654509699	54.648344019	54.647962328	54.647937102

3.3.7 (p. 126)

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	Exact
2.0	1.353191745	1.353193606	1.353193712	1.353193719

3.3.8 (p. 126)

x	$h = 0.05$	$h = 0.025$	$h = 0.0125$	Exact
1.50	10.498658198	10.499906266	10.499993820	10.500000000

3.3.9 (p. 126)

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	$h = 0.1$	$h = 0.05$	$h = 0.025$
3.0	1.456023907	1.456023403	1.456023379	0.0000124	0.000000611	0.0000000333
	Approximate Solutions			Residuals		

3.3.10 (p. 126)

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	$h = 0.1$	$h = 0.05$	$h = 0.025$
2.0	0.492663789	0.492663738	0.492663736	0.000000902	0.0000000508	0.00000000302
	Approximate Solutions			Residuals		

3.3.11 (p. 126)

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	"Exact"
1.0	0.659957046	0.659957646	0.659957686	0.659957689

3.3.12 (p. 126)

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	"Exact"
2.0	-0.750911103	-0.750912294	-0.750912367	-0.750912371

3.3.13 (p. 126) Applying variation of parameters to the given initial value problem yields

$y = ue^{-3x}$, where (A) $u' = 1 - 4x + 3x^2 - 4x^3, \quad u(0) = -3.$ Since $u^{(5)} = 0$, the Runge-Kutta method yields the exact solution of (A). Therefore the Euler semilinear method produces the exact solution of the given problem.

Runge-Kutta method				
x	$h = 0.1$	$h = 0.05$	$h = 0.025$	Exact
0.0	-3.000000000	-3.000000000	-3.000000000	-3.000000000
0.1	-2.162598011	-2.162526572	-2.162522707	-2.162522468
0.2	-1.577172164	-1.577070939	-1.577065457	-1.577065117
0.3	-1.163350794	-1.163242678	-1.163236817	-1.163236453
0.4	-0.868030294	-0.867927182	-0.867921588	-0.867921241
0.5	-0.655542739	-0.655450183	-0.655445157	-0.655444845
0.6	-0.501535352	-0.501455325	-0.501450977	-0.501450707
0.7	-0.389127673	-0.389060213	-0.389056546	-0.389056318
0.8	-0.306468018	-0.306412184	-0.306409148	-0.306408959
0.9	-0.245153433	-0.245107859	-0.245105379	-0.245105226
1.0	-0.199187198	-0.199150401	-0.199148398	-0.199148273

Runge-Kutta semilinear method				
x	$h = 0.1$	$h = 0.05$	$h = 0.025$	Exact
0.0	-3.000000000	-3.000000000	-3.000000000	-3.000000000
0.1	-2.162522468	-2.162522468	-2.162522468	-2.162522468
0.2	-1.577065117	-1.577065117	-1.577065117	-1.577065117
0.3	-1.163236453	-1.163236453	-1.163236453	-1.163236453
0.4	-0.867921241	-0.867921241	-0.867921241	-0.867921241
0.5	-0.655444845	-0.655444845	-0.655444845	-0.655444845
0.6	-0.501450707	-0.501450707	-0.501450707	-0.501450707
0.7	-0.389056318	-0.389056318	-0.389056318	-0.389056318
0.8	-0.306408959	-0.306408959	-0.306408959	-0.306408959
0.9	-0.245105226	-0.245105226	-0.245105226	-0.245105226
1.0	-0.199148273	-0.199148273	-0.199148273	-0.199148273

3.3.14 (p. 126)

Runge-Kutta method				
x	$h = 0.1$	$h = 0.05$	$h = 0.025$	"Exact"
3.0	15.281660036	15.281981407	15.282003300	15.282004826

Runge-Kutta semilinear method				
x	$h = 0.1$	$h = 0.05$	$h = 0.025$	"Exact"
3.0	15.282005990	15.282004899	15.282004831	15.282004826

3.3.15 (p. 127)

Runge-Kutta method				
x	$h = 0.2$	$h = 0.1$	$h = 0.05$	"Exact"
2.0	0.904678156	0.904295772	0.904277759	0.904276722

Runge-Kutta semilinear method				
x	$h = 0.2$	$h = 0.1$	$h = 0.05$	"Exact"
2.0	0.904592215	0.904297062	0.904278004	0.904276722

3.3.16 (p. 127)

Runge-Kutta method				
x	$h = 0.2$	$h = 0.1$	$h = 0.05$	"Exact"
3.0	0.967523147	0.967523152	0.967523153	0.967523153

Runge-Kutta semilinear method				
x	$h = 0.2$	$h = 0.1$	$h = 0.05$	"Exact"
3.0	0.967523147	0.967523152	0.967523153	0.967523153

3.3.17 (p. 127)

Runge-Kutta method				
x	$h = 0.0500$	$h = 0.0250$	$h = 0.0125$	"Exact"
1.50	0.343839158	0.343784814	0.343780796	0.343780513

Runge-Kutta semilinear method				
x	$h = 0.0500$	$h = 0.0250$	$h = 0.0125$	"Exact"
1.00	0.000000000	0.000000000	0.000000000	0.000000000
1.05	0.028121022	0.028121010	0.028121010	0.028121010
1.10	0.055393494	0.055393466	0.055393465	0.055393464
1.15	0.082164048	0.082163994	0.082163990	0.082163990
1.20	0.108862698	0.108862597	0.108862591	0.108862590
1.25	0.136058715	0.136058528	0.136058517	0.136058516
1.30	0.164564862	0.164564496	0.164564473	0.164564471
1.35	0.195651074	0.195650271	0.195650219	0.195650216
1.40	0.231542288	0.231540164	0.231540027	0.231540017
1.45	0.276818775	0.276811011	0.276810491	0.276810456
1.50	0.343839124	0.343784811	0.343780796	0.343780513

3.3.18 (p. 127)

Runge-Kutta method				
x	$h = 0.2$	$h = 0.1$	$h = 0.05$	"Exact"
2.0	0.732633229	0.732638318	0.732638609	0.732638628

Runge-Kutta semilinear method				
x	$h = 0.2$	$h = 0.1$	$h = 0.05$	"Exact"
2.0	0.732639212	0.732638663	0.732638630	0.732638628

3.3.19 (p. 127)

Runge-Kutta method				
x	$h = 0.0500$	$h = 0.0250$	$h = 0.0125$	"Exact"
1.50	2.244025683	2.244024088	2.244023989	2.244023982

Runge-Kutta semilinear method				
x	$h = 0.0500$	$h = 0.0250$	$h = 0.0125$	"Exact"
1.50	2.244025081	2.244024051	2.244023987	2.244023982

3.3.20 (p. 127)

Runge-Kutta method				
x	$h = 0.1$	$h = 0.05$	$h = 0.025$	"Exact"
1.0	0.056426886	0.056416137	0.056415552	0.056415515

Runge-Kutta semilinear method				
x	$h = 0.1$	$h = 0.05$	$h = 0.025$	"Exact"
1.0	0.056415185	0.056415495	0.056415514	0.056415515

3.3.21 (p. 127)

Runge-Kutta method				
x	$h = 0.1$	$h = 0.05$	$h = 0.025$	"Exact"
1.0	54.695901186	54.727111858	54.729426250	54.729594761

Runge-Kutta semilinear method				
x	$h=0.1$	$h=0.05$	$h=0.025$	"Exact"
1.0	54.729099966	54.729561720	54.729592658	54.729594761

3.3.22 (p. 127)

Runge-Kutta method				
x	$h=0.1$	$h=0.05$	$h=0.025$	"Exact"
3.0	1.361384082	1.361383812	1.361383809	1.361383810

Runge-Kutta semilinear method				
x	$h=0.1$	$h=0.05$	$h=0.025$	"Exact"
3.0	1.361456502	1.361388196	1.361384079	1.361383810

3.3.24 (p. 127)

x	$h=.1$	$h=.05$	$h=.025$	Exact
2.00	-1.000000000	-1.000000000	-1.000000000	-1.000000000

3.3.25 (p. 127)

x	$h=.1$	$h=.05$	$h=.025$	"Exact"
1.00	1.000000000	1.000000000	1.000000000	1.000000000

3.3.26 (p. 127)

x	$h=.1$	$h=.05$	$h=.025$	Exact
1.50	4.142171279	4.142170553	4.142170508	4.142170505

3.3.27 (p. 127)

x	$h=.1$	$h=.05$	$h=.025$	Exact
3.0	16.666666988	16.666666687	16.666666668	16.666666667

Section 4.1 Answers, pp. 191–??

4.1.1 (p. 191) $Q = 20e^{-(t\ln 2)/3200}$ g **4.1.2 (p. 191)** $\frac{2\ln 10}{\ln 2}$ days **4.1.3 (p. 191)** $\tau = 10\dfrac{\ln 2}{\ln 4/3}$ minutes
4.1.4 (p. 191) $\tau\dfrac{\ln(p_0/p_1)}{\ln 2}$ **4.1.5 (p. 191)** $\dfrac{t_p}{t_q} = \dfrac{\ln p}{\ln q}$ **4.1.6 (p. 191)** $k = \dfrac{1}{t_2 - t_1}\ln\dfrac{Q_1}{Q_2}$ **4.1.7 (p. 191)** 20 g
4.1.8 (p. 191) $\dfrac{50\ln 2}{3}$ yrs **4.1.9 (p. 191)** $\dfrac{25}{2}\ln 2\%$
4.1.10 (p. 192) **(a)** $= 20\ln 3$ yr **(b)**. $Q_0 = 100000e^{-.5}$ **4.1.11 (p. 192)** **(a)** $Q(t) = 5000 - 4750e^{-t/10}$ **(b)** 5000 lbs
4.1.12 (p. 192) $\dfrac{1}{25}$ yrs; **4.1.13 (p. 192)** $V = V_0e^{t\ln 10/2}$ 4 hours
4.1.14 (p. 192) $\dfrac{1500\ln\frac{4}{3}}{\ln 2}$ yrs; $2^{-4/3}Q_0$ **4.1.15 (p. 192)** $W(t) = 20 - 19e^{-t/20}$; $\lim_{t\to\infty} W(t) = 20$ ounces
4.1.16 (p. ??) $S(t) = 10(1 + e^{-t/10})$; $\lim_{t\to\infty} S(t) = 10$ g **4.1.17 (p. ??)** 10 gallons
4.1.18 (p. ??) $V(t) = 15000 + 10000e^{t/20}$ **4.1.19 (p. ??)** $W(t) = 4\times 10^6(t+1)^2$ dollars t years from now
4.1.20 (p. ??) $p = \dfrac{100}{25 - 24e^{-t/2}}$ **4.1.21 (p. ??)** **(a)** $P(t) = 1000e^{.06t} + 50\dfrac{e^{.06t}-1}{e^{.06/52}-1}$ **(b)** 5.64×10^{-4}
4.1.22 (p. ??) **(a)** $P' = rP - 12M$ **(b)** $P = \dfrac{12M}{r}(1 - e^{rt}) + P_0e^{rt}$ **(c)** $M \approx \dfrac{rP_0}{12(1 - e^{-rN})}$

(d) For **(i)** approximate $M = \$402.25$, exact $M = \$402.80$
for **(ii)** approximate $M = \$1206.05$, exact $M = \$1206.93$.

4.1.23 (p. ??) **(a)** $T(\alpha) = -\frac{1}{r}\ln\left(1-\left(1-e^{-rN}\right)/\alpha\right)$ years

$S(\alpha) = \frac{P_0}{(1-e^{-rN})}\left[rN + \alpha\ln\left(1-(1-e^{-rN})/\alpha\right)\right]$

(b) $T(1.05) = 13.69$ yrs, $S(1.05) = \$3579.94$ $T(1.10) = 12.61$ yrs, $S(1.10) = \$6476.63$ $T(1.15) = 11.70$ yrs, $S(1.15) = \$8874.98$.

4.1.24 (p. ??) $P_0 = \begin{cases} \dfrac{S_0(1-e^{(a-r)T})}{r-a} & \text{if } a \neq r, \\ S_0T & \text{if } a = r. \end{cases}$

Section 4.2 Answers, pp. 202–??

4.2.1 (p. 202) $\approx 15.15°$F **4.2.2 (p. 202)** $T = -10 + 110e^{-t\ln\frac{11}{9}}$ **4.2.3 (p. 202)** $\approx 24.33°$F
4.2.4 (p. 202) **(a)** 91.30°F **(b)** 8.99 minutes after being placed outside **(c)** never
4.2.5 (p. 202) **(a)** 12:11:32 **(b)** 12:47:33 **4.2.6 (p. 202)** $(85/3)°$C **4.2.7 (p. 202)** 32°F **4.2.8 (p. 202)** $Q(t) = 40(1-e^{-3t/40})$
4.2.9 (p. 203) $Q(t) = 30 - 20e^{-t/10}$ **4.2.10 (p. 203)** $K(t) = .3 - .2e^{-t/20}$ **4.2.11 (p. 203)** $Q(50) = 47.5$ (pounds)
4.2.12 (p. 203) 50 gallons **4.2.13 (p. 203)** $\min q_2 = q_1/\bar{c}$ **4.2.14 (p. 203)** $Q = t + 300 - \frac{234\times 10^5}{(t+300)^2}, \quad 0 \leqslant t \leqslant 300$
4.2.15 (p. 203) **(a)** $Q' + \frac{2}{25}Q = 6 - 2e^{-t/25}$ **(b)** $Q = 75 - 50e^{-t/25} - 25e^{-2t/25}$ **(c)** 75
4.2.16 (p. 203) **(a)** $T = T_m + (T_0 - T_m)e^{-kt} + \frac{k(S_0 - T_m)}{(k-k_m)}\left(e^{-k_m t} - e^{-kt}\right)$
(b) $T = T_m + k(S_0 - T_m)te^{-kt} + (T_0 - T_m)e^{-kt}$ **(c)** $\lim_{t\to\infty} T(t) = \lim_{t\to\infty} S(t) = T_m$
4.2.17 (p. 203) **(a)** $T' = -k\left(1+\frac{a}{a_m}\right)T + k\left(T_{m0} + \frac{a}{a_m}T_0\right)$ **(b)** $T = \frac{aT_0 + a_mT_{m0}}{a+a_m} + \frac{a_m(T_0 - T_{m0})}{a+a_m}e^{-k(1+a/a_m)t}$,
$T_m = \frac{aT_0 + a_mT_{m0}}{a+a_m} + \frac{a(T_{m0} - T_0)}{a+a_m}e^{-k(1+a/a_m)t}$; **(c)** $\lim_{t\to\infty} T(t) = \lim_{t\to\infty} T_m(t) = \frac{aT_0 + a_mT_{m0}}{a+a_m}$
4.2.18 (p. 203) $V = \frac{a}{b}\frac{V_0}{V_0 - (V_0 - a/b)\,e^{-at}}, \quad \lim_{t\to\infty} V(t) = a/b$
4.2.19 (p. 203) $c_1 = c\left(1 - e^{-rt/W}\right)$, $c_2 = c\left(1 - e^{-rt/W} - \frac{r}{W}te^{-rt/W}\right)$.
4.2.20 (p. ??) **(a)** $c_n = c\left(1 - e^{-rt/W}\sum_{j=0}^{n-1}\frac{1}{j!}\left(\frac{rt}{W}\right)^j\right)$ **(b)** c **(c)** 0
4.2.21 (p. ??) Let $c_\infty = \frac{c_1W_1 + c_2W_2}{W_1+W_2}$, $\alpha = \frac{c_2W_2^2 - c_1W_1^2}{W_1+W_2}$, and $\beta = \frac{W_1+W_2}{W_1W_2}$. Then:
(a) $c_1(t) = c_\infty + \frac{\alpha}{W_1}e^{-r\beta t}$, $c_2(t) = c_\infty - \frac{\alpha}{W_2}e^{-r\beta t}$
(b) $\lim_{t\to\infty} c_1(t) = \lim_{t\to\infty} c_2(t) = c_\infty$

Section 4.3 Answers, pp. 207–208

4.3.1 (p. 207) $v = -\frac{384}{5}\left(1 - e^{-5t/12}\right)$; $-\frac{384}{5}$ ft/s **4.3.2 (p. 207)** $k = 12; \quad v = -16(1-e^{-2t})$
4.3.3 (p. 207) $v = 25(1-e^{-t})$;25 ft/s **4.3.4 (p. 207)** $v = 20 - 27e^{-t/40}$ **4.3.5 (p. 207)** ≈ 17.10 ft
4.3.6 (p. 207) $v = -\frac{40(13 + 3e^{-4t/5})}{13 - 3e^{-4t/5}}$; -40 ft/s **4.3.7 (p. 207)** $v = -128(1 - e^{-t/4})$
4.3.9 (p. 207) $T = \frac{m}{k}\ln\left(1 + \frac{v_0k}{mg}\right); \quad y_m = y_0 + \frac{m}{k}\left[v_0 - \frac{mg}{k}\ln\left(1+\frac{v_0k}{mg}\right)\right]$
4.3.10 (p. 207) $v = -\frac{64(1-e^{-t})}{1+e^{-t}}$; -64 ft/s

4.3.11 (p. 207) $v = \alpha \dfrac{v_0(1+e^{-\beta t}) - \alpha(1-e^{-\beta t})}{\alpha(1+e^{-\beta t}) - v_0(1-e^{-\beta t})}$; $-\alpha$, where $\alpha = \sqrt{\dfrac{mg}{k}}$ and $\beta = 2\sqrt{\dfrac{kg}{m}}$.

4.3.12 (p. 207) $T = \sqrt{\dfrac{m}{kg}} \tan^{-1}\left(v_0\sqrt{\dfrac{k}{mg}}\right)$ $v = -\sqrt{\dfrac{mg}{k}}$; $\dfrac{1-e^{-2\sqrt{\frac{gk}{m}}(t-T)}}{1+e^{-2\sqrt{\frac{gk}{m}}(t-T)}}$

4.3.13 (p. 208) $s' = mg - \dfrac{as}{s+1}$; $a_0 = mg$. **4.3.14 (p. 208)** **(a)** $ms' = mg - f(s)$

4.3.15 (p. 208) **(a)** $v' = -9.8 + v^4/81$ **(b)** $v_T \approx -5.308$ m/s

4.3.16 (p. 208) **(a)** $v' = -32 + 8\sqrt{|v|}$; $v_T = -16$ ft/s **(b)** From Exercise 4.3.14**(c)**, v_T is the negative number such that $-32 + 8\sqrt{|v_T|} = 0$; thus, $v_T = -16$ ft/s.

4.3.17 (p. 208) ≈ 6.76 miles/s **4.3.18 (p. 208)** ≈ 1.47 miles/s **4.3.20 (p. 208)** $\alpha = \dfrac{gR^2}{(y_m + R)^2}$

Section 4.4 Answers, pp. 213–??

4.4.1 (p. 213) $\overline{y} = 0$ is a stable equilibrium; trajectories are $v^2 + \dfrac{y^4}{4} = c$

4.4.2 (p. 213) $\overline{y} = 0$ is an unstable equilibrium; trajectories are $v^2 + \dfrac{2y^3}{3} = c$

4.4.3 (p. 213) $\overline{y} = 0$ is a stable equilibrium; trajectories are $v^2 + \dfrac{2|y|^3}{3} = c$

4.4.4 (p. 213) $\overline{y} = 0$ is a stable equilibrium; trajectories are $v^2 - e^{-y}(y+1) = c$

4.4.5 (p. 213) equilibria: 0 (stable) and $-2, 2$ (unstable); trajectories: $2v^2 - y^4 + 8y^2 = c$; separatrix: $2v^2 - y^4 + 8y^2 = 16$

4.4.6 (p. 213) equilibria: 0 (unstable) and $-2, 2$ (stable); trajectories: $2v^2 + y^4 - 8y^2 = c$; separatrix: $2v^2 + y^4 - 8y^2 = 0$

4.4.7 (p. 213) equilibria: $0, -2, 2$ (stable), $-1, 1$ (unstable); trajectories: $6v^2 + y^2(2y^4 - 15y^2 + 24) = c$; separatrix: $6v^2 + y^2(2y^4 - 15y^2 + 24) = 11$

4.4.8 (p. 213) equilibria: $0, 2$ (stable) and $-2, 1$ (unstable); trajectories: $30v^2 + y^2(12y^3 - 15y^2 - 80y + 120) = c$; separatrices: $30v^2 + y^2(12y^3 - 15y^2 - 80y + 120) = 496$ and $30v^2 + y^2(12y^3 - 15y^2 - 80y + 120) = 37$

4.4.9 (p. 214) No equilibria if $a < 0$; 0 is unstable if $a = 0$; $\sqrt{a}$ is stable and $-\sqrt{a}$ is unstable if $a > 0$.

* **4.4.10 (p. 214)** 0 is a stable equilibrium if $a \leqslant 0$; $-\sqrt{a}$ and $\sqrt{a}$ are stable and 0 is unstable if $a > 0$.

4.4.11 (p. 214) 0 is unstable if $a \leqslant 0$; $-\sqrt{a}$ and $\sqrt{a}$ are unstable and 0 is stable if $a > 0$.

4.4.12 (p. 214) 0 is stable if $a \leqslant 0$; 0 is stable and $-\sqrt{a}$ and $\sqrt{a}$ are unstable if $a \leqslant 0$.

4.4.22 (p. ??) An equilibrium solution $\overline{y}$ of $y'' + p(y) = 0$ is unstable if there's an $\epsilon > 0$ such that, for every $\delta > 0$, there's a solution of (A) with $\sqrt{(y(0) - \overline{y})^2 + v^2(0)} < \delta$, but $\sqrt{(y(t) - \overline{y})^2 + v^2(t)} \geqslant \epsilon$ for some $t > 0$.

Section 4.5 Answers, pp. ??–??

4.5.1 (p. ??) $y' = -\dfrac{2xy}{x^2 + 3y^2}$ **4.5.2 (p. ??)** $y' = -\dfrac{y^2}{(xy-1)}$ **4.5.3 (p. ??)** $y' = -\dfrac{y(x^2 + y^2 - 2x^2 \ln|xy|)}{x(x^2 + y^2 - 2y^2 \ln|xy|)}$.

4.5.4 (p. ??) $xy' - y = -\dfrac{x^{1/2}}{2}$ **4.5.5 (p. ??)** $y' + 2xy = 4xe^{x^2}$ **4.5.6 (p. ??)** $xy' + y = 4x^3$

4.5.7 (p. ??) $y' - y = \cos x - \sin x$ **4.5.8 (p. ??)** $(1 + x^2)y' - 2xy = (1-x)^2 e^x$

4.5.10 (p. ??) $y'g - yg' = f'g - fg'$. **4.5.11 (p. ??)** $(x - x_0)y' = y - y_0$ **4.5.12 (p. ??)** $y'(y^2 - x^2 + 1) + 2xy = 0$ **4.5.13 (p. ??)** $2x(y-1)y' - y^2 + x^2 + 2y = 0$

4.5.14 (p. ??) **(a)** $y = -81 + 18x, (9, 81) \quad y = -1 + 2x, (1, 1)$
(b) $y = -121 + 22x, (11, 121) \quad y = -1 + 2x, (1, 1)$
(c) $y = -100 - 20x, (-10, 100) \quad y = -4 - 4x, (-2, 4)$
(d) $y = -25 - 10x, (-5, 25) \quad y = -1 - 2x, (-1, 1)$
4.5.15 (p. ??) **(e)** $y = \frac{5 + 3x}{4}, (-3/5, 4/5)\ y = -\frac{5 - 4x}{3}, (4/5, -3/5)$
4.5.17 (p. ??) **(a)** $y = -\frac{1}{2}(1 + x), (1, -1); \quad y = \frac{5}{2} + \frac{x}{10}, (25, 5)$
(b) $y = \frac{1}{4}(4 + x), (4, 2) \quad y = -\frac{1}{4}(4 + x), (4, -2);$
(c) $y = \frac{1}{2}(1 + x), (1, 1) \quad y = \frac{7}{2} + \frac{x}{14}, (49, 7)$
(d) $y = -\frac{1}{2}(1 + x), (1, -1) \quad y = -\frac{5}{2} - \frac{x}{10}, (25, -5)$
4.5.18 (p. ??) $y = 2x^2$ **4.5.19 (p. ??)** $y = \frac{cx}{\sqrt{|x^2 - 1|}}$ **4.5.20 (p. ??)** $y = y_1 + c(x - x_1)$
4.5.21 (p. ??) $y = -\frac{x^3}{2} - \frac{x}{2}$ **4.5.22 (p. ??)** $y = -x \ln|x| + cx$ **4.5.23 (p. ??)** $y = \sqrt{2x + 4}$
4.5.24 (p. ??) $y = \sqrt{x^2 - 3}$ **4.5.25 (p. ??)** $y = kx^2$ **4.5.26 (p. ??)** $(y - x)^3(y + x) = k$
4.5.27 (p. ??) $y^2 = -x + k$ **4.5.28 (p. ??)** $y^2 = -\frac{1}{2}\ln(1 + 2x^2) + k$
4.5.29 (p. ??) $y^2 = -2x - \ln(x - 1)^2 + k$ **4.5.30 (p. ??)** $y = 1 + \sqrt{\frac{9 - x^2}{2}}$; those with $c > 0$
4.5.33 (p. ??) $\tan^{-1}\frac{y}{x} - \frac{1}{2}\ln(x^2 + y^2) = k$ **4.5.34 (p. ??)** $\frac{1}{2}\ln(x^2 + y^2) + (\tan\alpha)\tan^{-1}\frac{y}{x} = k$

Section 5.1 Answers, pp. ??–??

5.1.1 (p. ??) **(c)** $y = -2e^{2x} + e^{5x}$ **(d)** $y = (5k_0 - k_1)\frac{e^{2x}}{3} + (k_1 - 2k_0)\frac{e^{5x}}{3}$.
5.1.2 (p. ??) **(c)** $y = e^x(3\cos x - 5\sin x)$ **(d)** $y = e^x\left(k_0\cos x + (k_1 - k_0)\sin x\right)$
5.1.3 (p. ??) **(c)** $y = e^x(7 - 3x)$ **(d)** $y = e^x\left(k_0 + (k_1 - k_0)x\right)$
5.1.4 (p. ??) (a) $y = \frac{c_1}{x - 1} + \frac{c_2}{x + 1}$ (b) $y = \frac{2}{x - 1} - \frac{3}{x + 1}$; $(-1, 1)$
5.1.5 (p. ??) **(a)** e^x **(b)** $e^{2x}\cos x$ **(c)** $x^2 + 2x - 2$ **(d)** $-\frac{5}{6}x^{-5/6}$ **(e)** $-\frac{1}{x^2}$ **(f)** $(x\ln|x|)^2$ **(g)** $\frac{e^{2x}}{2\sqrt{x}}$
5.1.6 (p. ??) 0 **5.1.7 (p. ??)** $W(x) = (1 - x^2)^{-1}$ **5.1.8 (p. ??)** $W(x) = \frac{1}{x}$ **5.1.10 (p. ??)** $y_2 = e^{-x}$
5.1.11 (p. ??) $y_2 = xe^{3x}$ **5.1.12 (p. ??)** $y_2 = xe^{ax}$ **5.1.13 (p. ??)** $y_2 = \frac{1}{x}$ **5.1.14 (p. ??)** $y_2 = x\ln x$
5.1.15 (p. ??) $y_2 = x^a \ln x$ **5.1.16 (p. ??)** $y_2 = x^{1/2}e^{-2x}$ **5.1.17 (p. ??)** $y_2 = x$ **5.1.18 (p. ??)** $y_2 = x\sin x$ **5.1.19 (p. ??)** $y_2 = x^{1/2}\cos x$ **5.1.20 (p. ??)** $y_2 = xe^{-x}$ **5.1.21 (p. ??)** $y_2 = \frac{1}{x^2 - 4}$
5.1.22 (p. ??) $y_2 = e^{2x}$
5.1.23 (p. ??) $y_2 = x^2$ **5.1.35 (p. ??)** **(a)** $y'' - 2y' + 5y = 0$ **(b)** $(2x - 1)y'' - 4xy' + 4y = 0$ **(c)** $x^2y'' - xy' + y = 0$
(d) $x^2y'' + xy' + y = 0$ **(e)** $y'' - y = 0$ **(f)** $xy'' - y' = 0$
5.1.37 (p. ??) **(c)** $y = k_0y_1 + k_1y_2$ **5.1.38 (p. ??)** $y_1 = 1, y_2 = x - x_0; y = k_0 + k_1(x - x_0)$
5.1.39 (p. ??) $y_1 = \cosh(x - x_0), y_2 = \sinh(x - x_0); y = k_0\cosh(x - x_0) + k_1\sinh(x - x_0)$
5.1.40 (p. ??) $y_1 = \cos\omega(x - x_0), y_2 = \frac{1}{\omega}\sin\omega(x - x_0)\ y = k_0\cos\omega(x - x_0) + \frac{k_1}{\omega}\sin\omega(x - x_0)$

5.1.41 (p. ??) $y_1 = \dfrac{1}{1-x^2}$, $y_2 = \dfrac{x}{1-x^2}$ $y = \dfrac{k_0 + k_1 x}{1-x^2}$
5.1.42 (p. ??) **(c)** $k_0 = k_1 = 0$; $y = \begin{cases} c_1x^2 + c_2x^3, x \geqslant 0, \\ c_1x^2 + c_3x^3, x < 0 \end{cases}$
(d) $(0, \infty)$ if $x_0 > 0$, $(-\infty, 0)$ if $x_0 < 0$
5.1.43 (p. ??) **(c)** $k_0 = 0$, k_1 arbitrary $y = k_1x + c_2x^2$
5.1.44 (p. ??) **(c)** $k_0 = k_1 = 0$ $y = \begin{cases} a_1x^3 + a_2x^4, x \geqslant 0, \\ b_1x^3 + b_2x^4, x < 0 \end{cases}$
(d) $(0, \infty)$ if $x_0 > 0$, $(-\infty, 0)$ if $x_0 < 0$

Section 5.2 Answers, pp. ??–??

5.2.1 (p. ??) $y = c_1e^{-6x} + c_2e^{x}$ **5.2.2 (p. ??)** $y = e^{2x}(c_1 \cos x + c_2 \sin x)$ **5.2.3 (p. ??)** $y = c_1e^{-7x} + c_2e^{-x}$
5.2.4 (p. ??) $y = e^{2x}(c_1 + c_2x)$ **5.2.5 (p. ??)** $y = e^{-x}(c_1 \cos 3x + c_2 \sin 3x)$
5.2.6 (p. ??) $y = e^{-3x}(c_1 \cos x + c_2 \sin x)$ **5.2.7 (p. ??)** $y = e^{4x}(c_1 + c_2x)$ **5.2.8 (p. ??)** $y = c_1 + c_2e^{-x}$
5.2.9 (p. ??) $y = e^{x}(c_1 \cos \sqrt{2}x + c_2 \sin \sqrt{2}x)$ **5.2.10 (p. ??)** $y = e^{-3x}(c_1 \cos 2x + c_2 \sin 2x)$
5.2.11 (p. ??) $y = e^{-x/2}\left(c_1 \cos \dfrac{3x}{2} + c_2 \sin \dfrac{3x}{2}\right)$ **5.2.12 (p. ??)** $y = c_1e^{-x/5} + c_2e^{x/2}$
5.2.13 (p. ??) $y = e^{-7x}(2\cos x - 3\sin x)$ **5.2.14 (p. ??)** $y = 4e^{x/2} + 6e^{-x/3}$ **5.2.15 (p. ??)** $y = 3e^{x/3} - 4e^{-x/2}$
5.2.16 (p. ??) $y = \dfrac{e^{-x/2}}{3} + \dfrac{3e^{3x/2}}{4}$ **5.2.17 (p. ??)** $y = e^{3x/2}(3 - 2x)$ **5.2.18 (p. ??)** $y = 3e^{-4x} - 4e^{-3x}$
5.2.19 (p. ??) $y = 2xe^{3x}$ **5.2.20 (p. ??)** $y = e^{x/6}(3+2x)$ **5.2.21 (p. ??)** $y = e^{-2x}\left(3\cos \sqrt{6}x + \dfrac{2\sqrt{6}}{3} \sin \sqrt{6}x\right)$
5.2.23 (p. ??) $y = 2e^{-(x-1)} - 3e^{-2(x-1)}$ **5.2.24 (p. ??)** $y = \dfrac{1}{3}e^{-(x-2)} - \dfrac{2}{3}e^{7(x-2)}$
5.2.25 (p. ??) $y = e^{7(x-1)}\left(2 - 3(x-1)\right)$ **5.2.26 (p. ??)** $y = e^{-(x-2)/3}\left(2 - 4(x-2)\right)$
5.2.27 (p. ??) $y = 2\cos \dfrac{2}{3}\left(x - \dfrac{\pi}{4}\right) - 3\sin \dfrac{2}{3}\left(x - \dfrac{\pi}{4}\right)$ **5.2.28 (p. ??)** $y = 2\cos \sqrt{3}\left(x - \dfrac{\pi}{3}\right) - \dfrac{1}{\sqrt{3}} \sin \sqrt{3}\left(x - \dfrac{\pi}{3}\right)$
5.2.30 (p. ??) $y = \dfrac{k_0}{r_2 - r_1}\left(r_2e^{r_1(x-x_0)} - r_1e^{r_2(x-x_0)}\right) + \dfrac{k_1}{r_2 - r_1}\left(e^{r_2(x-x_0)} - e^{r_1(x-x_0)}\right)$
5.2.31 (p. ??) $y = e^{r_1(x-x_0)}\left[k_0 + (k_1 - r_1k_0)(x - x_0)\right]$
5.2.32 (p. ??) $y = e^{\lambda(x-x_0)}\left[k_0 \cos \omega(x - x_0) + \left(\dfrac{k_1 - \lambda k_0}{\omega}\right) \sin \omega(x - x_0)\right]$

Section 5.3 Answers, pp. ??–??

5.3.1 (p. ??) $y_p = -1 + 2x + 3x^2$; $y = -1 + 2x + 3x^2 + c_1e^{-6x} + c_2e^{x}$
5.3.2 (p. ??) $y_p = 1 + x$; $y = 1 + x + e^{2x}(c_1 \cos x + c_2 \sin x)$
5.3.3 (p. ??) $y_p = -x + x^3$; $y = -x + x^3 + c_1e^{-7x} + c_2e^{-x}$
5.3.4 (p. ??) $y_p = 1 - x^2$; $y = 1 - x^2 + e^{2x}(c_1 + c_2x)$
5.3.5 (p. ??) $y_p = 2x + x^3$; $y = 2x + x^3 + e^{-x}(c_1 \cos 3x + c_2 \sin 3x)$;
$y = 2x + x^3 + e^{-x}(2\cos 3x + 3\sin 3x)$
5.3.6 (p. ??) $y_p = 1 + 2x$; $y = 1 + 2x + e^{-3x}(c_1 \cos x + c_2 \sin x)$; $y = 1 + 2x + e^{-3x}(\cos x - \sin x)$
5.3.8 (p. ??) $y_p = \dfrac{2}{x}$ **5.3.9 (p. ??)** $y_p = 4x^{1/2}$ **5.3.10 (p. ??)** $y_p = \dfrac{x^3}{2}$ **5.3.11 (p. ??)** $y_p = \dfrac{1}{x^3}$
5.3.12 (p. ??) $y_p = 9x^{1/3}$ **5.3.13 (p. ??)** $y_p = \dfrac{2x^4}{13}$ **5.3.16 (p. ??)** $y_p = \dfrac{e^{3x}}{3}$; $y = \dfrac{e^{3x}}{3} + c_1e^{-6x} + c_2e^{x}$ **5.3.17 (p. ??)** $y_p = e^{2x}$; $y = e^{2x}(1 + c_1 \cos x + c_2 \sin x)$

5.3.18 (p. ??) $y = -2e^{-2x}$; $y = -2e^{-2x} + c_1e^{-7x} + c_2e^{-x}$; $y = -2e^{-2x} - e^{-7x} + e^{-x}$
5.3.19 (p. ??) $y_p = e^x$; $y = e^x + e^{2x}(c_1 + c_2x)$; $y = e^x + e^{2x}(1 - 3x)$
5.3.20 (p. ??) $y_p = \frac{4}{45}e^{x/2}$; $y = \frac{4}{45}e^{x/2} + e^{-x}(c_1 \cos 3x + c_2 \sin 3x)$
5.3.21 (p. ??) $y_p = e^{-3x}$; $y = e^{-3x}(1 + c_1 \cos x + c_2 \sin x)$
5.3.24 (p. ??) $y_p = \cos x - \sin x$; $y = \cos x - \sin x + e^{4x}(c_1 + c_2x)$
5.3.25 (p. ??) $y_p = \cos 2x - 2\sin 2x$; $y = \cos 2x - 2\sin 2x + c_1 + c_2e^{-x}$
5.3.26 (p. ??) $y_p = \cos 3x$; $y = \cos 3x + e^x(c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x)$
5.3.27 (p. ??) $y_p = \cos x + \sin x$; $y = \cos x + \sin x + e^{-3x}(c_1 \cos 2x + c_2 \sin 2x)$
5.3.28 (p. ??) $y_p = -2\cos 2x + \sin 2x$; $y = -2\cos 2x + \sin 2x + c_1e^{-4x} + c_2e^{-3x}$
$y = -2\cos 2x + \sin 2x + 2e^{-4x} - 3e^{-3x}$
5.3.29 (p. ??) $y_p = \cos 3x - \sin 3x$; $y = \cos 3x - \sin 3x + e^{3x}(c_1 + c_2x)$
$y = \cos 3x - \sin 3x + e^{3x}(1 + 2x)$
5.3.30 (p. ??) $y = \frac{1}{\omega_0^2 - \omega^2}(M\cos\omega x + N\sin\omega x) + c_1\cos\omega_0x + c_2\sin\omega_0x$
5.3.33 (p. ??) $y_p = -1 + 2x + 3x^2 + \frac{e^{3x}}{3}$; $y = -1 + 2x + 3x^2 + \frac{e^{3x}}{3} + c_1e^{-6x} + c_2e^x$
5.3.34 (p. ??) $y_p = 1 + x + e^{2x}$; $y = 1 + x + e^{2x}(1 + c_1 \cos x + c_2 \sin x)$
5.3.35 (p. ??) $y_p = -x + x^3 - 2e^{-2x}$; $y = -x + x^3 - 2e^{-2x} + c_1e^{-7x} + c_2e^{-x}$
5.3.36 (p. ??) $y_p = 1 - x^2 + e^x$; $y = 1 - x^2 + e^x + e^{2x}(c_1 + c_2x)$
5.3.37 (p. ??) $y_p = 2x + x^3 + \frac{4}{45}e^{x/2}$; $y = 2x + x^3 + \frac{4}{45}e^{x/2} + e^{-x}(c_1 \cos 3x + c_2 \sin 3x)$
5.3.38 (p. ??) $y_p = 1 + 2x + e^{-3x}$; $y = 1 + 2x + e^{-3x}(1 + c_1 \cos x + c_2 \sin x)$

Section 5.4 Answers, pp. ??–??

5.4.1 (p. ??) $y_p = e^{3x}\left(-\frac{1}{4} + \frac{x}{2}\right)$ **5.4.2 (p. ??)** $y_p = e^{-3x}\left(1 - \frac{x}{4}\right)$ **5.4.3 (p. ??)** $y_p = e^x\left(2 - \frac{3x}{4}\right)$
5.4.4 (p. ??) $y_p = e^{2x}(1 - 3x + x^2)$ **5.4.5 (p. ??)** $y_p = e^{-x}(1 + x^2)$ **5.4.6 (p. ??)** $y_p = e^x(-2 + x + 2x^2)$
5.4.7 (p. ??) $y_p = xe^{-x}\left(\frac{1}{6} + \frac{x}{2}\right)$ **5.4.8 (p. ??)** $y_p = xe^x(1+2x)$ **5.4.9 (p. ??)** $y_p = xe^{3x}\left(-1 + \frac{x}{2}\right)$
5.4.10 (p. ??) $y_p = xe^{2x}(-2 + x)$ **5.4.11 (p. ??)** $y_p = x^2e^{-x}\left(1 + \frac{x}{2}\right)$ **5.4.12 (p. ??)** $y_p = x^2e^x\left(\frac{1}{2} - x\right)$
5.4.13 (p. ??) $y_p = \frac{x^2e^{2x}}{2}(1 - x + x^2)$ **5.4.14 (p. ??)** $y_p = \frac{x^2e^{-x/3}}{27}(3 - 2x + x^2)$
5.4.15 (p. ??) $y = \frac{e^{3x}}{4}(-1 + 2x) + c_1e^x + c_2e^{2x}$ **5.4.16 (p. ??)** $y = e^x(1 - 2x) + c_1e^{2x} + c_2e^{4x}$
5.4.17 (p. ??) $y = \frac{e^{2x}}{5}(1 - x) + e^{-3x}(c_1 + c_2x)$ **5.4.18 (p. ??)** $y = xe^x(1 - 2x) + c_1e^x + c_2e^{-3x}$
5.4.19 (p. ??) $y = e^x\left[x^2(1 - 2x) + c_1 + c_2x\right]$ **5.4.20 (p. ??)** $y = -e^{2x}(1 + x) + 2e^{-x} - e^{5x}$
5.4.21 (p. ??) $y = xe^{2x} + 3e^x - e^{-4x}$ **5.4.22 (p. ??)** $y = e^{-x}(2 + x - 2x^2) - e^{-3x}$
5.4.23 (p. ??) $y = e^{-2x}(3 - x) - 2e^{5x}$ **5.4.24 (p. ??)** $y_p = -\frac{e^x}{3}(1 - x) + e^{-x}(3 + 2x)$
5.4.25 (p. ??) $y_p = e^x(3 + 7x) + xe^{3x}$ **5.4.26 (p. ??)** $y_p = x^3e^{4x} + 1 + 2x + x^2$
5.4.27 (p. ??) $y_p = xe^{2x}(1 - 2x) + xe^x$ **5.4.28 (p. ??)** $y_p = e^x(1 + x) + x^2e^{-x}$
5.4.29 (p. ??) $y_p = x^2e^{-x} + e^{3x}(1 - x^2)$ **5.4.31 (p. ??)** $y_p = 2e^{2x}$ **5.4.32 (p. ??)** $y_p = 5xe^{4x}$
5.4.33 (p. ??) $y_p = x^2e^{4x}$ **5.4.34 (p. ??)** $y_p = -\frac{e^{3x}}{4}(1 + 2x - 2x^2)$ **5.4.35 (p. ??)** $y_p = xe^{3x}(4 - x + 2x^2)$

5.4.36 (p. ??) $y_p = x^2e^{-x/2}(-1+2x+3x^2)$

5.4.37 (p. ??) **(a)** $y = e^{-x}\left(\frac{4}{3}x^{3/2}+c_1x+c_2\right)$ **(b)** $y = e^{-3x}\left[\frac{x^2}{4}(2\ln x-3)+c_1x+c_2\right]$

(c) $y = e^{2x}\left[(x+1)\ln|x+1|+c_1x+c_2\right]$ **(d)** $y = e^{-x/2}\left(x\ln|x|+\frac{x^3}{6}+c_1x+c_2\right)$

5.4.39 (p. ??) **(a)** $e^x(3+x)+c$ **(b)** $-e^{-x}(1+x)^2+c$ **(c)** $-\frac{e^{-2x}}{8}(3+6x+6x^2+4x^3)+c$

(d) $e^x(1+x^2)+c$ **(e)** $e^{3x}(-6+4x+9x^2)+c$ **(f)** $-e^{-x}(1-2x^3+3x^4)+c$

5.4.40 (p. ??) $\frac{(-1)^k k! e^{\alpha x}}{\alpha^{k+1}}\sum_{r=0}^{k}\frac{(-\alpha x)^r}{r!}+c$

Section 5.5 Answers, pp. ??–??

5.5.1 (p. ??) $y_p = \cos x + 2\sin x$ **5.5.2 (p. ??)** $y_p = \cos x + (2-2x)\sin x$

5.5.3 (p. ??) $y_p = e^x(-2\cos x + 3\sin x)$

5.5.4 (p. ??) $y_p = \frac{e^{2x}}{2}(\cos 2x - \sin 2x)$ **5.5.5 (p. ??)** $y_p = -e^x(x\cos x - \sin x)$

5.5.6 (p. ??) $y_p = e^{-2x}(1-2x)(\cos 3x - \sin 3x)$ **5.5.7 (p. ??)** $y_p = x(\cos 2x - 3\sin 2x)$

5.5.8 (p. ??) $y_p = -x\left[(2-x)\cos x + (3-2x)\sin x\right]$ **5.5.9 (p. ??)** $y_p = x\left[x\cos\left(\frac{x}{2}\right) - 3\sin\left(\frac{x}{2}\right)\right]$

5.5.10 (p. ??) $y_p = xe^{-x}(3\cos x + 4\sin x)$ **5.5.11 (p. ??)** $y_p = xe^x\left[(-1+x)\cos 2x + (1+x)\sin 2x\right]$

5.5.12 (p. ??) $y_p = -(14-10x)\cos x - (2+8x-4x^2)\sin x.$

5.5.13 (p. ??) $y_p = (1+2x+x^2)\cos x + (1+3x^2)\sin x$ **5.5.14 (p. ??)** $y_p = \frac{x^2}{2}(\cos 2x - \sin 2x)$

5.5.15 (p. ??) $y_p = e^x(x^2\cos x + 2\sin x)$ **5.5.16 (p. ??)** $y_p = e^x(1-x^2)(\cos x + \sin x)$

5.5.17 (p. ??) $y_p = e^x(x^2-x^3)(\cos x + \sin x)$ **5.5.18 (p. ??)** $y_p = e^{-x}\left[(1+2x)\cos x - (1-3x)\sin x\right]$

5.5.19 (p. ??) $y_p = x(2\cos 3x - \sin 3x)$ **5.5.20 (p. ??)** $y_p = -x^3\cos x + (x+2x^2)\sin x$

5.5.21 (p. ??) $y_p = -e^{-x}\left[(x+x^2)\cos x - (1+2x)\sin x\right]$

5.5.22 (p. ??) $y = e^x(2\cos x + 3\sin x) + 3e^x - e^{6x}$ **5.5.23 (p. ??)** $y = e^x\left[(1+2x)\cos x + (1-3x)\sin x\right]$

5.5.24 (p. ??) $y = e^x(\cos x - 2\sin x) + e^{-3x}(\cos x + \sin x)$ **5.5.25 (p. ??)** $y = e^{3x}\left[(2+2x)\cos x - (1+3x)\sin x\right]$

5.5.26 (p. ??) $y = e^{3x}\left[(2+3x)\cos x + (4-x)\sin x\right] + 3e^x - 5e^{2x}$ **5.5.27 (p. ??)** $y_p = xe^{3x} - \frac{e^x}{5}(\cos x - 2\sin x)$

5.5.28 (p. ??) $y_p = x(\cos x + 2\sin x) - \frac{e^x}{2}(1-x) + \frac{e^{-x}}{2}$

5.5.29 (p. ??) $y_p = -\frac{xe^x}{2}(2+x) + 2xe^{2x} + \frac{1}{10}(3\cos x + \sin x)$

5.5.30 (p. ??) $y_p = xe^x(\cos x + x\sin x) + \frac{e^{-x}}{25}(4+5x) + 1 + x + \frac{x^2}{2}$

5.5.31 (p. ??) $y_p = \frac{x^2e^{2x}}{6}(3+x) - e^{2x}(\cos x - \sin x) + 3e^{3x} + \frac{1}{4}(2+x)$

5.5.32 (p. ??) $y = (1-2x+3x^2)e^{2x} + 4\cos x + 3\sin x$ **5.5.33 (p. ??)** $y = xe^{-2x}\cos x + 3\cos 2x$

5.5.34 (p. ??) $y = -\frac{3}{8}\cos 2x + \frac{1}{4}\sin 2x + e^{-x} - \frac{13}{8}e^{-2x} - \frac{3}{4}xe^{-2x}$

5.5.40 (p. ??) **(a)** $2x\cos x - (2-x^2)\sin x + c$ **(b)** $-\frac{e^x}{2}\left[(1-x^2)\cos x - (1-x)^2\sin x\right] + c$

(c) $-\frac{e^{-x}}{25}\left[(4+10x)\cos 2x - (3-5x)\sin 2x\right] + c$

(d) $-\frac{e^{-x}}{2}\left[(1+x)^2\cos x - (1-x^2)\sin x\right] + c$

(e) $-\frac{e^x}{2}\left[x(3-3x+x^2)\cos x - (3-3x+x^3)\sin x\right] + c$

(f) $-e^x\left[(1-2x)\cos x + (1+x)\sin x\right] + c$ **(g)** $e^{-x}\left[x\cos x + x(1+x)\sin x\right] + c$

Section 5.6 Answers, pp. ??–??

5.6.1 (p. ??) $y = 1 - 2x + c_1e^{-x} + c_2xe^x$; $\{e^{-x}, xe^x\}$ **5.6.2 (p. ??)** $y = \dfrac{4}{3x^2} + c_1x + \dfrac{c_2}{x}$; $\{x, 1/x\}$
5.6.3 (p. ??) $y = \dfrac{x(\ln|x|)^2}{2} + c_1x + c_2x\ln|x|$; $\{x, x\ln|x|\}$
5.6.4 (p. ??) $y = (e^{2x} + e^x)\ln(1 + e^{-x}) + c_1e^{2x} + c_2e^x$; $\{e^{2x}, e^x\}$
5.6.5 (p. ??) $y = e^x\left(\dfrac{4}{5}x^{7/2} + c_1 + c_2x\right)$; $\{e^x, xe^x\}$
5.6.6 (p. ??) $y = e^x(2x^{3/2} + x^{1/2}\ln x + c_1x^{1/2} + c_2x^{-1/2})$; $\{x^{1/2}e^x, x^{-1/2}e^{-x}\}$
5.6.7 (p. ??) $y = e^x(x\sin x + \cos x\ln|\cos x| + c_1\cos x + c_2\sin x)$; $\{e^x\cos x, e^x\sin x\}$
5.6.8 (p. ??) $y = e^{-x^2}(2e^{-2x} + c_1 + c_2x)$; $\{e^{-x^2}, xe^{-x^2}\}$
5.6.9 (p. ??) $y = 2x + 1 + c_1x^2 + \dfrac{c_2}{x^2}$; $\{x^2, 1/x^2\}$
5.6.10 (p. ??) $y = \dfrac{xe^{2x}}{9} + xe^{-x}(c_1 + c_2x)$; $\{xe^{-x}, x^2e^{-x}\}$
5.6.11 (p. ??) $y = xe^x\left(\dfrac{x}{3} + c_1 + \dfrac{c_2}{x^2}\right)$; $\{xe^x, e^x/x\}$
5.6.12 (p. ??) $y = -\dfrac{(2x-1)^2e^x}{8} + c_1e^x + c_2xe^{-x}$; $\{e^x, xe^{-x}\}$
5.6.13 (p. ??) $y = x^4 + c_1x^2 + c_2x^2\ln|x|$; $\{x^2, x^2\ln|x|\}$
5.6.14 (p. ??) $y = e^{-x}(x^{3/2} + c_1 + c_2x^{1/2})$; $\{e^{-x}, x^{1/2}e^{-x}\}$
5.6.15 (p. ??) $y = e^x(x + c_1 + c_2x^2)$; $\{e^x, x^2e^x\}$ **5.6.16 (p. ??)** $y = x^{1/2}\left(\dfrac{e^{2x}}{2} + c_1 + c_2e^x\right)$; $\{x^{1/2}, x^{1/2}e^x\}$
5.6.17 (p. ??) $y = -2x^2\ln x + c_1x^2 + c_2x^4$; $\{x^2, x^4\}$ **5.6.18 (p. ??)** $\{e^x, e^x/x\}$ **5.6.19 (p. ??)** $\{x^2, x^3\}$
5.6.20 (p. ??) $\{\ln|x|, x\ln|x|\}$ **5.6.21 (p. ??)** $\{\sin\sqrt{x}, \cos\sqrt{x}\}$ **5.6.22 (p. ??)** $\{e^x, x^3e^x\}$ **5.6.23 (p. ??)** $\{x^a, x^a\ln x\}$
5.6.24 (p. ??) $\{x\sin x, x\cos x\}$ **5.6.25 (p. ??)** $\{e^{2x}, x^2e^{2x}\}$ **5.6.26 (p. ??)** $\{x^{1/2}, x^{1/2}\cos x\}$
5.6.27 (p. ??) $\{x^{1/2}e^{2x}, x^{1/2}e^{-2x}\}$ **5.6.28 (p. ??)** $\{1/x, e^{2x}\}$ **5.6.29 (p. ??)** $\{e^x, x^2\}$ **5.6.30 (p. ??)** $\{e^{2x}, x^2e^{2x}\}$ **5.6.31 (p. ??)** $y = x^4 + 6x^2 - 8x^2\ln|x|$
5.6.32 (p. ??) $y = 2e^{2x} - xe^{-x}$ **5.6.33 (p. ??)** $y = \dfrac{(x+1)}{4}\left[-e^x(3-2x) + 7e^{-x}\right]$
5.6.34 (p. ??) $y = \dfrac{x^2}{4} + x$ **5.6.35 (p. ??)** $y = \dfrac{(x+2)^2}{6(x-2)} + \dfrac{2x}{x^2-4}$
5.6.38 (p. ??) **(a)** $y = \dfrac{-kc_1\sin kx + kc_2\cos kx}{c_1\cos kx + c_2\sin kx}$ **(b)** $y = \dfrac{c_1 + 2c_2e^x}{c_1 + c_2e^x}$

(c) $y = \dfrac{-6c_1 + c_2e^{7x}}{c_1 + c_2e^{7x}}$ **(d)** $y = -\dfrac{7c_1 + c_2e^{6x}}{c_1 + c_2e^{6x}}$
(e) $y = -\dfrac{(7c_1 - c_2)\cos x + (c_1 + 7c_2)\sin x}{c_1\cos x + c_2\sin x}$
(f) $y = \dfrac{-2c_1 + 3c_2e^{5x/6}}{6(c_1 + c_2e^{5x/6})}$ **(g)** $y = \dfrac{c_1 + c_2(x+6)}{6(c_1 + c_2x)}$
5.6.39 (p. ??) **(a)** $y = \dfrac{c_1 + c_2e^x(1+x)}{x(c_1 + c_2e^x)}$ **(b)** $y = \dfrac{-2c_1x + c_2(1-2x^2)}{c_1 + c_2x}$

(c) $y = \dfrac{-c_1 + c_2e^{2x}(x+1)}{c_1 + c_2xe^{2x}}$ **(d)** $y = \dfrac{2c_1 + c_2e^{-3x}(1-x)}{c_1 + c_2xe^{-3x}}$
(e) $y = \dfrac{(2c_2x - c_1)\cos x - (2c_1x + c_2)\sin x}{2x(c_1\cos x + c_2\sin x)}$ **(f)** $y = \dfrac{c_1 + 7c_2x^6}{x(c_1 + c_2x^6)}$

Section 5.7 Answers, pp. ??–??

5.7.1 (p. ??) $y_p = \dfrac{-\cos 3x \ln|\sec 3x + \tan 3x|}{9}$ **5.7.2 (p. ??)** $y_p = -\dfrac{\sin 2x \ln|\cos 2x|}{4} + \dfrac{x\cos 2x}{2}$
5.7.3 (p. ??) $y_p = 4e^x(1+e^x)\ln(1+e^{-x})$ **5.7.4 (p. ??)** $y_p = 3e^x(\cos x \ln|\cos x| + x\sin x)$
5.7.5 (p. ??) $y_p = \dfrac{8}{5}x^{7/2}e^x$ **5.7.6 (p. ??)** $y_p = e^x \ln(1-e^{-2x}) - e^{-x}\ln(e^{2x}-1)$ **5.7.7 (p. ??)** $y_p = \dfrac{2(x^2-3)}{3}$
5.7.8 (p. ??) $y_p = \dfrac{e^{2x}}{x}$ **5.7.9 (p. ??)** $y_p = x^{1/2}e^x \ln x$ **5.7.10 (p. ??)** $y_p = e^{-x(x+2)}$
5.7.11 (p. ??) $y_p = -4x^{5/2}$ **5.7.12 (p. ??)** $y_p = -2x^2\sin x - 2x\cos x$ **5.7.13 (p. ??)** $y_p = -\dfrac{xe^{-x}(x+1)}{2}$
5.7.14 (p. ??) $y_p = -\dfrac{\sqrt{x}\cos\sqrt{x}}{2}$ **5.7.15 (p. ??)** $y_p = \dfrac{3x^4e^x}{2}$ **5.7.16 (p. ??)** $y_p = x^{a+1}$
5.7.17 (p. ??) $y_p = \dfrac{x^2\sin x}{2}$ **5.7.18 (p. ??)** $y_p = -2x^2$ **5.7.19 (p. ??)** $y_p = -e^{-x}\sin x$
5.7.20 (p. ??) $y_p = -\dfrac{\sqrt{x}}{2}$ **5.7.21 (p. ??)** $y_p = \dfrac{x^{3/2}}{4}$ **5.7.22 (p. ??)** $y_p = -3x^2$
5.7.23 (p. ??) $y_p = \dfrac{x^3e^x}{2}$ **5.7.24 (p. ??)** $y_p = -\dfrac{4x^{3/2}}{15}$ **5.7.25 (p. ??)** $y_p = x^3e^x$ **5.7.26 (p. ??)** $y_p = xe^x$
5.7.27 (p. ??) $y_p = x^2$ **5.7.28 (p. ??)** $y_p = xe^x(x-2)$ **5.7.29 (p. ??)** $y_p = \sqrt{x}e^x(x-1)/4$
5.7.30 (p. ??) $y = \dfrac{e^{2x}(3x^2-2x+6)}{6} + \dfrac{xe^{-x}}{3}$ **5.7.31 (p. ??)** $y = (x-1)^2\ln(1-x) + 2x^2 - 5x + 3$
5.7.32 (p. ??) $y = (x^2-1)e^x - 5(x-1)$ **5.7.33 (p. ??)** $y = \dfrac{x(x^2+6)}{3(x^2-1)}$ **5.7.34 (p. ??)** $y = -\dfrac{x^2}{2} + x + \dfrac{1}{2x^2}$ **5.7.35 (p. ??)** $y = \dfrac{x^2(4x+9)}{6(x+1)}$
5.7.38 (p. ??) **(a)** $y = k_0\cosh x + k_1\sinh x + \int_0^x \sinh(x-t)f(t)\,dt$
(b) $y' = k_0\sinh x + k_1\cosh x + \int_0^x \cosh(x-t)f(t)\,dt$
5.7.39 (p. ??) **(a)** $y(x) = k_0\cos x + k_1\sin x + \int_0^x \sin(x-t)f(t)\,dt$
(b) $y'(x) = -k_0\sin x + k_1\cos x + \int_0^x \cos(x-t)f(t)\,dt$

Section 6.1 Answers, pp. ??–??

6.1.1 (p. ??) $y = 3\cos 4\sqrt{6}t - \dfrac{1}{2\sqrt{6}}\sin 4\sqrt{6}t$ ft **6.1.2 (p. ??)** $y = -\dfrac{1}{4}\cos 8\sqrt{5}t - \dfrac{1}{4\sqrt{5}}\sin 8\sqrt{5}t$ ft
6.1.3 (p. ??) $y = 1.5\cos 14\sqrt{10}t$ cm
6.1.4 (p. ??) $y = \dfrac{1}{4}\cos 8t - \dfrac{1}{16}\sin 8t$ ft; $R = \dfrac{\sqrt{17}}{16}$ ft; $\omega_0 = 8$ rad/s; $T = \pi/4$ s;
$\phi \approx -.245$ rad $\approx -14.04°$;
6.1.5 (p. ??) $y = 10\cos 14t + \dfrac{25}{14}\sin 14t$ cm; $R = \dfrac{5}{14}\sqrt{809}$ cm; $\omega_0 = 14$ rad/s; $T = \pi/7$ s;
$\phi \approx .177$ rad $\approx 10.12°$
6.1.6 (p. ??) $y = -\dfrac{1}{4}\cos\sqrt{70}\,t + \dfrac{2}{\sqrt{70}}\sin\sqrt{70}\,t$ m; $R = \dfrac{1}{4}\sqrt{\dfrac{67}{35}}$ m $\omega_0 = \sqrt{70}$ rad/s;
$T = 2\pi/\sqrt{70}$ s; $\phi \approx 2.38$ rad $\approx 136.28°$

6.1.7 (p. ??) $y = \frac{2}{3}\cos 16t - \frac{1}{4}\sin 16t$ ft **6.1.8 (p. ??)** $y = \frac{1}{2}\cos 8t - \frac{3}{8}\sin 8t$ ft **6.1.9 (p. ??)** .72 m
6.1.10 (p. ??) $y = \frac{1}{3}\sin t + \frac{1}{2}\cos 2t + \frac{5}{6}\sin 2t$ ft **6.1.11 (p. ??)** $y = \frac{16}{5}\left(4\sin\frac{t}{4} - \sin t\right)$
6.1.12 (p. ??) $y = -\frac{1}{16}\sin 8t + \frac{1}{3}\cos 4\sqrt{2}t - \frac{1}{8\sqrt{2}}\sin 4\sqrt{2}t$
6.1.13 (p. ??) $y = -t\cos 8t - \frac{1}{6}\cos 8t + \frac{1}{8}\sin 8t$ ft **6.1.14 (p. ??)** $T = 4\sqrt{2}$ s
6.1.15 (p. ??) $\omega = 8$ rad/s $y = -\frac{t}{16}(-\cos 8t + 2\sin 8t) + \frac{1}{128}\sin 8t$ ft
6.1.16 (p. ??) $\omega = 4\sqrt{6}$ rad/s; $y = -\frac{t}{\sqrt{6}}\left[\frac{8}{3}\cos 4\sqrt{6}t + 4\sin 4\sqrt{6}t\right] + \frac{1}{9}\sin 4\sqrt{6}t$ ft
6.1.17 (p. ??) $y = \frac{t}{2}\cos 2t - \frac{t}{4}\sin 2t + 3\cos 2t + 2\sin 2t$ m
6.1.18 (p. ??) $y = y_0\cos\omega_0 t + \frac{\nu_0}{\omega_0}\sin\omega_0 t$; $R = \frac{1}{\omega_0}\sqrt{(\omega_0 y_0)^2 + (\nu_0)^2}$;
$\cos\phi = \frac{y_0\omega_0}{\sqrt{(\omega_0 y_0)^2 + (\nu_0)^2}}$; $\sin\phi = \frac{\nu_0}{\sqrt{(\omega_0 y_0)^2 + (\nu_0)^2}}$
6.1.19 (p. ??) The object with the longer period weighs four times as much as the other.
6.1.20 (p. ??) $T_2 = \sqrt{2}T_1$, where T_1 is the period of the smaller object.
6.1.21 (p. ??) $k_1 = 9k_2$, where k_1 is the spring constant of the system with the shorter period.

Section 6.2 Answers, pp. ??–??

6.2.1 (p. ??) $y = \frac{e^{-2t}}{2}(3\cos 2t - \sin 2t)$ ft; $\sqrt{\frac{5}{2}}e^{-2t}$ ft
6.2.2 (p. ??) $y = -e^{-t}\left(3\cos 3t + \frac{1}{3}\sin 3t\right)$ ft $\frac{\sqrt{82}}{3}e^{-t}$ ft
6.2.3 (p. ??) $y = e^{-16t}\left(\frac{1}{4} + 10t\right)$ ft **6.2.4 (p. ??)** $y = -\frac{e^{-3t}}{4}(5\cos t + 63\sin t)$ ft
6.2.5 (p. ??) $0 \leqslant c < 8$ lb-sec/ft **6.2.6 (p. ??)** $y = \frac{1}{2}e^{-3t}\left(\cos\sqrt{91}t + \frac{11}{\sqrt{91}}\sin\sqrt{91}t\right)$ ft
6.2.7 (p. ??) $y = -\frac{e^{-4t}}{3}(2 + 8t)$ ft **6.2.8 (p. ??)** $y = e^{-10t}\left(9\cos 4\sqrt{6}t + \frac{45}{2\sqrt{6}}\sin 4\sqrt{6}t\right)$ cm
6.2.9 (p. ??) $y = e^{-3t/2}\left(\frac{3}{2}\cos\frac{\sqrt{41}}{2}t + \frac{9}{2\sqrt{41}}\sin\frac{\sqrt{41}}{2}t\right)$ ft
6.2.10 (p. ??) $y = e^{-\frac{3}{2}t}\left(\frac{1}{2}\cos\frac{\sqrt{119}}{2}t - \frac{9}{2\sqrt{119}}\sin\frac{\sqrt{119}}{2}t\right)$ ft
6.2.11 (p. ??) $y = e^{-8t}\left(\frac{1}{4}\cos 8\sqrt{2}t - \frac{1}{4\sqrt{2}}\sin 8\sqrt{2}t\right)$ ft
6.2.12 (p. ??) $y = e^{-t}\left(-\frac{1}{3}\cos 3\sqrt{11}t + \frac{14}{9\sqrt{11}}\sin 3\sqrt{11}t\right)$ ft
6.2.13 (p. ??) $y_p = \frac{22}{61}\cos 2t + \frac{2}{61}\sin 2t$ ft **6.2.14 (p. ??)** $y = -\frac{2}{3}(e^{-8t} - 2e^{-4t})$
6.2.15 (p. ??) $y = e^{-2t}\left(\frac{1}{10}\cos 4t - \frac{1}{5}\sin 4t\right)$ m **6.2.16 (p. ??)** $y = e^{-3t}(10\cos t - 70\sin t)$ cm
6.2.17 (p. ??) $y_p = -\frac{2}{15}\cos 3t + \frac{1}{15}\sin 3t$ ft
6.2.18 (p. ??) $y_p = \frac{11}{100}\cos 4t + \frac{27}{100}\sin 4t$ cm **6.2.19 (p. ??)** $y_p = \frac{42}{73}\cos t + \frac{39}{73}\sin t$ ft

6.2.20 (p. ??) $y = -\frac{1}{2}\cos 2t + \frac{1}{4}\sin 2t$ m **6.2.21 (p. ??)** $y_p = \frac{1}{c\omega_0}(-\beta\cos\omega_0 t + \alpha\sin\omega_0 t)$
6.2.24 (p. ??) $y = e^{-ct/2m}\left(y_0\cos\omega_1 t + \frac{1}{\omega_1}(v_0 + \frac{cy_0}{2m})\sin\omega_1 t\right)$
6.2.25 (p. ??) $y = \frac{r_2 y_0 - v_0}{r_2 - r_1}e^{r_1 t} + \frac{v_0 - r_1 y_0}{r_2 - r_1}e^{r_2 t}$ **6.2.26 (p. ??)** $y = e^{r_1 t}(y_0 + (v_0 - r_1 y_0)t)$

Section 6.3 Answers, pp. ??–??

6.3.1 (p. ??) $I = e^{-15t}\left(2\cos 5\sqrt{15}t - \frac{6}{\sqrt{31}}\sin 5\sqrt{31}t\right)$
6.3.2 (p. ??) $I = e^{-20t}(2\cos 40t - 101\sin 40t)$ **6.3.3 (p. ??)** $I = -\frac{200}{3}e^{-10t}\sin 30t$
6.3.4 (p. ??) $I = -10e^{-30t}(\cos 40t + 18\sin 40t)$ **6.3.5 (p. ??)** $I = -e^{-40t}(2\cos 30t - 86\sin 30t)$
6.3.6 (p. ??) $I_p = -\frac{1}{3}(\cos 10t + 2\sin 10t)$ **6.3.7 (p. ??)** $I_p = \frac{20}{37}(\cos 25t - 6\sin 25t)$
6.3.8 (p. ??) $I_p = \frac{3}{13}(8\cos 50t - \sin 50t)$ **6.3.9 (p. ??)** $I_p = \frac{20}{123}(17\sin 100t - 11\cos 100t)$
6.3.10 (p. ??) $I_p = -\frac{45}{52}(\cos 30t + 8\sin 30t)$
6.3.12 (p. ??) $\omega_0 = 1/\sqrt{LC}$ maximum amplitude $= \sqrt{U^2 + V^2}/R$

Section 6.4 Answers, pp. ??–??

6.4.1 (p. ??) If $e = 1$, then $Y^2 = \rho(\rho - 2X)$; if $e \neq 1$ $\left(X + \frac{e\rho}{1-e^2}\right)^2 + \frac{Y^2}{1-e^2} = \frac{\rho^2}{(1-e^2)^2}$ if ;
$e < 1$ let $X_0 = -\frac{e\rho}{1-e^2}$, $a = \frac{\rho}{1-e^2}$, $b = \frac{\rho}{\sqrt{1-e^2}}$.

6.4.2 (p. ??) Let $h = r_0^2\theta_0'$; then $\rho = \frac{h^2}{k}$, $e = \left[\left(\frac{\rho}{r_0} - 1\right)^2 + \left(\frac{\rho r_0'}{h}\right)^2\right]^{1/2}$. If $e = 0$, then
θ_0 is undefined, but also irrelevant if $e \neq 0$ then $\phi = \theta_0 - \alpha$, where $-\pi \leqslant \alpha < \pi$,
$\cos\alpha = \frac{1}{e}\left(\frac{\rho}{r_0} - 1\right)$ and $\sin\alpha = \frac{\rho r_0'}{eh}$.

6.4.3 (p. ??) **(a)** $e = \frac{\gamma_2 - \gamma_1}{\gamma_1 + \gamma_2}$ **(b)** $r_0 = R\gamma_1$, $r_0' = 0$, θ_0 arbitrary, $\theta_0' = \left[\frac{2g\gamma_2}{R\gamma_1^3(\gamma_1 + \gamma_2)}\right]^{1/2}$
6.4.4 (p. ??) $f(r) = -mh^2\left(\frac{6c}{r^4} + \frac{1}{r^3}\right)$ **6.4.5 (p. ??)** $f(r) = -\frac{mh^2(\gamma^2 + 1)}{r^3}$
6.4.6 (p. ??) **(a)** $\frac{d^2u}{d\theta^2} + \left(1 - \frac{k}{h^2}\right)u = 0$, $u(\theta_0) = \frac{1}{r_0}$, $\frac{du(\theta_0)}{d\theta} = -\frac{r_0'}{h}$. **(b)** with $\gamma =$
$\left|1 - \frac{k}{h^2}\right|^{1/2}$: (i) $r = r_0\left(\cosh\gamma(\theta - \theta_0) - \frac{r_0 r_0'}{\gamma h}\sinh\gamma(\theta - \theta_0)\right)^{-1}$ (ii) $r = r_0\left(1 - \frac{r_0 r_0'}{h}(\theta - \theta_0)\right)^{-1}$;
(iii) $r = r_0\left(\cos\gamma(\theta - \theta_0) - \frac{r_0 r_0'}{\gamma h}\sin\gamma(\theta - \theta_0)\right)^{-1}$

Section 7.1 Answers, pp. ??–??

7.1.1 (p. ??) **(a)** $R = 2$; $I = (-1, 3)$; **(b)** $R = 1/2$; $I = (3/2, 5/2)$ **(c)** $R = 0$; **(d)** $R = 16$; $I = (-14, 18)$ **(e)** $R = \infty$; $I = (-\infty, \infty)$ **(f)** $R = 4/3$; $I = (-25/3, -17/3)$
7.1.3 (p. ??) **(a)** $R = 1$; $I = (0, 2)$ **(b)** $R = \sqrt{2}$; $I = (-2 - \sqrt{2}, -2 + \sqrt{2})$; **(c)** $R = \infty$; $I = (-\infty, \infty)$ **(d)** $R = 0$ **(e)** $R = \sqrt{3}$; $I = (-\sqrt{3}, \sqrt{3})$ **(f)** $R = 1$ $I = (0, 2)$

7.1.5 (p. ??) **(a)** $R = 3$; $I = (0, 6)$ **(b)** $R = 1$; $I = (-1, 1)$ **(c)** $R = 1/\sqrt{3}$
$I = (3 - 1/\sqrt{3}, 3 + 1/\sqrt{3})$ **(d)** $R = \infty$; $I = (-\infty, \infty)$ **(e)** $R = 0$ **(f)** $R = 2$;
$I = (-1, 3)$
7.1.11 (p. ??) $b_n = 2(n+2)(n+1)a_{n+2} + (n+1)na_{n+1} + (n+3)a_n$
7.1.12 (p. ??) $b_0 = 2a_2 - 2a_0$ $b_n = (n+2)(n+1)a_{n+2} + [3n(n-1)-2]a_n + 3(n-1)a_{n-1},\ n \geqslant 1$
7.1.13 (p. ??) $b_n = (n+2)(n+1)a_{n+2} + 2(n+1)a_{n+1} + (2n^2 - 5n + 4)a_n$
7.1.14 (p. ??) $b_n = (n+2)(n+1)a_{n+2} + 2(n+1)a_{n+1} + (n^2 - 2n + 3)a_n$
7.1.15 (p. ??) $b_n = (n+2)(n+1)a_{n+2} + (3n^2 - 5n + 4)a_n$
7.1.16 (p. ??) $b_0 = -2a_2 + 2a_1 + a_0$,
$b_n = -(n+2)(n+1)a_{n+2} + (n+1)(n+2)a_{n+1} + (2n+1)a_n + a_{n-1},\ n \geqslant 2$
7.1.17 (p. ??) $b_0 = 8a_2 + 4a_1 - 6a_0$,
$b_n = 4(n+2)(n+1)a_{n+2} + 4(n+1)^2 a_{n+1} + (n^2 + n - 6)a_n - 3a_{n-1},\ n \geqslant 1$
7.1.21 (p. ??) $b_0 = (r+1)(r+2)a_0$,
$b_n = (n+r+1)(n+r+2)a_n - (n+r-2)^2 a_{n-1},\ n \geqslant 1.$
7.1.22 (p. ??) $b_0 = (r-2)(r+2)a_0$,
$b_n = (n+r-2)(n+r+2)a_n + (n+r+2)(n+r-3)a_{n-1},\ n \geqslant 14$
7.1.23 (p. ??) $b_0 = (r-1)^2 a_0,\ b_1 = r^2 a_1 + (r+2)(r+3)a_0$,
$b_n = (n+r-1)^2 a_n + (n+r+1)(n+r+2)a_{n-1} + (n+r-1)a_{n-2},\ n \geqslant 2$
7.1.24 (p. ??) $b_0 = r(r+1)a_0,\ b_1 = (r+1)(r+2)a_1 + 3(r+1)(r+2)a_0$,
$b_n = (n+r)(n+r+1)a_n + 3(n+r)(n+r+1)a_{n-1} + (n+r)a_{n-2},\ n \geqslant 2$
7.1.25 (p. ??) $b_0 = (r+2)(r+1)a_0 \quad b_1 = (r+3)(r+2)a_1$,
$b_n = (n+r+2)(n+r+1)a_n + 2(n+r-1)(n+r-3)a_{n-2},\ n \geqslant 2$
7.1.26 (p. ??) $b_0 = 2(r+1)(r+3)a_0,\ b_1 = 2(r+2)(r+4)a_1$,
$b_n = 2(n+r+1)(n+r+3)a_n + (n+r-3)(n+r)a_{n-2},\ n \geqslant 2$

Section 7.2 Answers, pp. ??–??

7.2.1 (p. ??) $y = a_0 \sum_{m=0}^{\infty} (-1)^m (2m+1) x^{2m} + a_1 \sum_{m=0}^{\infty} (-1)^m (m+1) x^{2m+1}$

7.2.2 (p. ??) $y = a_0 \sum_{m=0}^{\infty} (-1)^{m+1} \frac{x^{2m}}{2m-1} + a_1 x$

7.2.3 (p. ??) $y = a_0(1 - 10x^2 + 5x^4) + a_1 \left(x - 2x^3 + \frac{1}{5} x^5 \right)$

7.2.4 (p. ??) $y = a_0 \sum_{m=0}^{\infty} (m+1)(2m+1) x^{2m} + \frac{a_1}{3} \sum_{m=0}^{\infty} (m+1)(2m+3) x^{2m+1}$

7.2.5 (p. ??) $y = a_0 \sum_{m=0}^{\infty} (-1)^m \left[\prod_{j=0}^{m-1} \frac{4j+1}{2j+1} \right] x^{2m} + a_1 \sum_{m=0}^{\infty} (-1)^m \left[\prod_{j=0}^{m-1} (4j+3) \right] \frac{x^{2m+1}}{2^m m!}$

7.2.6 (p. ??) $y = a_0 \sum_{m=0}^{\infty} (-1)^m \left[\prod_{j=0}^{m-1} \frac{(4j+1)^2}{2j+1} \right] \frac{x^{2m}}{8^m m!} + a_1 \sum_{m=0}^{\infty} (-1)^m \left[\prod_{j=0}^{m-1} \frac{(4j+3)^2}{2j+3} \right] \frac{x^{2m+1}}{8^m m!}$

7.2.7 (p. ??) $y = a_0 \sum_{m=0}^{\infty} \frac{2^m m!}{\prod_{j=0}^{m-1}(2j+1)} x^{2m} + a_1 \sum_{m=0}^{\infty} \frac{\prod_{j=0}^{m-1}(2j+3)}{2^m m!} x^{2m+1}$

7.2.8 (p. ??) $y = a_0 \left(1 - 14x^2 + \frac{35}{3} x^4 \right) + a_1 \left(x - 3x^3 + \frac{3}{5} x^5 + \frac{1}{35} x^7 \right)$

7.2.9 (p. ??) **(a)** $y = a_0 \sum_{m=0}^{\infty} (-1)^m \frac{x^{2m}}{\prod_{j=0}^{m-1}(2j+1)} + a_1 \sum_{m=0}^{\infty} (-1)^m \frac{x^{2m+1}}{2^m m!}$

7.2.10 (p. ??) **(a)** $y = a_0 \sum_{m=0}^{\infty} (-1)^m \left[\prod_{j=0}^{m-1} \frac{4j+3}{2j+1} \right] \frac{x^{2m}}{2^m m!} + a_1 \sum_{m=0}^{\infty} (-1)^m \left[\prod_{j=0}^{m-1} \frac{4j+5}{2j+3} \right] \frac{x^{2m+1}}{2^m m!}$

7.2.11 (p. ??) $y = 2 - x - x^2 + \frac{1}{3}x^3 + \frac{5}{12}x^4 - \frac{1}{6}x^5 - \frac{17}{72}x^6 + \frac{13}{126}x^7 + \cdots$

7.2.12 (p. ??) $y = 1 - x + 3x^2 - \frac{5}{2}x^3 + 5x^4 - \frac{21}{8}x^5 + 3x^6 - \frac{11}{16}x^7 + \cdots$

7.2.13 (p. ??) $y = 2 - x - 2x^2 + \frac{1}{3}x^3 + 3x^4 - \frac{5}{6}x^5 - \frac{49}{5}x^6 + \frac{45}{14}x^7 + \cdots$

7.2.16 (p. ??) $y = a_0 \sum_{m=0}^{\infty} \frac{(x-3)^{2m}}{(2m)!} + a_1 \sum_{m=0}^{\infty} \frac{(x-3)^{2m+1}}{(2m+1)!}$

7.2.17 (p. ??) $y = a_0 \sum_{m=0}^{\infty} \frac{(x-3)^{2m}}{2^m m!} + a_1 \sum_{m=0}^{\infty} \frac{(x-3)^{2m+1}}{\prod_{j=0}^{m-1}(2j+3)}$

7.2.18 (p. ??) $y = a_0 \sum_{m=0}^{\infty} \left[\prod_{j=0}^{m-1} (2j+3) \right] \frac{(x-1)^{2m}}{m!} + a_1 \sum_{m=0}^{\infty} \frac{4^m (m+1)!}{\prod_{j=0}^{m-1}(2j+3)} (x-1)^{2m+1}$

7.2.19 (p. ??) $y = a_0 \left(1 - 6(x-2)^2 + \frac{4}{3}(x-2)^4 + \frac{8}{135}(x-2)^6 \right) + a_1 \left((x-2) - \frac{10}{9}(x-2)^3 \right)$

7.2.20 (p. ??) $y = a_0 \sum_{m=0}^{\infty} (-1)^m \left[\prod_{j=0}^{m-1} (2j+1) \right] \frac{3^m}{4^m m!} (x+1)^{2m} + a_1 \sum_{m=0}^{\infty} (-1)^m \frac{3^m m!}{\prod_{j=0}^{m-1}(2j+3)} (x+1)^{2m+1}$

7.2.21 (p. ??) $y = -1 + 2x + \frac{3}{8}x^2 - \frac{1}{3}x^3 - \frac{3}{128}x^4 - \frac{1}{1024}x^6 + \cdots$

7.2.22 (p. ??) $y = -2 + 3(x-3) + 3(x-3)^2 - 2(x-3)^3 - \frac{5}{4}(x-3)^4 + \frac{3}{5}(x-3)^5 + \frac{7}{24}(x-3)^6 - \frac{4}{35}(x-3)^7 + \cdots$

7.2.23 (p. ??) $y = -1 + (x-1) + 3(x-1)^2 - \frac{5}{2}(x-1)^3 - \frac{27}{4}(x-1)^4 + \frac{21}{4}(x-1)^5 + \frac{27}{2}(x-1)^6 - \frac{81}{8}(x-1)^7 + \cdots$

7.2.24 (p. ??) $y = 4 - 6(x-3) - 2(x-3)^2 + (x-3)^3 + \frac{3}{2}(x-3)^4 - \frac{5}{4}(x-3)^5 - \frac{49}{20}(x-3)^6 + \frac{135}{56}(x-3)^7 + \cdots$

7.2.25 (p. ??) $y = 3 - 4(x-4) + 15(x-4)^2 - 4(x-4)^3 + \frac{15}{4}(x-4)^4 - \frac{1}{5}(x-4)^5$

7.2.26 (p. ??) $y = 3 - 3(x+1) - 30(x+1)^2 + \frac{20}{3}(x+1)^3 + 20(x+1)^4 - \frac{4}{3}(x+1)^5 - \frac{8}{9}(x+1)^6$

7.2.27 (p. ??) **(a)**$y = a_0 \sum_{m=0}^{\infty} (-1)^m x^{2m} + a_1 \sum_{m=0}^{\infty} (-1)^m x^{2m+1}$ **(b)**$y = \frac{a_0 + a_1 x}{1+x^2}$

7.2.33 (p. ??) $y = a_0 \sum_{m=0}^{\infty} \frac{x^{3m}}{3^m m! \prod_{j=0}^{m-1}(3j+2)} + a_1 \sum_{m=0}^{\infty} \frac{x^{3m+1}}{3^m m! \prod_{j=0}^{m-1}(3j+4)}$

7.2.34 (p. ??) $y = a_0 \sum_{m=0}^{\infty} \left(\frac{2}{3}\right)^m \left[\prod_{j=0}^{m-1} (3j+2) \right] \frac{x^{3m}}{m!} + a_1 \sum_{m=0}^{\infty} \frac{6^m m!}{\prod_{j=0}^{m-1}(3j+4)} x^{3m+1}$

7.2.35 (p. ??) $y = a_0 \sum_{m=0}^{\infty} (-1)^m \frac{3^m m!}{\prod_{j=0}^{m-1}(3j+2)} x^{3m} + a_1 \sum_{m=0}^{\infty} (-1)^m \left[\prod_{j=0}^{m-1} (3j+4) \right] \frac{x^{3m+1}}{3^m m!}$

7.2.36 (p. ??) $y = a_0(1 - 4x^3 + 4x^6) + a_1 \sum_{m=0}^{\infty} 2^m \left[\prod_{j=0}^{m-1} \frac{3j-5}{3j+4} \right] x^{3m+1}$

7.2.37 (p. ??) $y = a_0 \left(1 + \frac{21}{2}x^3 + \frac{42}{5}x^6 + \frac{7}{20}x^9 \right) + a_1 \left(x + 4x^4 + \frac{10}{7}x^7 \right)$

7.2.39 (p. ??) $y = a_0 \sum_{m=0}^{\infty} (-2)^m \left[\prod_{j=0}^{m-1} \frac{5j+1}{5j+4}\right] x^{5m} + a_1 \sum_{m=0}^{\infty} \left(-\frac{2}{5}\right)^m \left[\prod_{j=0}^{m-1} (5j+2)\right] \frac{x^{5m+1}}{m!}$

7.2.40 (p. ??) $y = a_0 \sum_{m=0}^{\infty} (-1)^m \frac{x^{4m}}{4^m m! \prod_{j=0}^{m-1}(4j+3)} + a_1 \sum_{m=0}^{\infty} (-1)^m \frac{x^{4m+1}}{4^m m! \prod_{j=0}^{m-1}(4j+5)}$

7.2.41 (p. ??) $y = a_0 \sum_{m=0}^{\infty} (-1)^m \frac{x^{7m}}{\prod_{j=0}^{m-1}(7j+6)} + a_1 \sum_{m=0}^{\infty} (-1)^m \frac{x^{7m+1}}{7^m m!}$

7.2.42 (p. ??) $y = a_0 \left(1 - \frac{9}{7}x^8\right) + a_1 \left(x - \frac{7}{9}x^9\right)$

7.2.43 (p. ??) $y = a_0 \sum_{m=0}^{\infty} x^{6m} + a_1 \sum_{m=0}^{\infty} x^{6m+1}$

7.2.44 (p. ??) $y = a_0 \sum_{m=0}^{\infty} (-1)^m \frac{x^{6m}}{\prod_{j=0}^{m-1}(6j+5)} + a_1 \sum_{m=0}^{\infty} (-1)^m \frac{x^{6m+1}}{6^m m!}$

Section 7.3 Answers, pp. ??–??

7.3.1 (p. ??) $y = 2 - 3x - 2x^2 + \frac{7}{2}x^3 - \frac{55}{12}x^4 + \frac{59}{8}x^5 - \frac{83}{6}x^6 + \frac{9547}{336}x^7 + \cdots$

7.3.2 (p. ??) $y = -1 + 2x - 4x^3 + 4x^4 + 4x^5 - 12x^6 + 4x^7 + \cdots$

7.3.3 (p. ??) $y = 1 + x^2 - \frac{2}{3}x^3 + \frac{11}{6}x^4 - \frac{9}{5}x^5 + \frac{329}{90}x^6 - \frac{1301}{315}x^7 + \cdots$

7.3.4 (p. ??) $y = x - x^2 - \frac{7}{2}x^3 + \frac{15}{2}x^4 + \frac{45}{8}x^5 - \frac{261}{8}x^6 + \frac{207}{16}x^7 + \cdots$

7.3.5 (p. ??) $y = 4 + 3x - \frac{15}{4}x^2 + \frac{1}{4}x^3 + \frac{11}{16}x^4 - \frac{5}{16}x^5 + \frac{1}{20}x^6 + \frac{1}{120}x^7 + \cdots$

7.3.6 (p. ??) $y = 7 + 3x - \frac{16}{3}x^2 + \frac{13}{3}x^3 - \frac{23}{9}x^4 + \frac{10}{9}x^5 - \frac{7}{27}x^6 - \frac{1}{9}x^7 + \cdots$

7.3.7 (p. ??) $y = 2 + 5x - \frac{7}{4}x^2 - \frac{3}{16}x^3 + \frac{37}{192}x^4 - \frac{7}{192}x^5 - \frac{1}{1920}x^6 + \frac{19}{11520}x^7 + \cdots$

7.3.8 (p. ??) $y = 1 - (x-1) + \frac{4}{3}(x-1)^3 - \frac{4}{3}(x-1)^4 - \frac{4}{5}(x-1)^5 + \frac{136}{45}(x-1)^6 - \frac{104}{63}(x-1)^7 + \cdots$

7.3.9 (p. ??) $y = 1 - (x+1) + 4(x+1)^2 - \frac{13}{3}(x+1)^3 + \frac{77}{6}(x+1)^4 - \frac{278}{15}(x+1)^5 + \frac{1942}{45}(x+1)^6 - \frac{23332}{315}(x+1)^7 +$

7.3.10 (p. ??) $y = 2 - (x-1) - \frac{1}{2}(x-1)^2 + \frac{5}{3}(x-1)^3 - \frac{19}{12}(x-1)^4 + \frac{7}{30}(x-1)^5 + \frac{59}{45}(x-1)^6 - \frac{1091}{630}(x-1)^7 + \cdots$

7.3.11 (p. ??) $y = -2 + 3(x+1) - \frac{1}{2}(x+1)^2 - \frac{2}{3}(x+1)^3 + \frac{5}{8}(x+1)^4 - \frac{11}{30}(x+1)^5 + \frac{29}{144}(x+1)^6 - \frac{101}{840}(x+1)^7 + \cdot\cdot$

7.3.12 (p. ??) $y = 1 - 2(x-1) - 3(x-1)^2 + 8(x-1)^3 - 4(x-1)^4 - \frac{42}{5}(x-1)^5 + 19(x-1)^6 - \frac{604}{35}(x-1)^7 + \cdots$

7.3.19 (p. ??) $y = 2 - 7x - 4x^2 - \frac{17}{6}x^3 - \frac{3}{4}x^4 - \frac{9}{40}x^5 + \cdots$

7.3.20 (p. ??) $y = 1 - 2(x-1) + \frac{1}{2}(x-1)^2 - \frac{1}{6}(x-1)^3 + \frac{5}{36}(x-1)^4 - \frac{73}{1080}(x-1)^5 + \cdots$

7.3.21 (p. ??) $y = 2 - (x+2) - \frac{7}{2}(x+2)^2 + \frac{4}{3}(x+2)^3 - \frac{1}{24}(x+2)^4 + \frac{1}{60}(x+2)^5 + \cdots$

7.3.22 (p. ??) $y = 2 - 2(x+3) - (x+3)^2 + (x+3)^3 - \frac{11}{12}(x+3)^4 + \frac{67}{60}(x+3)^5 + \cdots$

7.3.23 (p. ??) $y = -1 + 2x + \frac{1}{3}x^3 - \frac{5}{12}x^4 + \frac{2}{5}x^5 + \cdots$

7.3.24 (p. ??) $y = 2 - 3(x+1) + \frac{7}{2}(x+1)^2 - 5(x+1)^3 + \frac{197}{24}(x+1)^4 - \frac{287}{20}(x+1)^5 + \cdots$

7.3.25 (p. ??) $y = -2 + 3(x+2) - \frac{9}{2}(x+2)^2 + \frac{11}{6}(x+2)^3 + \frac{5}{24}(x+2)^4 + \frac{7}{20}(x+2)^5 + \cdots$

7.3.26 (p. ??) $y = 2 - 4(x-2) - \frac{1}{2}(x-2)^2 + \frac{2}{9}(x-2)^3 + \frac{49}{432}(x-2)^4 + \frac{23}{1080}(x-2)^5 + \cdots$

7.3.27 (p. ??) $y = 1 + 2(x+4) - \frac{1}{6}(x+4)^2 - \frac{10}{27}(x+4)^3 + \frac{19}{648}(x+4)^4 + \frac{13}{324}(x+4)^5 + \cdots$

7.3.28 (p. ??) $y = -1 + 2(x+1) - \frac{1}{4}(x+1)^2 + \frac{1}{2}(x+1)^3 - \frac{65}{96}(x+1)^4 + \frac{67}{80}(x+1)^5 + \cdots$

7.3.31 (p. ??) **(a)** $y = \frac{c_1}{1+x} + \frac{c_2}{1+2x}$ **(b)** $y = \frac{c_1}{1-2x} + \frac{c_2}{1-3x}$ **(c)** $y = \frac{c_1}{1-2x} + \frac{c_2 x}{(1-2x)^2}$
(d) $y = \frac{c_1}{2+x} + \frac{c_2 x}{(2+x)^2}$ **(e)** $y = \frac{c_1}{2+x} + \frac{c_2}{2+3x}$

7.3.32 (p. ??) $y = 1 - 2x - \frac{3}{2}x^2 + \frac{5}{3}x^3 + \frac{17}{24}x^4 - \frac{11}{20}x^5 + \cdots$

7.3.33 (p. ??) $y = 1 - 2x - \frac{5}{2}x^2 + \frac{2}{3}x^3 - \frac{3}{8}x^4 + \frac{1}{3}x^5 + \cdots$

7.3.34 (p. ??) $y = 6 - 2x + 9x^2 + \frac{2}{3}x^3 - \frac{23}{4}x^4 - \frac{3}{10}x^5 + \cdots$

7.3.35 (p. ??) $y = 2 - 5x + 2x^2 - \frac{10}{3}x^3 + \frac{3}{2}x^4 - \frac{25}{12}x^5 + \cdots$

7.3.36 (p. ??) $y = 3 + 6x - 3x^2 + x^3 - 2x^4 - \frac{17}{20}x^5 + \cdots$

7.3.37 (p. ??) $y = 3 - 2x - 3x^2 + \frac{3}{2}x^3 + \frac{3}{2}x^4 - \frac{49}{80}x^5 + \cdots$

7.3.38 (p. ??) $y = -2 + 3x + \frac{4}{3}x^2 - x^3 - \frac{19}{54}x^4 + \frac{13}{60}x^5 + \cdots$

7.3.39 (p. ??) $y_1 = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{m!} = e^{-x^2}, \quad y_2 = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{m!} = xe^{-x^2}$

7.3.40 (p. ??) $y = -2 + 3x + x^2 - \frac{1}{6}x^3 - \frac{3}{4}x^4 + \frac{31}{120}x^5 + \cdots$

7.3.41 (p. ??) $y = 2 + 3x - \frac{7}{2}x^2 - \frac{5}{6}x^3 + \frac{41}{24}x^4 + \frac{41}{120}x^5 + \cdots$

7.3.42 (p. ??) $y = -3 + 5x - 5x^2 + \frac{23}{6}x^3 - \frac{23}{12}x^4 + \frac{11}{30}x^5 + \cdots$

7.3.43 (p. ??) $y = -2 + 3(x-1) + \frac{3}{2}(x-1)^2 - \frac{17}{12}(x-1)^3 - \frac{1}{12}(x-1)^4 + \frac{1}{8}(x-1)^5 + \cdots$

7.3.44 (p. ??) $y = 2 - 3(x+2) + \frac{1}{2}(x+2)^2 - \frac{1}{3}(x+2)^3 + \frac{31}{24}(x+2)^4 - \frac{53}{120}(x+2)^5 + \cdots$

7.3.45 (p. ??) $y = 1 - 2x + \frac{3}{2}x^2 - \frac{11}{6}x^3 + \frac{15}{8}x^4 - \frac{71}{60}x^5 + \cdots$

7.3.46 (p. ??) $y = 2 - (x+2) - \frac{7}{2}(x+2)^2 - \frac{43}{6}(x+2)^3 - \frac{203}{24}(x+2)^4 - \frac{167}{30}(x+2)^5 + \cdots$

7.3.47 (p. ??) $y = 2 - x - x^2 + \frac{7}{6}x^3 - x^4 + \frac{89}{120}x^5 + \cdots$

7.3.48 (p. ??) $y = 1 + \frac{3}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 - \frac{1}{8}(x-1)^5 + \cdots$

7.3.49 (p. ??) $y = 1 - 2(x-3) + \frac{1}{2}(x-3)^2 - \frac{1}{6}(x-3)^3 + \frac{1}{4}(x-3)^4 - \frac{1}{6}(x-3)^5 + \cdots$

Section 7.4 Answers, pp. ??–??

7.4.1 (p. ??) $y = c_1x^{-4} + c_2x^{-2}$ **7.4.2 (p. ??)** $y = c_1x + c_2x^7$
7.4.3 (p. ??) $y = x(c_1 + c_2 \ln x)$ **7.4.4 (p. ??)** $y = x^{-2}(c_1 + c_2 \ln x)$
7.4.5 (p. ??) $y = c_1 \cos(\ln x) + c_2 \sin(\ln x)$ **7.4.6 (p. ??)** $y = x^2[c_1 \cos(3 \ln x) + c_2 \sin(3 \ln x)]$
7.4.7 (p. ??) $y = c_1x + \frac{c_2}{x^3}$ **7.4.8 (p. ??)** $y = c_1x^{2/3} + c_2x^{3/4}$ **7.4.9 (p. ??)** $y = x^{-1/2}(c_1 + c_2 \ln x)$
7.4.10 (p. ??) $y = c_1x + c_2x^{1/3}$ **7.4.11 (p. ??)** $y = c_1x^2 + c_2x^{1/2}$ **7.4.12 (p. ??)** $y = \frac{1}{x}[c_1 \cos(2 \ln x) + c_2 \sin(2 \ln x]$

7.4.13 (p. ??) $y = x^{-1/3}(c_1 + c_2 \ln x)$ **7.4.14 (p. ??)** $y = x\,[c_1 \cos(3\ln x) + c_2 \sin(3\ln x)]$
7.4.15 (p. ??) $y = c_1x^3 + \dfrac{c_2}{x^2}$ **7.4.16 (p. ??)** $y = \dfrac{c_1}{x} + c_2x^{1/2}$ **7.4.17 (p. ??)** $y = x^2(c_1 + c_2 \ln x)$
7.4.18 (p. ??) $y = \dfrac{1}{x^2}\left[c_1 \cos\left(\dfrac{1}{\sqrt{2}}\ln x\right) + c_2 \sin\left(\dfrac{1}{\sqrt{2}}\ln x\right)\right]$

Section 7.5 Answers, pp. ??–??

7.5.1 (p. ??) $y_1 = x^{1/2}\left(1 - \frac{1}{5}x - \frac{2}{35}x^2 + \frac{31}{315}x^3 + \cdots\right)$ $y_2 = x^{-1}\left(1 + x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \cdots\right)$;

7.5.2 (p. ??) $y_1 = x^{1/3}\left(1 - \frac{2}{3}x + \frac{8}{9}x^2 - \frac{40}{81}x^3 + \cdots\right)$; $y_2 = 1 - x + \frac{6}{5}x^2 - \frac{4}{5}x^3 + \cdots$

7.5.3 (p. ??) $y_1 = x^{1/3}\left(1 - \frac{4}{7}x - \frac{7}{45}x^2 + \frac{970}{2457}x^3 + \cdots\right)$; $y_2 = x^{-1}\left(1 - x^2 + \frac{2}{3}x^3 + \cdots\right)$

7.5.4 (p. ??) $y_1 = x^{1/4}\left(1 - \frac{1}{2}x - \frac{19}{104}x^2 + \frac{1571}{10608}x^3 + \cdots\right)$; $y_2 = x^{-1}\left(1 + 2x - \frac{11}{6}x^2 - \frac{1}{7}x^3 + \cdots\right)$

7.5.5 (p. ??) $y_1 = x^{1/3}\left(1 - x + \frac{28}{31}x^2 - \frac{1111}{1333}x^3 + \cdots\right)$; $y_2 = x^{-1/4}\left(1 - x + \frac{7}{8}x^2 - \frac{19}{24}x^3 + \cdots\right)$;

7.5.6 (p. ??) $y_1 = x^{1/5}\left(1 - \frac{6}{25}x - \frac{1217}{625}x^2 + \frac{41972}{46875}x^3 + \cdots\right)$; $y_2 = x - \frac{1}{4}x^2 - \frac{35}{18}x^3 + \frac{11}{12}x^4 + \cdots$

7.5.7 (p. ??) $y_1 = x^{3/2}\left(1 - x + \frac{11}{26}x^2 - \frac{109}{1326}x^3 + \cdots\right)$; $y_2 = x^{1/4}\left(1 + 4x - \frac{131}{24}x^2 + \frac{39}{14}x^3 + \cdots\right)$

7.5.8 (p. ??) $y_1 = x^{1/3}\left(1 - \frac{1}{3}x + \frac{2}{15}x^2 - \frac{5}{63}x^3 + \cdots\right)$; $y_2 = x^{-1/6}\left(1 - \frac{1}{12}x^2 + \frac{1}{18}x^3 + \cdots\right)$

7.5.9 (p. ??) $y_1 = 1 - \frac{1}{14}x^2 + \frac{1}{105}x^3 + \cdots$; $y_2 = x^{-1/3}\left(1 - \frac{1}{18}x - \frac{71}{405}x^2 + \frac{719}{34992}x^3 + \cdots\right)$

7.5.10 (p. ??) $y_1 = x^{1/5}\left(1 + \frac{3}{17}x - \frac{7}{153}x^2 - \frac{547}{5661}x^3 + \cdots\right)$; $y_2 = x^{-1/2}\left(1 + x + \frac{14}{13}x^2 - \frac{556}{897}x^3 + \cdots\right)$

7.5.14 (p. ??) $y_1 = x^{1/2}\sum_{n=0}^{\infty}\frac{(-2)^n}{\prod_{j=1}^{n}(2j+3)}x^n$; $y_2 = x^{-1}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}x^n$

7.5.15 (p. ??) $y_1 = x^{1/3}\sum_{n=0}^{\infty}\frac{(-1)^n\prod_{j=1}^{n}(3j+1)}{9^n n!}x^n$; x^{-1}

7.5.16 (p. ??) $y_1 = x^{1/2}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n n!}x^n$; $y_2 = \frac{1}{x^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{\prod_{j=1}^{n}(2j-5)}x^n$

7.5.17 (p. ??) $y_1 = x\sum_{n=0}^{\infty}\frac{(-1)^n}{\prod_{j=1}^{n}(3j+4)}x^n$; $y_2 = x^{-1/3}\sum_{n=0}^{\infty}\frac{(-1)^n}{3^n n!}x^n$

7.5.18 (p. ??) $y_1 = x\sum_{n=0}^{\infty}\frac{2^n}{n!\prod_{j=1}^{n}(2j+1)}x^n$; $y_2 = x^{1/2}\sum_{n=0}^{\infty}\frac{2^n}{n!\prod_{j=1}^{n}(2j-1)}x^n$

7.5.19 (p. ??) $y_1 = x^{1/3}\sum_{n=0}^{\infty}\frac{1}{n!\prod_{j=1}^{n}(3j+2)}x^n$; $y_2 = x^{-1/3}\sum_{n=0}^{\infty}\frac{1}{n!\prod_{j=1}^{n}(3j-2)}x^n$

7.5.20 (p. ??) $y_1 = x\left(1 + \frac{2}{7}x + \frac{1}{70}x^2\right)$; $y_2 = x^{-1/3}\sum_{n=0}^{\infty}\frac{(-1)^n}{3^n n!}\left(\prod_{j=1}^{n}\frac{3j-13}{3j-4}\right)x^n$

7.5.21 (p. ??) $y_1 = x^{1/2}\sum_{n=0}^{\infty}(-1)^n\left(\prod_{j=1}^{n}\frac{2j+1}{6j+1}\right)$; x^n $y_2 = x^{1/3}\sum_{n=0}^{\infty}\frac{(-1)^n}{9^n n!}\left(\prod_{j=1}^{n}(3j+1)\right)x^n$

7.5.22 (p. ??) $y_1 = x\sum_{n=0}^{\infty}\frac{(-1)^n(n+2)!}{2\prod_{j=1}^{n}(4j+3)};x^n\ y_2 = x^{1/4}\sum_{n=0}^{\infty}\frac{(-1)^n}{16^n n!}\prod_{j=1}^{n}(4j+5)x^n$

7.5.23 (p. ??) $y_1 = x^{-1/2}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\prod_{j=1}^{n}(2j+1)}x^n; y_2 = x^{-1}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\prod_{j=1}^{n}(2j-1)}x^n$

7.5.24 (p. ??) $y_1 = x^{1/3}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left(\frac{2}{9}\right)^n\left(\prod_{j=1}^{n}(6j+5)\right)x^n; y_2 = x^{-1}\sum_{n=0}^{\infty}(-1)^n 2^n\left(\prod_{j=1}^{n}\frac{2j-1}{3j-4}\right)x^n$

7.5.25 (p. ??) $y_1 = 4x^{1/3}\sum_{n=0}^{\infty}\frac{1}{6^n n!(3n+4)}x^n; x^{-1}$

7.5.28 (p. ??) $y_1 = x^{1/2}\left(1-\frac{9}{40}x+\frac{5}{128}x^2-\frac{245}{39936}x^3+\cdots\right);\ y_2 = x^{1/4}\left(1-\frac{25}{96}x+\frac{675}{14336}x^2-\frac{38025}{5046272}x^3+\cdots\right)$

7.5.29 (p. ??) $y_1 = x^{1/3}\left(1+\frac{32}{117}x-\frac{28}{1053}x^2+\frac{4480}{540189}x^3+\cdots\right);\ y_2 = x^{-3}\left(1+\frac{32}{7}x+\frac{48}{7}x^2\right)$

7.5.30 (p. ??) $y_1 = x^{1/2}\left(1-\frac{5}{8}x+\frac{55}{96}x^2-\frac{935}{1536}x^3+\cdots\right);\ y_2 = x^{-1/2}\left(1+\frac{1}{4}x-\frac{5}{32}x^2-\frac{55}{384}x^3+\cdots\right).$

7.5.31 (p. ??) $y_1 = x^{1/2}\left(1-\frac{3}{4}x+\frac{5}{96}x^2+\frac{5}{4224}x^3+\cdots\right);\ y_2 = x^{-2}\left(1+8x+60x^2-160x^3+\cdots\right)$

7.5.32 (p. ??) $y_1 = x^{-1/3}\left(1-\frac{10}{63}x+\frac{200}{7371}x^2-\frac{17600}{3781323}x^3;+\cdots\right);\ y_2 = x^{-1/2}\left(1-\frac{3}{20}x+\frac{9}{352}x^2-\frac{105}{23936}x^3+\cdots\right)$

7.5.33 (p. ??) $y_1 = x^{1/2}\sum_{m=0}^{\infty}\frac{(-1)^m}{8^m m!}\left(\prod_{j=1}^{m}\frac{4j-3}{8j+1}\right)x^{2m}; y_2 = x^{1/4}\sum_{m=0}^{\infty}\frac{(-1)^m}{16^m m!}\left(\prod_{j=1}^{m}\frac{8j-7}{8j-1}\right)x^{2m}$

7.5.34 (p. ??) $y_1 = x^{1/2}\sum_{m=0}^{\infty}\left(\prod_{j=1}^{m}\frac{8j-3}{8j+1}\right)x^{2m}; y_2 = x^{1/4}\sum_{m=0}^{\infty}\frac{1}{2^m m!}\left(\prod_{j=1}^{m}(2j-1)\right)x^{2m}$

7.5.35 (p. ??) $y_1 = x^4\sum_{m=0}^{\infty}(-1)^m(m+1)x^{2m}; y_2 = -x\sum_{m=0}^{\infty}(-1)^m(2m-1)x^{2m}$

7.5.36 (p. ??) $y_1 = x^{1/3}\sum_{m=0}^{\infty}\frac{(-1)^m}{18^m m!}\left(\prod_{j=1}^{m}(6j-17)\right)x^{2m}; y_2 = 1+\frac{4}{5}x^2+\frac{8}{55}x^4$

7.5.37 (p. ??) $y_1 = x^{1/4}\sum_{m=0}^{\infty}\left(\prod_{j=1}^{m}\frac{8j+1}{8j+5}\right)x^{2m}; y_2 = x^{-1}\sum_{m=0}^{\infty}\frac{\prod_{j=1}^{m}(2j-1)}{2^m m!}x^{2m}$

7.5.38 (p. ??) $y_1 = x^{1/2}\sum_{m=0}^{\infty}\frac{1}{8^m m!}\left(\prod_{j=1}^{m}(4j-1)\right)x^{2m}; y_2 = x^{1/3}\sum_{m=0}^{\infty}2^m\left(\prod_{j=1}^{m}\frac{3j-1}{12j-1}\right)x^{2m}$

7.5.39 (p. ??) $y_1 = x^{7/2}\sum_{m=0}^{\infty}(-1)^m\frac{\prod_{j=1}^{m}(4j+5)}{8^m m!}x^{2m}; y_2 = x^{1/2}\sum_{m=0}^{\infty}\frac{(-1)^m}{4^m}\left(\prod_{j=1}^{m}\frac{4j-1}{2j-3}\right)x^{2m}$

7.5.40 (p. ??) $y_1 = x^{1/2}\sum_{m=0}^{\infty}\frac{(-1)^m}{4^m}\left(\prod_{j=1}^{m}\frac{4j-1}{2j+1}\right)x^{2m}; y_2 = x^{-1/2}\sum_{m=0}^{\infty}\frac{(-1)^m}{8^m m!}\left(\prod_{j=1}^{m}(4j-3)\right)x^{2m}$

7.5.41 (p. ??) $y_1 = x^{1/2}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\left(\prod_{j=1}^{m}(2j+1)\right)x^{2m}; y_2 = \frac{1}{x^2}\sum_{m=0}^{\infty}(-2)^m\left(\prod_{j=1}^{m}\frac{4j-3}{4j-5}\right)x^{2m}$

7.5.42 (p. ??) $y_1 = x^{1/3}\sum_{m=0}^{\infty}(-1)^m\left(\prod_{j=1}^{m}\frac{3j-4}{3j+2}\right)x^{2m}$; $y_2 = x^{-1}(1+x^2)$

7.5.43 (p. ??) $y_1 = \sum_{m=0}^{\infty}(-1)^m\frac{2^m(m+1)!}{\prod_{j=1}^{m}(2j+3)}x^{2m}$; $y_2 = \frac{1}{x^3}\sum_{m=0}^{\infty}(-1)^m\frac{\prod_{j=1}^{m}(2j-1)}{2^m m!}x^{2m}$

7.5.44 (p. ??) $y_1 = x^{1/2}\sum_{m=0}^{\infty}\frac{(-1)^m}{8^m m!}\left(\prod_{j=1}^{m}\frac{(4j-3)^2}{4j+3}\right)x^{2m}$; $y_2 = x^{-1}\sum_{m=0}^{\infty}\frac{(-1)^m}{2^m m!}\left(\prod_{j=1}^{m}\frac{(2j-3)^2}{4j-3}\right)x^{2m}$

7.5.45 (p. ??) $y_1 = x\sum_{m=0}^{\infty}(-2)^m\left(\prod_{j=1}^{m}\frac{2j+1}{4j+5}\right)x^{2m}$; $y_2 = x^{-3/2}\sum_{m=0}^{\infty}\frac{(-1)^m}{4^m m!}\left(\prod_{j=1}^{m}(4j-3)\right)x^{2m}$

7.5.46 (p. ??) $y_1 = x^{1/3}\sum_{m=0}^{\infty}\frac{(-1)^m}{2^m\prod_{j=1}^{m}(3j+1)}x^{2m}$; $y_2 = x^{-1/3}\sum_{m=0}^{\infty}\frac{(-1)^m}{6^m m!}x^{2m}$

7.5.47 (p. ??) $y_1 = x^{1/2}\left(1-\frac{6}{13}x^2+\frac{36}{325}x^4-\frac{216}{12025}x^6+\cdots\right)$; $y_2 = x^{1/3}\left(1-\frac{1}{2}x^2+\frac{1}{8}x^4-\frac{1}{48}x^6+\cdots\right)$

7.5.48 (p. ??) $y_1 = x^{1/4}\left(1-\frac{13}{64}x^2+\frac{273}{8192}x^4-\frac{2639}{524288}x^6+\cdots\right)$; $y_2 = x^{-1}\left(1-\frac{1}{3}x^2+\frac{2}{33}x^4-\frac{2}{209}x^6+\cdots\right)$

7.5.49 (p. ??) $y_1 = x^{1/3}\left(1-\frac{3}{4}x^2+\frac{9}{14}x^4-\frac{81}{140}x^6+\cdots\right)$; $y_2 = x^{-1/3}\left(1-\frac{2}{3}x^2+\frac{5}{9}x^4-\frac{40}{81}x^6+\cdots\right)$

7.5.50 (p. ??) $y_1 = x^{1/2}\left(1-\frac{3}{2}x^2+\frac{15}{8}x^4-\frac{35}{16}x^6+\cdots\right)$; $y_2 = x^{-1/2}\left(1-2x^2+\frac{8}{3}x^4-\frac{16}{5}x^6+\cdots\right)$

7.5.51 (p. ??) $y_1 = x^{1/4}\left(1-x^2+\frac{3}{2}x^4-\frac{5}{2}x^6+\cdots\right)$; $y_2 = x^{-1/2}\left(1-\frac{2}{5}x^2+\frac{36}{65}x^4-\frac{408}{455}x^6+\cdots\right)$

7.5.53 (p. ??) **(a)** $y_1 = x^{\nu}\sum_{m=0}^{\infty}\frac{(-1)^m}{4^m m!\prod_{j=1}^{m}(j+\nu)}x^{2m}$; $y_2 = x^{-\nu}\sum_{m=0}^{\infty}\frac{(-1)^m}{4^m m!\prod_{j=1}^{m}(j-\nu)}x^{2m}$

$y_1 = \frac{\sin x}{\sqrt{x}}$; $y_2 = \frac{\cos x}{\sqrt{x}}$

7.5.61 (p. ??) $y_1 = \frac{x^{1/2}}{1+x}$; $y_2 = \frac{x}{1+x}$ **7.5.62 (p. ??)** $y_1 = \frac{x^{1/3}}{1+2x^2}$; $y_2 = \frac{x^{1/2}}{1+2x^2}$

7.5.63 (p. ??) $y_1 = \frac{x^{1/4}}{1-3x}$; $y_2 = \frac{x^2}{1-3x}$ **7.5.64 (p. ??)** $y_1 = \frac{x^{1/3}}{5+x}$; $y_2 = \frac{x^{-1/3}}{5+x}$

7.5.65 (p. ??) $y_1 = \frac{x^{1/4}}{2-x^2}$; $y_2 = \frac{x^{-1/2}}{2-x^2}$ **7.5.66 (p. ??)** $y_1 = \frac{x^{1/2}}{1+3x+x^2}$; $y_2 = \frac{x^{3/2}}{1+3x+x^2}$

7.5.67 (p. ??) $y_1 = \frac{x}{(1+x)^2}$; $y_2 = \frac{x^{1/3}}{(1+x)^2}$ **7.5.68 (p. ??)** $y_1 = \frac{x}{3+2x+x^2}$; $y_2 = \frac{x^{1/4}}{3+2x+x^2}$

Section 7.6 Answers, pp. ??–??

7.6.1 (p. ??) $y_1 = x\left(1-x+\frac{3}{4}x^2-\frac{13}{36}x^3+\cdots\right)$; $y_2 = y_1\ln x + x^2\left(1-x+\frac{65}{108}x^2+\cdots\right)$

7.6.2 (p. ??) $y_1 = x^{-1}\left(1-2x+\frac{9}{2}x^2-\frac{20}{3}x^3+\cdots\right)$; $y_2 = y_1\ln x + 1 - \frac{15}{4}x + \frac{133}{18}x^2+\cdots$

7.6.3 (p. ??) $y_1 = 1+x-x^2+\frac{1}{3}x^3+\cdots$; $y_2 = y_1\ln x - x\left(3-\frac{1}{2}x-\frac{31}{18}x^2+\cdots\right)$

7.6.4 (p. ??) $y_1 = x^{1/2}\left(1-2x+\frac{5}{2}x^2-2x^3+\cdots\right)$; $y_2 = y_1\ln x + x^{3/2}\left(1-\frac{9}{4}x+\frac{17}{6}x^2+\cdots\right)$

7.6.5 (p. ??) $y_1 = x\left(1-4x+\frac{19}{2}x^2-\frac{49}{3}x^3+\cdots\right)$; $y_2 = y_1\ln x + x^2\left(3-\frac{43}{4}x+\frac{208}{9}x^2+\cdots\right)$

7.6.6 (p. ??) $y_1 = x^{-1/3}\left(1 - x + \frac{5}{6}x^2 - \frac{1}{2}x^3 + \cdots\right)$; $y_2 = y_1 \ln x + x^{2/3}\left(1 - \frac{11}{12}x + \frac{25}{36}x^2 + \cdots\right)$

7.6.7 (p. ??) $y_1 = 1 - 2x + \frac{7}{4}x^2 - \frac{7}{9}x^3 + \cdots$; $y_2 = y_1 \ln x + x\left(3 - \frac{15}{4}x + \frac{239}{108}x^2 + \cdots\right)$

7.6.8 (p. ??) $y_1 = x^{-2}\left(1 - 2x + \frac{5}{2}x^2 - 3x^3 + \cdots\right)$; $y_2 = y_1 \ln x + \frac{3}{4} - \frac{13}{6}x + \cdots$

7.6.9 (p. ??) $y_1 = x^{-1/2}\left(1 - x + \frac{1}{4}x^2 + \frac{1}{18}x^3 + \cdots\right)$; $y_2 = y_1 \ln x + x^{1/2}\left(\frac{3}{2} - \frac{13}{16}x + \frac{1}{54}x^2 + \cdots\right)$

7.6.10 (p. ??) $y_1 = x^{-1/4}\left(1 - \frac{1}{4}x - \frac{7}{32}x^2 + \frac{23}{384}x^3 + \cdots\right)$; $y_2 = y_1 \ln x + x^{3/4}\left(\frac{1}{4} + \frac{5}{64}x - \frac{157}{2304}x^2 + \cdots\right)$

7.6.11 (p. ??) $y_1 = x^{-1/3}\left(1 - x + \frac{7}{6}x^2 - \frac{23}{18}x^3 + \cdots\right)$; $y_2 = y_1 \ln x - x^{5/3}\left(\frac{1}{12} - \frac{13}{108}x \cdots\right)$

7.6.12 (p. ??) $y_1 = x^{1/2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(n!)^2}x^n$; $y_2 = y_1 \ln x - 2x^{1/2}\sum_{n=1}^{\infty}\frac{(-1)^n}{(n!)^2}\left(\sum_{j=1}^{n}\frac{1}{j}\right)x^n$;

7.6.13 (p. ??) $y_1 = x^{1/6}\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n\frac{\prod_{j=1}^{n}(3j+1)}{n!}x^n$;

$$y_2 = y_1 \ln x - x^{1/6}\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n\frac{\prod_{j=1}^{n}(3j+1)}{n!}\left(\sum_{j=1}^{n}\frac{1}{j(3j+1)}\right)x^n$$

7.6.14 (p. ??) $y_1 = x^2\sum_{n=0}^{\infty}(-1)^n(n+1)^2x^n$; $y_2 = y_1 \ln x - 2x^2\sum_{n=1}^{\infty}(-1)^n n(n+1)x^n$

7.6.15 (p. ??) $y_1 = x^3\sum_{n=0}^{\infty}2^n(n+1)x^n$; $y_2 = y_1 \ln x - x^3\sum_{n=1}^{\infty}2^n n x^n$

7.6.16 (p. ??) $y_1 = x^{1/5}\sum_{n=0}^{\infty}\frac{(-1)^n\prod_{j=1}^{n}(5j+1)}{125^n(n!)^2}x^n$;

$$y_2 = y_1 \ln x - x^{1/5}\sum_{n=1}^{\infty}\frac{(-1)^n\prod_{j=1}^{n}(5j+1)}{125^n(n!)^2}\left(\sum_{j=1}^{n}\frac{5j+2}{j(5j+1)}\right)x^n$$

7.6.17 (p. ??) $y_1 = x^{1/2}\sum_{n=0}^{\infty}\frac{(-1)^n\prod_{j=1}^{n}(2j-3)}{4^n n!}x^n$;

$$y_2 = y_1 \ln x + 3x^{1/2}\sum_{n=1}^{\infty}\frac{(-1)^n\prod_{j=1}^{n}(2j-3)}{4^n n!}\left(\sum_{j=1}^{n}\frac{1}{j(2j-3)}\right)x^n$$

7.6.18 (p. ??) $y_1 = x^{1/3}\sum_{n=0}^{\infty}\frac{(-1)^n\prod_{j=1}^{n}(6j-7)^2}{81^n(n!)^2}x^n$;

$$y_2 = y_1 \ln x + 14x^{1/3}\sum_{n=1}^{\infty}\frac{(-1)^n\prod_{j=1}^{n}(6j-7)^2}{81^n(n!)^2}\left(\sum_{j=1}^{n}\frac{1}{j(6j-7)})\right)x^n$$

7.6.19 (p. ??) $y_1 = x^2\sum_{n=0}^{\infty}\frac{(-1)^n\prod_{j=1}^{n}(2j+5)}{(n!)^2}x^n$;

$$y_2 = y_1 \ln x - 2x^2\sum_{n=1}^{\infty}\frac{(-1)^n\prod_{j=1}^{n}(2j+5)}{(n!)^2}\left(\sum_{j=1}^{n}\frac{(j+5)}{j(2j+5)}\right)x^n$$

7.6.20 (p. ??) $y_1 = \frac{1}{x}\sum_{n=0}^{\infty} \frac{2^n \prod_{j=1}^{n}(2j-1)}{n!} x^n$;

$$y_2 = y_1 \ln x + \frac{1}{x}\sum_{n=1}^{\infty} \frac{2^n \prod_{j=1}^{n}(2j-1)}{n!}\left(\sum_{j=1}^{n} \frac{1}{j(2j-1)}\right) x^n$$

7.6.21 (p. ??) $y_1 = \frac{1}{x}\sum_{n=0}^{\infty} \frac{(-1)^n \prod_{j=1}^{n}(2j-5)}{n!} x^n$;

$$y_2 = y_1 \ln x + \frac{5}{x}\sum_{n=1}^{\infty} \frac{(-1)^n \prod_{j=1}^{n}(2j-5)}{n!}\left(\sum_{j=1}^{n} \frac{1}{j(2j-5)}\right) x^n$$

7.6.22 (p. ??) $y_1 = x^2\sum_{n=0}^{\infty} \frac{(-1)^n \prod_{j=1}^{n}(2j+3)}{2^n n!} x^n$;

$$y_2 = y_1 \ln x - 3x^2\sum_{n=0}^{\infty} \frac{(-1)^n \prod_{j=1}^{n}(2j+3)}{2^n n!}\left(\sum_{j=1}^{n} \frac{1}{j(2j+3)}\right) x^n$$

7.6.23 (p. ??) $y_1 = x^{-2}\left(1 + 3x + \frac{3}{2}x^2 - \frac{1}{2}x^3 + \cdots\right)$; $y_2 = y_1 \ln x - 5x^{-1}\left(1 + \frac{5}{4}x - \frac{1}{4}x^2 + \cdots\right)$

7.6.24 (p. ??) $y_1 = x^3(1 + 20x + 180x^2 + 1120x^3 + \cdots$; $y_2 = y_1 \ln x - x^4\left(26 + 324x + \frac{6968}{3}x^2 + \cdots\right)$

7.6.25 (p. ??) $y_1 = x\left(1 - 5x + \frac{85}{4}x^2 - \frac{3145}{36}x^3 + \cdots\right)$; $y_2 = y_1 \ln x + x^2\left(2 - \frac{39}{4}x + \frac{4499}{108}x^2 + \cdots\right)$

7.6.26 (p. ??) $y_1 = 1 - x + \frac{3}{4}x^2 - \frac{7}{12}x^3 + \cdots$; $y_2 = y_1 \ln x + x\left(1 - \frac{3}{4}x + \frac{5}{9}x^2 + \cdots\right)$

7.6.27 (p. ??) $y_1 = x^{-3}(1 + 16x + 36x^2 + 16x^3 + \cdots)$; $y_2 = y_1 \ln x - x^{-2}\left(40 + 150x + \frac{280}{3}x^2 + \cdots\right)$

7.6.28 (p. ??) $y_1 = x\sum_{m=0}^{\infty} \frac{(-1)^m}{2^m m!} x^{2m}$; $y_2 = y_1 \ln x - \frac{x}{2}\sum_{m=1}^{\infty} \frac{(-1)^m}{2^m m!}\left(\sum_{j=1}^{m} \frac{1}{j}\right) x^{2m}$

7.6.29 (p. ??) $y_1 = x^2\sum_{m=0}^{\infty} (-1)^m (m+1) x^{2m}$; $y_2 = y_1 \ln x - \frac{x^2}{2}\sum_{m=1}^{\infty} (-1)^m m x^{2m}$

7.6.30 (p. ??) $y_1 = x^{1/2}\sum_{m=0}^{\infty} \frac{(-1)^m}{4^m m!} x^{2m}$; $y_2 = y_1 \ln x - \frac{x^{1/2}}{2}\sum_{m=1}^{\infty} \frac{(-1)^m}{4^m m!}\left(\sum_{j=1}^{m} \frac{1}{j}\right) x^{2m}$

7.6.31 (p. ??) $y_1 = x\sum_{m=0}^{\infty} \frac{(-1)^m \prod_{j=1}^{m}(2j-1)}{2^m m!} x^{2m}$;

$$y_2 = y_1 \ln x + \frac{x}{2}\sum_{m=1}^{\infty} \frac{(-1)^m \prod_{j=1}^{m}(2j-1)}{2^m m!}\left(\sum_{j=1}^{m} \frac{1}{j(2j-1)}\right) x^{2m}$$

7.6.32 (p. ??) $y_1 = x^{1/2}\sum_{m=0}^{\infty} \frac{(-1)^m \prod_{j=1}^{m}(4j-1)}{8^m m!} x^{2m}$;

$$y_2 = y_1 \ln x + \frac{x^{1/2}}{2}\sum_{m=1}^{\infty} \frac{(-1)^m \prod_{j=1}^{m}(4j-1)}{8^m m!}\left(\sum_{j=1}^{m} \frac{1}{j(4j-1)}\right) x^{2m}$$

7.6.33 (p. ??) $y_1 = x\sum_{m=0}^{\infty} \frac{(-1)^m \prod_{j=1}^{m}(2j+1)}{2^m m!} x^{2m}$;

$$y_2 = y_1 \ln x - \frac{x}{2} \sum_{m=1}^{\infty} \frac{(-1)^m \prod_{j=1}^{m}(2j+1)}{2^m m!} \left(\sum_{j=1}^{m} \frac{1}{j(2j+1)} \right) x^{2m}$$

7.6.34 (p. ??) $y_1 = x^{-1/4} \sum_{m=0}^{\infty} \frac{(-1)^m \prod_{j=1}^{m}(8j-13)}{(32)^m m!} x^{2m}$;

$$y_2 = y_1 \ln x + \frac{13}{2} x^{-1/4} \sum_{m=1}^{\infty} \frac{(-1)^m \prod_{j=1}^{m}(8j-13)}{(32)^m m!} \left(\sum_{j=1}^{m} \frac{1}{j(8j-13)} \right) x^{2m}$$

7.6.35 (p. ??) $y_1 = x^{1/3} \sum_{m=0}^{\infty} \frac{(-1)^m \prod_{j=1}^{m}(3j-1)}{9^m m!} x^{2m}$;

$$y_2 = y_1 \ln x + \frac{x^{1/3}}{2} \sum_{m=1}^{\infty} \frac{(-1)^m \prod_{j=1}^{m}(3j-1)}{9^m m!} \left(\sum_{j=1}^{m} \frac{1}{j(3j-1)} \right) x^{2m}$$

7.6.36 (p. ??) $y_1 = x^{1/2} \sum_{m=0}^{\infty} \frac{(-1)^m \prod_{j=1}^{m}(4j-3)(4j-1)}{4^m (m!)^2} x^{2m}$;

$$y_2 = y_1 \ln x + x^{1/2} \sum_{m=1}^{\infty} \frac{(-1)^m \prod_{j=1}^{m}(4j-3)(4j-1)}{4^m (m!)^2} \left(\sum_{j=1}^{m} \frac{8j-3}{j(4j-3)(4j-1)} \right) x^{2m}$$

7.6.37 (p. ??) $y_1 = x^{5/3} \sum_{m=0}^{\infty} \frac{(-1)^m}{3^m m!} x^{2m}$; $y_2 = y_2 1 \ln x - \frac{x^{5/3}}{2} \sum_{m=1}^{\infty} \frac{(-1)^m}{3^m m!} \left(\sum_{j=1}^{m} \frac{1}{j} \right) x^{2m}$

7.6.38 (p. ??) $y_1 = \frac{1}{x} \sum_{m=0}^{\infty} \frac{(-1)^m \prod_{j=1}^{m}(4j-7)}{2^m m!} x^{2m}$;

$$y_2 = y_1 \ln x + \frac{7}{2x} \sum_{m=1}^{\infty} \frac{(-1)^m \prod_{j=1}^{m}(4j-7)}{2^m m!} \left(\sum_{j=1}^{m} \frac{1}{j(4j-7)} \right) x^{2m}$$

7.6.39 (p. ??) $y_1 = x^{-1} \left(1 - \frac{3}{2}x^2 + \frac{15}{8}x^4 - \frac{35}{16}x^6 + \cdots \right)$

; $y_2 = y_1 \ln x + x \left(\frac{1}{4} - \frac{13}{32}x^2 + \frac{101}{192}x^4 + \cdots \right)$

7.6.40 (p. ??) $y_1 = x \left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{1}{48}x^6 + \cdots \right)$; $y_2 = y_1 \ln x + x^3 \left(\frac{1}{4} - \frac{3}{32}x^2 + \frac{11}{576}x^4 + \cdots \right)$

7.6.41 (p. ??) $y_1 = x^{-2} \left(1 - \frac{3}{4}x^2 - \frac{9}{64}x^4 - \frac{25}{256}x^6 + \cdots \right)$; $y_2 = y_1 \ln x + \frac{1}{2} - \frac{21}{128}x^2 - \frac{215}{1536}x^4 + \cdots$

7.6.42 (p. ??) $y_1 = x^{-3} \left(1 - \frac{17}{8}x^2 + \frac{85}{256}x^4 - \frac{85}{18432}x^6 + \cdots \right)$; $y_2 = y_1 \ln x + x^{-1} \left(\frac{25}{8} - \frac{471}{512}x^2 + \frac{1583}{110592}x^4 + \cdots \right)$

7.6.43 (p. ??) $y_1 = x^{-1} \left(1 - \frac{3}{4}x^2 + \frac{45}{64}x^4 - \frac{175}{256}x^6 + \cdots \right)$; $y_2 = y_1 \ln x - x \left(\frac{1}{4} - \frac{33}{128}x^2 + \frac{395}{1536}x^4 + \cdots \right)$

7.6.44 (p. ??) $y_1 = \frac{1}{x}$; $y_2 = y_1 \ln x - 6 + 6x - \frac{8}{3}x^2$

7.6.45 (p. ??) $y_1 = 1 - x$; $y_2 = y_1 \ln x + 4x$

7.6.46 (p. ??) $y_1 = \frac{(x-1)^2}{x}$; $y_2 = y_1 \ln x + 3 - 3x + 2 \sum_{n=2}^{\infty} \frac{1}{n(n^2-1)} x^n$

7.6.47 (p. ??) $y_1 = x^{1/2}(x+1)^2$; $y_2 = y_1 \ln x - x^{3/2} \left(3 + 3x + 2 \sum_{n=2}^{\infty} \frac{(-1)^n}{n(n^2-1)} x^n \right)$

7.6.48 (p. ??) $y_1 = x^2(1-x)^3$; $y_2 = y_1 \ln x + x^3\left(4 - 7x + \frac{11}{3}x^2 - 6\sum_{n=3}^{\infty}\frac{1}{n(n-2)(n^2-1)}x^n\right)$

7.6.49 (p. ??) $y_1 = x - 4x^3 + x^5$; $y_2 = y_1 \ln x + 6x^3 - 3x^5$

7.6.50 (p. ??) $y_1 = x^{1/3}\left(1 - \frac{1}{6}x^2\right)$; $y_2 = y_1 \ln x + x^{7/3}\left(\frac{1}{4} - \frac{1}{12}\sum_{m=1}^{\infty}\frac{1}{6^m m(m+1)(m+1)!}x^{2m}\right)$

7.6.51 (p. ??) $y_1 = (1+x^2)^2$; $y_2 = y_1 \ln x - \frac{3}{2}x^2 - \frac{3}{2}x^4 + \sum_{m=3}^{\infty}\frac{(-1)^m}{m(m-1)(m-2)}x^{2m}$

7.6.52 (p. ??) $y_1 = x^{-1/2}\left(1 - \frac{1}{2}x^2 + \frac{1}{32}x^4\right)$; $y_2 = y_1 \ln x + x^{3/2}\left(\frac{5}{8} - \frac{9}{128}x^2 + \sum_{m=2}^{\infty}\frac{1}{4^{m+1}(m-1)m(m+1)(m+1)!}\right.$

7.6.56 (p. ??) $y_1 = \sum_{m=0}^{\infty}\frac{(-1)^m}{4^m(m!)^2}x^{2m}$; $y_2 = y_1 \ln x - \sum_{m=1}^{\infty}\frac{(-1)^m}{4^m(m!)^2}\left(\sum_{j=1}^{m}\frac{1}{j}\right)x^{2m}$

7.6.58 (p. ??) $\frac{x^{1/2}}{1+x}$; $\frac{x^{1/2}\ln x}{1+x}$ **7.6.59 (p. ??)** $\frac{x^{1/3}}{3+x}$; $\frac{x^{1/3}\ln x}{3+x}$

7.6.60 (p. ??) $\frac{x}{2-x^2}$; $\frac{x\ln x}{2-x^2}$ **7.6.61 (p. ??)** $\frac{x^{1/4}}{1+x^2}$; $\frac{x^{1/4}\ln x}{1+x^2}$

7.6.62 (p. ??) $\frac{x}{4+3x}$; $\frac{x\ln x}{4+3x}$ **7.6.63 (p. ??)** $\frac{x^{1/2}}{1+3x+x^2}$; $\frac{x^{1/2}\ln x}{1+3x+x^2}$

7.6.64 (p. ??) $\frac{x}{(1-x)^2}$; $\frac{x\ln x}{(1-x)^2}$ **7.6.65 (p. ??)** $\frac{x^{1/3}}{1+x+x^2}$; $\frac{x^{1/3}\ln x}{1+x+x^2}$

Section 7.7 Answers, pp. ??–??

7.7.1 (p. ??) $y_1 = 2x^3 \sum_{n=0}^{\infty} \frac{(-4)^n}{n!(n+2)!} x^n$; $y_2 = x + 4x^2 - 8\left(y_1 \ln x - 4\sum_{n=1}^{\infty} \frac{(-4)^n}{n!(n+2)!}\left(\sum_{j=1}^{n} \frac{j+1}{j(j+2)}\right) x^n\right)$

7.7.2 (p. ??) $y_1 = x \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+1)!} x^n$; $y_2 = 1 - y_1 \ln x + x \sum_{n=1}^{\infty} \frac{(-1)^n}{n!(n+1)!}\left(\sum_{j=1}^{n} \frac{2j+1}{j(j+1)}\right) x^n$

7.7.3 (p. ??) $y_1 = x^{1/2}$; $y_2 = x^{-1/2} + y_1 \ln x + x^{1/2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} x^n$

7.7.4 (p. ??) $y_1 = x \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n = xe^{-x}$; $y_2 = 1 - y_1 \ln x + x \sum_{n=1}^{\infty} \frac{(-1)^n}{n!}\left(\sum_{j=1}^{n} \frac{1}{j}\right) x^n$

7.7.5 (p. ??) $y_1 = x^{1/2} \sum_{n=0}^{\infty} \left(-\frac{3}{4}\right)^n \frac{\prod_{j=1}^{n}(2j+1)}{n!} x^n$;

$$y_2 = x^{-1/2} - \frac{3}{4}\left(y_1 \ln x - x^{1/2} \sum_{n=1}^{\infty} \left(-\frac{3}{4}\right)^n \frac{\prod_{j=1}^{n}(2j+1)}{n!}\left(\sum_{j=1}^{n} \frac{1}{j(2j+1)}\right) x^n\right)$$

7.7.6 (p. ??) $y_1 = x \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n = xe^{-x}$; $y_2 = x^{-2}\left(1 + \frac{1}{2}x + \frac{1}{2}x^2\right) - \frac{1}{2}\left(y_1 \ln x - x \sum_{n=1}^{\infty} \frac{(-1)^n}{n!}\left(\sum_{j=1}^{n} \frac{1}{j}\right) x^n\right)$

7.7.7 (p. ??) $y_1 = 6x^{3/2} \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n n!(n+3)!} x^n$;

$$y_2 = x^{-3/2}\left(1 + \frac{1}{8}x + \frac{1}{64}x^2\right) - \frac{1}{768}\left(y_1 \ln x - 6x^{3/2} \sum_{n=1}^{\infty} \frac{(-1)^n}{4^n n!(n+3)!}\left(\sum_{j=1}^{n} \frac{2j+3}{j(j+3)}\right) x^n\right)$$

7.7.8 (p. ??) $y_1 = \frac{120}{x^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+5)!} x^n$;

$$y_2 = x^{-7}\left(1 + \frac{1}{4}x + \frac{1}{24}x^2 + \frac{1}{144}x^3 + \frac{1}{576}x^4\right) - \frac{1}{2880}\left(y_1 \ln x - \frac{120}{x^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n!(n+5)!}\left(\sum_{j=1}^{n} \frac{2j+5}{j(j+5)}\right) x^n\right)$$

7.7.9 (p. ??) $y_1 = \frac{x^{1/2}}{6} \sum_{n=0}^{\infty} (-1)^n (n+1)(n+2)(n+3) x^n$;

$$y_2 = x^{-5/2}\left(1 + \frac{1}{2}x + x^2\right) - 3y_1 \ln x + \frac{3}{2}x^{1/2} \sum_{n=1}^{\infty} (-1)^n (n+1)(n+2)(n+3)\left(\sum_{j=1}^{n} \frac{1}{j(j+3)}\right) x^n$$

7.7.10 (p. ??) $y_1 = x^4\left(1 - \frac{2}{5}x\right)$ $y_2 = 1 + 10x + 50x^2 + 200x^3 - 300\left(y_1 \ln x + \frac{27}{25}x^5 - \frac{1}{30}x^6\right)$

7.7.11 (p. ??) $y_1 = x^3$; $y_2 = x^{-3}\left(1 - \frac{6}{5}x + \frac{3}{4}x^2 - \frac{1}{3}x^3 + \frac{1}{8}x^4 - \frac{1}{20}x^5\right) - \frac{1}{120}\left(y_1 \ln x + x^3 \sum_{n=1}^{\infty} \frac{(-1)^n 6!}{n(n+6)!} x^n\right)$

7.7.12 (p. ??) $y_1 = x^2 \sum_{n=0}^{\infty} \frac{1}{n!}\left(\prod_{j=1}^{n} \frac{2j+3}{j+4}\right) x^n$;

$$y_2 = x^{-2}\left(1 + x + \frac{1}{4}x^2 - \frac{1}{12}x^3\right) - \frac{1}{16}y_1 \ln x + \frac{x^2}{8} \sum_{n=1}^{\infty} \frac{1}{n!}\left(\prod_{j=1}^{n} \frac{2j+3}{j+4}\right)\left(\sum_{j=1}^{n} \frac{(j^2+3j+6)}{j(j+4)(2j+3)}\right) x^n$$

7.7.13 (p. ??) $y_1 = x^5 \sum_{n=0}^{\infty} (-1)^n (n+1)(n+2)x^n$; $y_2 = 1 - \frac{x}{2} + \frac{x^2}{6}$

7.7.14 (p. ??) $y_1 = \frac{1}{x} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\prod_{j=1}^{n} \frac{(j+3)(2j-3)}{j+6} \right) x^n$; $y_2 = x^{-7} \left(1 + \frac{26}{5}x + \frac{143}{20}x^2 \right)$

7.7.15 (p. ??) $y_1 = x^{7/2} \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n (n+4)!} x^n$; $y_2 = x^{-1/2} \left(1 - \frac{1}{2}x + \frac{1}{8}x^2 - \frac{1}{48}x^3 \right)$

7.7.16 (p. ??) $y_1 = x^{10/3} \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)}{9^n} \left(\prod_{j=1}^{n} \frac{3j+7}{j+4} \right) x^n$; $y_2 = x^{-2/3} \left(1 + \frac{4}{27}x - \frac{1}{243}x^2 \right)$

7.7.17 (p. ??) $y_1 = x^3 \sum_{n=0}^{7} (-1)^n (n+1) \left(\prod_{j=1}^{n} \frac{j-8}{j+6} \right) x^n$; $y_2 = x^{-3} \left(1 + \frac{52}{5}x + \frac{234}{5}x^2 + \frac{572}{5}x^3 + 143x^4 \right)$

7.7.18 (p. ??) $y_1 = x^3 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\prod_{j=1}^{n} \frac{(j+3)^2}{j+5} \right) x^n$; $y_2 = x^{-2} \left(1 + \frac{1}{4}x \right)$

7.7.19 (p. ??) $y_1 = x^6 \sum_{n=0}^{4} (-1)^n 2^n \left(\prod_{j=1}^{n} \frac{j-5}{j+5} \right) x^n$; $y_2 = x(1 + 18x + 144x^2 + 672x^3 + 2016x^4)$

7.7.20 (p. ??) $y_1 = x^6 \left(1 + \frac{2}{3}x + \frac{1}{7}x^2 \right)$; $y_2 = x \left(1 + \frac{21}{4}x + \frac{21}{2}x^2 + \frac{35}{4}x^3 \right)$

7.7.21 (p. ??) $y_1 = x^{7/2} \sum_{n=0}^{\infty} (-1)^n (n+1)x^n$; $y_2 = x^{-7/2} \left(1 - \frac{5}{6}x + \frac{2}{3}x^2 - \frac{1}{2}x^3 + \frac{1}{3}x^4 - \frac{1}{6}x^5 \right)$

7.7.22 (p. ??) $y_1 = \frac{x^{10}}{6} \sum_{n=0}^{\infty} (-1)^n 2^n (n+1)(n+2)(n+3)x^n$;

$$y_2 = \left(1 - \frac{4}{3}x + \frac{5}{3}x^2 - \frac{40}{21}x^3 + \frac{40}{21}x^4 - \frac{32}{21}x^5 + \frac{16}{21}x^6 \right)$$

7.7.23 (p. ??) $y_1 = x^6 \sum_{m=0}^{\infty} \frac{(-1)^m \prod_{j=1}^{m}(2j+5)}{2^m m!} x^{2m}$;

$$y_2 = x^2 \left(1 + \frac{3}{2}x^2 \right) - \frac{15}{2} y_1 \ln x + \frac{75}{2} x^6 \sum_{m=1}^{\infty} \frac{(-1)^m \prod_{j=1}^{m}(2j+5)}{2^{m+1} m!} \left(\sum_{j=1}^{m} \frac{1}{j(2j+5)} \right) x^{2m}$$

7.7.24 (p. ??) $y_1 = x^6 \sum_{m=0}^{\infty} \frac{(-1)^m}{2^m m!} x^{2m} = x^6 e^{-x^2/2}$;

$$y_2 = x^2 \left(1 + \frac{1}{2}x^2 \right) - \frac{1}{2} y_1 \ln x + \frac{x^6}{4} \sum_{m=1}^{\infty} \frac{(-1)^m}{2^m m!} \left(\sum_{j=1}^{m} \frac{1}{j} \right) x^{2m}$$

7.7.25 (p. ??) $y_1 = 6x^6 \sum_{m=0}^{\infty} \frac{(-1)^m}{4^m m!(m+3)!} x^{2m}$;

$$y_2 = 1 + \frac{1}{8}x^2 + \frac{1}{64}x^4 - \frac{1}{384} \left(y_1 \ln x - 3x^6 \sum_{m=1}^{\infty} \frac{(-1)^m}{4^m m!(m+3)!} \left(\sum_{j=1}^{m} \frac{2j+3}{j(j+3)} \right) x^{2m} \right)$$

7.7.26 (p. ??) $y_1 = \frac{x}{2} \sum_{m=0}^{\infty} \frac{(-1)^m (m+2)}{m!} x^{2m}$;

$$y_2 = x^{-1} - 4y_1 \ln x + x \sum_{m=1}^{\infty} \frac{(-1)^m (m+2)}{m!} \left(\sum_{j=1}^{m} \frac{j^2+4j+2}{j(j+1)(j+2)} \right) x^{2m}$$

7.7.27 (p. ??) $y_1 = 2x^3 \sum_{m=0}^{\infty} \frac{(-1)^m}{4^m m!(m+2)!} x^{2m}$;

$$y_2 = x^{-1}\left(1 + \frac{1}{4}x^2\right) - \frac{1}{16}\left(y_1 \ln x - 2x^3 \sum_{m=1}^{\infty} \frac{(-1)^m}{4^m m!(m+2)!} \left(\sum_{j=1}^{m} \frac{j+1}{j(j+2)} \right) x^{2m} \right)$$

7.7.28 (p. ??) $y_1 = x^{-1/2} \sum_{m=0}^{\infty} \frac{(-1)^m \prod_{j=1}^{m}(2j-1)}{8^m m!(m+1)!} x^{2m}$;

$$y_2 = x^{-5/2} + \frac{1}{4} y_1 \ln x - x^{-1/2} \sum_{m=1}^{\infty} \frac{(-1)^m \prod_{j=1}^{m}(2j-1)}{8^{m+1} m!(m+1)!} \left(\sum_{j=1}^{m} \frac{2j^2-2j-1}{j(j+1)(2j-1)} \right) x^{2m}$$

7.7.29 (p. ??) $y_1 = x \sum_{m=0}^{\infty} \frac{(-1)^m}{2^m m!} x^{2m} = x e^{-x^2/2}$; $y_2 = x^{-1} - y_1 \ln x + \frac{x}{2} \sum_{m=1}^{\infty} \frac{(-1)^m}{2^m m!} \left(\sum_{j=1}^{m} \frac{1}{j} \right) x^{2m}$

7.7.30 (p. ??) $y_1 = x^2 \sum_{m=0}^{\infty} \frac{1}{m!} x^{2m} = x^2 e^{x^2}$; $y_2 = x^{-2}(1-x^2) - 2y_1 \ln x + x^2 \sum_{m=1}^{\infty} \frac{1}{m!} \left(\sum_{j=1}^{m} \frac{1}{j} \right) x^{2m}$

7.7.31 (p. ??) $y_1 = 6x^{5/2} \sum_{m=0}^{\infty} \frac{(-1)^m}{16^m m!(m+3)!} x^{2m}$;

$$y_2 = x^{-7/2}\left(1 + \frac{1}{32}x^2 + \frac{1}{1024}x^4\right) - \frac{1}{24576}\left(y_1 \ln x - 3x^{5/2} \sum_{m=1}^{\infty} \frac{(-1)^m}{16^m m!(m+3)!} \left(\sum_{j=1}^{m} \frac{2j+3}{j(j+3)} \right) x^{2m} \right)$$

7.7.32 (p. ??) $y_1 = 2x^{13/3} \sum_{m=0}^{\infty} \frac{\prod_{j=1}^{m}(3j+1)}{9^m m!(m+2)!} x^{2m}$;

$$y_2 = x^{1/3}\left(1 + \frac{2}{9}x^2\right) + \frac{2}{81}\left(y_1 \ln x - x^{13/3} \sum_{m=0}^{\infty} \frac{\prod_{j=1}^{m}(3j+1)}{9^m m!(m+2)!} \left(\sum_{j=1}^{m} \frac{3j^2+2j+2}{j(j+2)(3j+1)} \right) x^{2m} \right)$$

7.7.33 (p. ??) $y_1 = x^2$; $y_2 = x^{-2}(1+2x^2) - 2\left(y_1 \ln x + x^2 \sum_{m=1}^{\infty} \frac{1}{m(m+2)!} x^{2m} \right)$

7.7.34 (p. ??) $y_1 = x^2\left(1 - \frac{1}{2}x^2\right)$; $y_2 = x^{-2}\left(1 + \frac{9}{2}x^2\right) - \frac{27}{2}\left(y_1 \ln x + \frac{7}{12}x^4 - x^2 \sum_{m=2}^{\infty} \frac{\left(\frac{3}{2}\right)^m}{m(m-1)(m+2)!} x^{2m} \right)$

7.7.35 (p. ??) $y_1 = \sum_{m=0}^{\infty} (-1)^m (m+1) x^{2m}$; $y_2 = x^{-4}$

7.7.36 (p. ??) $y_1 = x^{5/2} \sum_{m=0}^{\infty} \frac{(-1)^m}{(m+1)(m+2)(m+3)} x^{2m}$; $y_2 = x^{-7/2}(1+x^2)^2$

7.7.37 (p. ??) $y_1 = \frac{x^7}{5} \sum_{m=0}^{\infty} (-1)^m (m+5) x^{2m}$; $y_2 = x^{-1}\left(1 - 2x^2 + 3x^4 - 4x^6\right)$

7.7.38 (p. ??) $y_1 = x^3 \sum_{m=0}^{\infty} (-1)^m \frac{m+1}{2^m} \left(\prod_{j=1}^{m} \frac{2j+1}{j+5} \right) x^{2m}$; $y_2 = x^{-7}\left(1 + \frac{21}{8}x^2 + \frac{35}{16}x^4 + \frac{35}{64}x^6\right)$

7.7.39 (p. ??) $y_1 = 2x^4 \sum_{m=0}^{\infty} (-1)^m \frac{\prod_{j=1}^{m}(4j+5)}{2^m(m+2)!} x^{2m}$; $y_2 = 1 - \frac{1}{2}x^2$

7.7.40 (p. ??) $y_1 = x^{3/2} \sum_{m=0}^{\infty} \frac{(-1)^m \prod_{j=1}^{m}(2j-1)}{2^{m-1}(m+2)!} x^{2m}$; $y_2 = x^{-5/2}\left(1 + \frac{3}{2}x^2\right)$

7.7.42 (p. ??) $y_1 = x^{\nu} \sum_{m=0}^{\infty} \frac{(-1)^m}{4^m m! \prod_{j=1}^{m}(j+\nu)} x^{2m}$;

$$y_2 = x^{-\nu} \sum_{m=0}^{\nu-1} \frac{(-1)^m}{4^m m! \prod_{j=1}^{m}(j-\nu)} x^{2m} - \frac{2}{4^{\nu}\nu!(\nu-1)!}\left(y_1 \ln x - \frac{x^{\nu}}{2} \sum_{m=1}^{\infty} \frac{(-1)^m}{4^m m! \prod_{j=1}^{m}(j+\nu)} \left(\sum_{j=1}^{m} \frac{2j+\nu}{j(j+\nu)}\right) x^{2m}\right)$$

Section 8.1 Answers, pp. 225–228

8.1.1 (p. 225) **(a)** $\frac{1}{s^2}$ **(b)** $\frac{1}{(s+1)^2}$ **(c)** $\frac{b}{s^2-b^2}$ **(d)** $\frac{-2s+5}{(s-1)(s-2)}$ **(e)** $\frac{2}{s^3}$

8.1.2 (p. 225) **(a)** $\frac{s^2+2}{[(s-1)^2+1]\,[(s+1)^2+1]}$ **(b)** $\frac{2}{s(s^2+4)}$ **(c)** $\frac{s^2+8}{s(s^2+16)}$ **(d)** $\frac{s^2-2}{s(s^2-4)}$
(e) $\frac{4s}{(s^2-4)^2}$ **(f)** $\frac{1}{s^2+4}$ **(g)** $\frac{1}{\sqrt{2}}\frac{s+1}{s^2+1}$ **(h)** $\frac{5s}{(s^2+4)(s^2+9)}$ **(i)** $\frac{s^3+2s^2+4s+32}{(s^2+4)(s^2+16)}$

8.1.4 (p. 225) **(a)** $f(3-)=-1,\ f(3)=f(3+)=1$ **(b)** $f(1-)=3,\ f(1)=4,\ f(1+)=1$
(c) $f\left(\frac{\pi}{2}-\right)=1,\ f\left(\frac{\pi}{2}\right)=f\left(\frac{\pi}{2}+\right)=2,\ f(\pi-)=0,\ f(\pi)=f(\pi+)=-1$
(d) $f(1-)=1,\ f(1)=2,\ f(1+)=1,\ f(2-)=0,\ f(2)=3,\ f(2+)=6$

8.1.5 (p. 226) **(a)** $\frac{1-e^{-(s+1)}}{s+1} + \frac{e^{-(s+2)}}{s+2}$ **(b)** $\frac{1}{s} + e^{-4s}\left(\frac{1}{s^2}+\frac{3}{s}\right)$ **(c)** $\frac{1-e^{-s}}{s^2}$ **(d)** $\frac{1-e^{-(s-1)}}{(s-1)^2}$

8.1.7 (p. 226) $\mathcal{L}(e^{\lambda t}\cos\omega t) = \frac{(s-\lambda)^2-\omega^2}{((s-\lambda)^2+\omega^2)^2}$ $\mathcal{L}(e^{\lambda t}\sin\omega t) = \frac{2\omega(s-\lambda)}{((s-\lambda)^2+\omega^2)^2}$

8.1.15 (p. 227) **(a)** $\tan^{-1}\frac{\omega}{s},\quad s>0$ **(b)** $\frac{1}{2}\ln\frac{s^2}{s^2+\omega^2},\quad s>0$ **(c)** $\ln\frac{s-b}{s-a},\quad s>\max(a,b)$
(d) $\frac{1}{2}\ln\frac{s^2}{s^2-1},\quad s>1$ **(e)** $\frac{1}{4}\ln\frac{s^2}{s^2-4},\quad s>2$

8.1.18 (p. 228) **(a)** $\frac{1}{s^2}\tanh\frac{s}{2}$ **(b)** $\frac{1}{s}\tanh\frac{s}{4}$ **(c)** $\frac{1}{s^2+1}\coth\frac{\pi s}{2}$ **(d)** $\frac{1}{(s^2+1)(1-e^{-\pi s})}$

Section 8.2 Answers, pp. 236–238

8.2.1 (p. 236) **(a)** $\frac{t^3e^{7t}}{2}$ **(b)** $2e^{2t}\cos 3t$ **(c)** $\frac{e^{-2t}}{4}\sin 4t$ **(d)** $\frac{2}{3}\sin 3t$ **(e)** $t\cos t$
(f) $\frac{e^{2t}}{2}\sinh 2t$ **(g)** $\frac{2te^{2t}}{3}\sin 9t$ **(h)** $\frac{2e^{3t}}{3}\sinh 3t$ **(i)** $e^{2t}t\cos t$

8.2.2 (p. 236) **(a)** $t^2e^{7t} + \frac{17}{6}t^3e^{7t}$ **(b)** $e^{2t}\left(\frac{1}{6}t^3 + \frac{1}{6}t^4 + \frac{1}{40}t^5\right)$ **(c)** $e^{-3t}\left(\cos 3t + \frac{2}{3}\sin 3t\right)$
(d) $2\cos 3t + \frac{1}{3}\sin 3t$ **(e)** $(1-t)e^{-t}$ **(f)** $\cosh 3t + \frac{1}{3}\sinh 3t$ **(g)** $\left(1-t-t^2-\frac{1}{6}t^3\right)e^{-t}$
(h) $e^t\left(2\cos 2t + \frac{5}{2}\sin 2t\right)$ **(i)** $1-\cos t$ **(j)** $3\cosh t + 4\sinh t$ **(k)** $3e^t + 4\cos 3t + \frac{1}{3}\sin 3t$
(l) $3te^{-2t} - 2\cos 2t - 3\sin 2t$

8.2.3 (p. 236) **(a)** $\frac{1}{4}e^{2t} - \frac{1}{4}e^{-2t} - e^{-t}$ **(b)** $\frac{1}{5}e^{-4t} - \frac{41}{5}e^t + 5e^{3t}$ **(c)** $-\frac{1}{2}e^{2t} - \frac{13}{10}e^{-2t} - \frac{1}{5}e^{3t}$
(d) $-\frac{2}{5}e^{-4t} - \frac{3}{5}e^t$ **(e)** $\frac{3}{20}e^{2t} - \frac{37}{12}e^{-2t} + \frac{1}{3}e^t + \frac{8}{5}e^{-3t}$ **(f)** $\frac{39}{10}e^t + \frac{3}{14}e^{3t} + \frac{23}{105}e^{-4t} - \frac{7}{3}e^{2t}$

8.2.4 (p. 237) **(a)** $\frac{4}{5}e^{-2t}-\frac{1}{2}e^{-t}-\frac{3}{10}\cos t+\frac{11}{10}\sin t$ **(b)** $\frac{2}{5}\sin t+\frac{6}{5}\cos t+\frac{7}{5}e^{-t}\sin t-\frac{6}{5}e^{-t}\cos t$
(c) $\frac{8}{13}e^{2t}-\frac{8}{13}e^{-t}\cos 2t+\frac{15}{26}e^{-t}\sin 2t$ **(d)** $\frac{1}{2}te^{t}+\frac{3}{8}e^{t}+e^{-2t}-\frac{11}{8}e^{-3t}$
(e) $\frac{2}{3}te^{t}+\frac{1}{9}e^{t}+te^{-2t}-\frac{1}{9}e^{-2t}$ **(f)** $-e^{t}+\frac{5}{2}te^{t}+\cos t-\frac{3}{2}\sin t$
8.2.5 (p. 237) **(a)** $\frac{3}{5}\cos 2t+\frac{1}{5}\sin 2t-\frac{3}{5}\cos 3t-\frac{2}{15}\sin 3t$ **(b)** $-\frac{4}{15}\cos t+\frac{1}{15}\sin t+\frac{4}{15}\cos 4t-\frac{1}{60}\sin 4t$
(c) $\frac{5}{3}\cos t+\sin t-\frac{5}{3}\cos 2t-\frac{1}{2}\sin 2t$ **(d)** $-\frac{1}{3}\cos\frac{t}{2}+\frac{2}{3}\sin\frac{t}{2}+\frac{1}{3}\cos t-\frac{1}{3}\sin t$
(e) $\frac{1}{15}\cos\frac{t}{4}-\frac{8}{15}\sin\frac{t}{4}-\frac{1}{15}\cos 4t+\frac{1}{30}\sin 4t$ **(f)** $\frac{2}{5}\cos\frac{t}{3}-\frac{3}{5}\sin\frac{t}{3}-\frac{2}{5}\cos\frac{t}{2}+\frac{2}{5}\sin\frac{t}{2}$
8.2.6 (p. 237) **(a)** $e^{t}(\cos 2t+\sin 2t)-e^{-t}\left(\cos 3t+\frac{4}{3}\sin 3t\right)$ **(b)** $e^{3t}\left(-\cos 2t+\frac{3}{2}\sin 2t\right)+e^{-t}\left(\cos 2t+\frac{1}{2}\sin 2t\right)$
(c) $e^{-2t}\left(\frac{1}{8}\cos t+\frac{1}{4}\sin t\right)-e^{2t}\left(\frac{1}{8}\cos 3t-\frac{1}{12}\sin 3t\right)$ **(d)** $e^{2t}\left(\cos t+\frac{1}{2}\sin t\right)-e^{3t}\left(\cos 2t-\frac{1}{4}\sin 2t\right)$
(e) $e^{t}\left(\frac{1}{5}\cos t+\frac{2}{5}\sin t\right)-e^{-t}\left(\frac{1}{5}\cos 2t+\frac{2}{5}\sin 2t\right)$ **(f)** $e^{t/2}\left(-\cos t+\frac{9}{8}\sin t\right)+e^{-t/2}\left(\cos t-\frac{1}{8}\sin t\right)$
8.2.7 (p. 237) **(a)** $1-\cos t$ **(b)** $\frac{e^{t}}{16}(1-\cos 4t)$ **(c)** $\frac{4}{9}e^{2t}+\frac{5}{9}e^{-t}\sin 3t-\frac{4}{9}e^{-t}\cos 3t$ **(d)** $3e^{t/2}-\frac{7}{2}e^{t}\sin 2t-3e^{t}\cos 2t$
(e) $\frac{1}{4}e^{3t}-\frac{1}{4}e^{-t}\cos 2t$ **(f)** $\frac{1}{9}e^{2t}-\frac{1}{9}e^{-t}\cos 3t+\frac{5}{9}e^{-t}\sin 3t$
8.2.8 (p. 237) **(a)** $-\frac{3}{10}\sin t+\frac{2}{5}\cos t-\frac{3}{4}e^{t}+\frac{7}{20}e^{3t}$ **(b)** $-\frac{3}{5}e^{-t}\sin t+\frac{1}{5}e^{-t}\cos t-\frac{1}{2}e^{-t}+\frac{3}{10}e^{t}$
(c) $-\frac{1}{10}e^{t}\sin t-\frac{7}{10}e^{t}\cos t+\frac{1}{5}e^{-t}+\frac{1}{2}e^{2t}$ **(d)** $-\frac{1}{2}e^{t}+\frac{7}{10}e^{-t}-\frac{1}{5}\cos 2t+\frac{3}{5}\sin 2t$
(e) $\frac{3}{10}+\frac{1}{10}e^{2t}+\frac{1}{10}e^{t}\sin 2t-\frac{2}{5}e^{t}\cos 2t$ **(f)** $-\frac{4}{9}e^{2t}\cos 3t+\frac{1}{3}e^{2t}\sin 3t-\frac{5}{9}e^{2t}+e^{t}$
8.2.9 (p. 238) $\frac{1}{a}e^{\frac{b}{a}t}f\left(\frac{t}{a}\right)$

Section 8.3 Answers, pp. 243–244

8.3.1 (p. 243) $y=\frac{1}{6}e^{t}-\frac{9}{2}e^{-t}+\frac{16}{3}e^{-2t}$ **8.3.2 (p. 243)** $y=-\frac{1}{3}+\frac{8}{15}e^{3t}+\frac{4}{5}e^{-2t}$
8.3.3 (p. 243) $y=-\frac{23}{15}e^{-2t}+\frac{1}{3}e^{t}+\frac{1}{5}e^{3t}$ **8.3.4 (p. 243)** $y=-\frac{1}{4}e^{2t}+\frac{17}{20}e^{-2t}+\frac{2}{5}e^{3t}$
8.3.5 (p. 243) $y=\frac{11}{15}e^{-2t}+\frac{1}{6}e^{t}+\frac{1}{10}e^{3t}$ **8.3.6 (p. 243)** $y=e^{t}+2e^{-2t}-2e^{-t}$
8.3.7 (p. 243) $y=\frac{5}{3}\sin t-\frac{1}{3}\sin 2t$ **8.3.8 (p. 243)** $y=4e^{t}-4e^{2t}+e^{3t}$
8.3.9 (p. 243) $y=-\frac{7}{2}e^{2t}+\frac{13}{3}e^{t}+\frac{1}{6}e^{4t}$ **8.3.10 (p. 244)** $y=\frac{5}{2}e^{t}-4e^{2t}+\frac{1}{2}e^{3t}$
8.3.11 (p. 244) $y=\frac{1}{3}e^{t}-2e^{-t}+\frac{5}{3}e^{-2t}$ **8.3.12 (p. 244)** $y=2-e^{-2t}+e^{t}$
8.3.13 (p. 244) $y=1-\cos 2t+\frac{1}{2}\sin 2t$ **8.3.14 (p. 244)** $y=-\frac{1}{3}+\frac{8}{15}e^{3t}+\frac{4}{5}e^{-2t}$
8.3.15 (p. 244) $y=\frac{1}{6}e^{t}-\frac{2}{3}e^{-2t}+\frac{1}{2}e^{-t}$ **8.3.16 (p. 244)** $y=-1+e^{t}+e^{-t}$
8.3.17 (p. 244) $y=\cos 2t-\sin 2t+\sin t$ **8.3.18 (p. 244)** $y=\frac{7}{3}-\frac{7}{2}e^{-t}+\frac{1}{6}e^{3t}$
8.3.19 (p. 244) $y=1+\cos t$ **8.3.20 (p. 244)** $y=t+\sin t$ **8.3.21 (p. 244)** $y=t-6\sin t+\cos t+\sin 2t$
8.3.22 (p. 244) $y=e^{-t}+4e^{-2t}-4e^{-3t}$ **8.3.23 (p. 244)** $y=-3\cos t-2\sin t+e^{-t}(2+5t)$
8.3.24 (p. 244) $y=-\sin t-2\cos t+3e^{3t}+e^{-t}$ **8.3.25 (p. 244)** $y=(3t+4)\sin t-(2t+6)\cos t$
8.3.26 (p. 244) $y=-(2t+2)\cos 2t+\sin 2t+3\cos t$ **8.3.27 (p. 244)** $y=e^{t}(\cos t-3\sin t)+e^{3t}$

8.3.28 (p. 244) $y = -1 + t + e^{-t}(3\cos t - 5\sin t)$ **8.3.29 (p. 244)** $y = 4\cos t - 3\sin t - e^{t}(3\cos t - 8\sin t)$
8.3.30 (p. 244) $y = e^{-t} - 2e^{t} + e^{-2t}(\cos 3t - 11/3\sin 3t)$
8.3.31 (p. 244) $y = e^{-t}(\sin t - \cos t) + e^{-2t}(\cos t + 4\sin t)$
8.3.32 (p. 244) $y = \frac{1}{5}e^{2t} - \frac{4}{3}e^{t} + \frac{32}{15}e^{-t/2}$ **8.3.33 (p. 244)** $y = \frac{1}{7}e^{2t} - \frac{2}{5}e^{t/2} + \frac{9}{35}e^{-t/3}$
8.3.34 (p. 244) $y = e^{-t/2}(5\cos(t/2) - \sin(t/2)) + 2t - 4$
8.3.35 (p. 244) $y = \frac{1}{17}\left(12\cos t + 20\sin t - 3e^{t/2}(4\cos t + \sin t)\right)$.
8.3.36 (p. 244) $y = \frac{e^{-t/2}}{10}(5t + 26) - \frac{1}{5}(3\cos t + \sin t)$ **8.3.37 (p. 244)** $y = \frac{1}{100}\left(3e^{3t} - e^{t/3}(3 + 310t)\right)$

Section 8.4 Answers, pp. 252–255

8.4.1 (p. 252) $1 + u(t-4)(t-1)$; $\frac{1}{s} + e^{-4s}\left(\frac{1}{s^2} + \frac{3}{s}\right)$ **8.4.2 (p. 252)** $t + u(t-1)(1-t)$; $\frac{1-e^{-s}}{s^2}$
8.4.3 (p. 252) $2t - 1 - u(t-2)(t-1)$; $\left(\frac{2}{s^2} - \frac{1}{s}\right) - e^{-2s}\left(\frac{1}{s^2} + \frac{1}{s}\right)$
8.4.4 (p. 252) $1 + u(t-1)(t+1)$; $\frac{1}{s} + e^{-s}\left(\frac{1}{s^2} + \frac{2}{s}\right)$
8.4.5 (p. 252) $t - 1 + u(t-2)(5-t)$; $\frac{1}{s^2} - \frac{1}{s} - e^{-2s}\left(\frac{1}{s^2} - \frac{3}{s}\right)$
8.4.6 (p. 252) $t^2(1 - u(t-1))$; $\frac{2}{s^3} - e^{-s}\left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right)$
8.4.7 (p. 252) $u(t-2)(t^2 + 3t)$; $e^{-2s}\left(\frac{2}{s^3} + \frac{7}{s^2} + \frac{10}{s}\right)$
8.4.8 (p. 252) $t^2 + 2 + u(t-1)(t - t^2 - 2)$; $\frac{2}{s^3} + \frac{2}{s} - e^{-s}\left(\frac{2}{s^3} + \frac{1}{s^2} + \frac{2}{s}\right)$
8.4.9 (p. 252) $te^{t} + u(t-1)(e^{t} - te^{t})$; $\frac{1 - e^{-(s-1)}}{(s-1)^2}$
8.4.10 (p. 252) $e^{-t} + u(t-1)(e^{-2t} - e^{-t})$; $\frac{1 - e^{-(s+1)}}{s+1} + \frac{e^{-(s+2)}}{s+2}$
8.4.11 (p. 252) $-t + 2u(t-2)(t-2) - u(t-3)(t-5)$; $-\frac{1}{s^2} + \frac{2e^{-2s}}{s^2} + e^{-3s}\left(\frac{2}{s} - \frac{1}{s^2}\right)$
8.4.12 (p. 252) $[u(t-1) - u(t-2)]\,t$; $e^{-s}\left(\frac{1}{s^2} + \frac{1}{s}\right) - e^{-2s}\left(\frac{1}{s^2} + \frac{2}{s}\right)$
8.4.13 (p. 252) $t + u(t-1)(t^2 - t) - u(t-2)t^2$; $\frac{1}{s^2} + e^{-s}\left(\frac{2}{s^3} + \frac{1}{s^2}\right) - e^{-2s}\left(\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}\right)$
8.4.14 (p. 252) $t + u(t-1)(2 - 2t) + u(t-2)(4 + t)$; $\frac{1}{s^2} - 2\frac{e^{-s}}{s^2} + e^{-2s}\left(\frac{1}{s^2} + \frac{6}{s}\right)$
8.4.15 (p. 252) $\sin t + u(t - \pi/2)\sin t + u(t - \pi)(\cos t - 2\sin t)$; $\frac{1 + e^{-\frac{\pi}{2}s}s - e^{-\pi s}(s-2)}{s^2+1}$
8.4.16 (p. 253) $2 - 2u(t-1)t + u(t-3)(5t-2)$; $\frac{2}{s} - e^{-s}\left(\frac{2}{s^2} + \frac{2}{s}\right) + e^{-3s}\left(\frac{5}{s^2} + \frac{13}{s}\right)$
8.4.17 (p. 253) $3 + u(t-2)(3t-1) + u(t-4)(t-2)$; $\frac{3}{s} + e^{-2s}\left(\frac{3}{s^2} + \frac{5}{s}\right) + e^{-4s}\left(\frac{1}{s^2} + \frac{2}{s}\right)$
8.4.18 (p. 253) $(t+1)^2 + u(t-1)(2t+3)$; $\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s} + e^{-s}\left(\frac{2}{s^2} + \frac{5}{s}\right)$

8.4.19 (p. 253) $u(t-2)e^{2(t-2)} = \begin{cases} 0, & 0 \leqslant t < 2, \\ e^{2(t-2)}, & t \geqslant 2. \end{cases}$

8.4.20 (p. 253) $u(t-1)\left(1-e^{-(t-1)}\right) = \begin{cases} 0, & 0 \leqslant t < 1, \\ 1-e^{-(t-1)}, & t \geqslant 1. \end{cases}$

8.4.21 (p. 253) $u(t-1)\dfrac{(t-1)^2}{2} + u(t-2)(t-2) = \begin{cases} 0, & 0 \leqslant t < 1, \\ \dfrac{(t-1)^2}{2}, & 1 \leqslant t < 2, \\ \dfrac{t^2-3}{2}, & t \geqslant 2. \end{cases}$

8.4.22 (p. 253) $2+t+u(t-1)(4-t)+u(t-3)(t-2) = \begin{cases} 2+t, & 0 \leqslant t < 1, \\ 6, & 1 \leqslant t < 3, \\ t+4, & t \geqslant 3. \end{cases}$

8.4.23 (p. 253) $5-t+u(t-3)(7t-15)+\dfrac{3}{2}u(t-6)(t-6)^2 = \begin{cases} 5-t, & 0 \leqslant t < 3, \\ 6t-10, & 3 \leqslant t < 6, \\ 44-12t+\frac{3}{2}t^2, & t \geqslant 6. \end{cases}$

8.4.24 (p. 253) $u(t-\pi)e^{-2(t-\pi)}(2\cos t - 5\sin t) = \begin{cases} 0, & 0 \leqslant t < \pi, \\ e^{-2(t-\pi)}(2\cos t - 5\sin t), & t \geqslant \pi. \end{cases}$

8.4.25 (p. 253) $1-\cos t + u(t-\pi/2)(3\sin t + \cos t) = \begin{cases} 1-\cos t, & 0 \leqslant t < \dfrac{\pi}{2}, \\ 1+3\sin t, & t \geqslant \dfrac{\pi}{2}. \end{cases}$

8.4.26 (p. 253) $u(t-2)\left(4e^{-(t-2)} - 4e^{2(t-2)} + 2e^{(t-2)}\right) = \begin{cases} 0, & 0 \leqslant t < 2, \\ 4e^{-(t-2)} - 4e^{2(t-2)} + 2e^{(t-2)}, & t \geqslant 2. \end{cases}$

8.4.27 (p. 253) $1+t+u(t-1)(2t+1)+u(t-3)(3t-5) = \begin{cases} t+1, & 0 \leqslant t < 1, \\ 3t+2, & 1 \leqslant t < 3, \\ 6t-3, & t \geqslant 3. \end{cases}$

8.4.28 (p. 253) $1-t^2+u(t-2)\left(-\dfrac{t^2}{2}+2t+1\right)+u(t-4)(t-4) = \begin{cases} 1-t^2, & 0 \leqslant t < 2 \\ -\dfrac{3t^2}{2}+2t+2, & 2 \leqslant t < 4, \\ -\dfrac{3t^2}{2}+3t-2, & t \geqslant 4. \end{cases}$

8.4.29 (p. 253) $\dfrac{e^{-\tau s}}{s}$ **8.4.30 (p. 253)** For each t only finitely many terms are nonzero.

8.4.33 (p. 255) $1+\sum_{m=1}^{\infty} u(t-m)$; $\dfrac{1}{s(1-e^{-s})}$ **8.4.34 (p. 255)** $1+2\sum_{m=1}^{\infty}(-1)^m u(t-m)$; $\dfrac{1}{s}$; $\dfrac{1-e^{-s}}{1+e^{-s}}$

8.4.35 (p. 255) $1+\sum_{m=1}^{\infty}(2m+1)u(t-m)$; $\dfrac{e^{-s}(1+e^{-s})}{s(1-e^{-s})^2}$ **8.4.36 (p. 255)** $\sum_{m=1}^{\infty}(-1)^m(2m-1)u(t-m)$; $\dfrac{1}{s}\dfrac{(1-e^s)}{(1+e^s)^2}$

Section 8.5 Answers, pp. 261–264

8.5.1 (p. 261) $y = 3(1-\cos t) - 3u(t-\pi)(1+\cos t)$

8.5.2 (p. 261) $y = 3 - 2\cos t + 2u(t-4)\left(t-4-\sin(t-4)\right)$ **8.5.3 (p. 261)** $y = -\dfrac{15}{2} + \dfrac{3}{2}e^{2t} - 2t + \dfrac{u(t-1)}{2}(e^{2(t-1)} - 2t$

8.5.4 (p. 261) $y = \frac{1}{2}e^t + \frac{13}{6}e^{-t} + \frac{1}{3}e^{2t} + u(t-2)\left(-1 + \frac{1}{2}e^{t-2} + \frac{1}{2}e^{-(t-2)} + \frac{1}{2}e^{t+2} - \frac{1}{6}e^{-(t-6)} - \frac{1}{3}e^{2t}\right)$

8.5.5 (p. 261) $y = -7e^t + 4e^{2t} + u(t-1)\left(\frac{1}{2} - e^{t-1} + \frac{1}{2}e^{2(t-1)}\right) - 2u(t-2)\left(\frac{1}{2} - e^{t-2} + \frac{1}{2}e^{2(t-2)}\right)$

8.5.6 (p. 261) $y = \frac{1}{3}\sin 2t - 3\cos 2t + \frac{1}{3}\sin t - 2u(t-\pi)\left(\frac{1}{3}\sin t + \frac{1}{6}\sin 2t\right) + u(t-2\pi)\left(\frac{1}{3}\sin t - \frac{1}{6}\sin 2t\right)$

8.5.7 (p. 262) $y = \frac{1}{4} - \frac{31}{12}e^{4t} + \frac{16}{3}e^t + u(t-1)\left(\frac{2}{3}e^{t-1} - \frac{1}{6}e^{4(t-1)} - \frac{1}{2}\right) + u(t-2)\left(\frac{1}{4} + \frac{1}{12}e^{4(t-2)} - \frac{1}{3}e^{t-2}\right)$

8.5.8 (p. 262) $y = \frac{1}{8}(\cos t - \cos 3t) - \frac{1}{8}u\left(t - \frac{3\pi}{2}\right)\left(\sin t - \cos t + \sin 3t - \frac{1}{3}\cos 3t\right)$

8.5.9 (p. 262) $y = \frac{t}{4} - \frac{1}{8}\sin 2t + \frac{1}{8}u\left(t - \frac{\pi}{2}\right)(\pi\cos 2t - \sin 2t + 2\pi - 2t)$

8.5.10 (p. 262) $y = t - \sin t - 2u(t-\pi)(t + \sin t + \pi\cos t)$

8.5.11 (p. 262) $y = u(t-2)\left(t - \frac{1}{2} + \frac{e^{2(t-2)}}{2} - 2e^{t-2}\right)$

8.5.12 (p. 262) $y = t + \sin t + \cos t - u(t-2\pi)(3t - 3\sin t - 6\pi\cos t)$

8.5.13 (p. 262) $y = \frac{1}{2} + \frac{1}{2}e^{-2t} - e^{-t} + u(t-2)\left(2e^{-(t-2)} - e^{-2(t-2)} - 1\right)$

8.5.14 (p. 262) $y = -\frac{1}{3} - \frac{1}{6}e^{3t} + \frac{1}{2}e^t + u(t-1)\left(\frac{2}{3} + \frac{1}{3}e^{3(t-1)} - e^{t-1}\right)$

8.5.15 (p. 262) $y = \frac{1}{4}\left(e^t + e^{-t}(11 + 6t)\right) + u(t-1)(te^{-(t-1)} - 1)$

8.5.16 (p. 262) $y = e^t - e^{-t} - 2te^{-t} - u(t-1)\left(e^t - e^{-(t-2)} - 2(t-1)e^{-(t-2)}\right)$

8.5.17 (p. 262) $y = te^{-t} + e^{-2t} + u(t-1)\left(e^{-t}(2-t) - e^{-(2t-1)}\right)$

8.5.18 (p. 262) $y = y = \frac{t^2e^{2t}}{2} - te^{2t} - u(t-2)(t-2)^2e^{2t}$

8.5.19 (p. 262) $y = \frac{t^4}{12} + 1 - \frac{1}{12}u(t-1)(t^4 + 2t^3 - 10t + 7) + \frac{1}{6}u(t-2)(2t^3 + 3t^2 - 36t + 44)$

8.5.20 (p. 262) $y = \frac{1}{2}e^{-t}(3\cos t + \sin t) + \frac{1}{2}$

$-u(t-2\pi)\left(e^{-(t-2\pi)}\left((\pi-1)\cos t + \frac{2\pi-1}{2}\sin t\right) + 1 - \frac{t}{2}\right)$

$-\frac{1}{2}u(t-3\pi)\left(e^{-(t-3\pi)}(3\pi\cos t + (3\pi+1)\sin t) + t\right)$

8.5.21 (p. 262) $y = \frac{t^2}{2} + \sum_{m=1}^{\infty} u(t-m)\frac{(t-m)^2}{2}$

8.5.22 (p. 263) **(a)** $y = \begin{cases} 2m + 1 - \cos t, & 2m\pi \leqslant t < (2m+1)\pi \quad (m = 0, 1, \dots) \\ 2m, & (2m-1)\pi \leqslant t < 2m\pi \quad (m = 1, 2, \dots) \end{cases}$

(b) $y = (m+1)(t - \sin t - m\pi\cos t),\ 2m\pi \leqslant t < (2m+2)\pi \quad (m = 0, 1, \dots)$

(c) $y = (-1)^m - (2m+1)\cos t,\ m\pi \leqslant t < (m+1)\pi \quad (m = 0, 1, \dots)$

(d) $y = \frac{e^{m+1} - 1}{2(e-1)}(e^{t-m} + e^{-t}) - m - 1, \quad m \leqslant t < m + 1\ (m = 0, 1 \dots)$

(e) $y = \left(m + 1 - \left(\frac{e^{2(m+1)\pi} - 1}{e^{2\pi} - 1}\right)e^{-t}\right)\sin t\ 2m\pi \leqslant t < 2(m+1)\pi \quad (m = 0, 1, \dots)$

(f) $y = \frac{m+1}{2} - e^{t-m}\frac{e^{m+1} - 1}{e - 1} + \frac{1}{2}e^{2(t-m)}\frac{e^{2m+2} - 1}{e^2 - 1},\ m \leqslant t < m + 1 \quad (m = 0, 1, \dots)$

Section 8.6 Answers, pp. 274–278

8.6.1 (p. 274) **(a)** $\frac{1}{2}\int_0^t \tau\sin 2(t-\tau)\,d\tau$ **(b)** $\int_0^t e^{-2\tau}\cos 3(t-\tau)\,d\tau$
(c) $\frac{1}{2}\int_0^t \sin 2\tau\cos 3(t-\tau)\,d\tau$ **or** $\frac{1}{3}\int_0^t \sin 3\tau\cos 2(t-\tau)\,d\tau$ **(d)** $\int_0^t \cos\tau\sin(t-\tau)\,d\tau$
(e) $\int_0^t e^{a\tau}d\tau$ **(f)** $e^{-t}\int_0^t \sin(t-\tau)\,d\tau$ **(g)** $e^{-2t}\int_0^t \tau e^{\tau}\sin(t-\tau)\,d\tau$
(h) $\frac{e^{-2t}}{2}\int_0^t \tau^2(t-\tau)e^{3\tau}\,d\tau$ **(i)** $\int_0^t (t-\tau)e^{\tau}\cos\tau\,d\tau$ **(j)** $\int_0^t e^{-3\tau}\cos\tau\cos 2(t-\tau)\,d\tau$
(k) $\frac{1}{4!5!}\int_0^t \tau^4(t-\tau)^5e^{3\tau}\,d\tau$ **(l)** $\frac{1}{4}\int_0^t \tau^2e^{\tau}\sin 2(t-\tau)\,d\tau$
(m) $\frac{1}{2}\int_0^t \tau(t-\tau)^2e^{2(t-\tau)}\,d\tau$ **(n)** $\frac{1}{5!6!}\int_0^t (t-\tau)^5e^{2(t-\tau)}\tau^6\,d\tau$

8.6.2 (p. 274) **(a)** $\frac{as}{(s^2+a^2)(s^2+b^2)}$ **(b)** $\frac{a}{(s-1)(s^2+a^2)}$ **(c)** $\frac{as}{(s^2-a^2)^2}$ **(d)** $\frac{2\omega s(s^2-\omega^2)}{(s^2+\omega^2)^4}$
(e) $\frac{(s-1)\omega}{((s-1)^2+\omega^2)^2}$ **(f)** $\frac{2}{(s-2)^3(s-1)^2}$ **(g)** $\frac{s+1}{(s+2)^2\left[(s+1)^2+\omega^2\right]}$
(h) $\frac{1}{(s-3)\left((s-1)^2-1\right)}$ **(i)** $\frac{2}{(s-2)^2(s^2+4)}$ **(j)** $\frac{6}{s^4(s-1)}$ **(k)** $\frac{3\cdot 6!}{s^7\left[(s+1)^2+9\right]}$
(l) $\frac{12}{s^7}$ **(m)** $\frac{2\cdot 7!}{s^8\left[(s+1)^2+4\right]}$ **(n)** $\frac{48}{s^5(s^2+4)}$

8.6.3 (p. 275) **(a)** $y=\frac{2}{\sqrt{5}}\int_0^t f(t-\tau)e^{-3\tau/2}\sinh\frac{\sqrt{5}\tau}{2}\,d\tau$ **(b)** $y=\frac{1}{2}\int_0^t f(t-\tau)\sin 2\tau\,d\tau$
(c) $y=\int_0^t \tau e^{-\tau}f(t-\tau)\,d\tau$ **(d)** $y(t)=-\frac{1}{k}\sin kt+\cos kt+\frac{1}{k}\int_0^t f(t-\tau)\sin k\tau\,d\tau$
(e) $y=-2te^{-3t}+\int_0^t \tau e^{-3\tau}f(t-\tau)\,d\tau$ **(f)** $y=\frac{3}{2}\sinh 2t+\frac{1}{2}\int_0^t f(t-\tau)\sinh 2\tau\,d\tau$
(g) $y=e^{3t}+\int_0^t (e^{3\tau}-e^{2\tau})f(t-\tau)\,d\tau$ **(h)** $y=\frac{k_1}{\omega}\sin\omega t+k_0\cos\omega t+\frac{1}{\omega}\int_0^t f(t-\tau)\sin\omega\tau\,d\tau$

8.6.4 (p. 275) **(a)** $y=\sin t$**(b)** $y=te^{-t}$ **(c)** $y=1+2te^{t}$ **(d)** $y=t+\frac{t^2}{2}$
(e) $y=4+\frac{5}{2}t^2+\frac{1}{24}t^4$ **(f)** $y=1-t$

8.6.5 (p. 275) **(a)** $\frac{7!8!}{16!}t^{16}$ **(b)** $\frac{13!7!}{21!}t^{21}$ **(c)** $\frac{6!7!}{14!}t^{14}$ **(d)** $\frac{1}{2}(e^{-t}+\sin t-\cos t)$ **(e)** $\frac{1}{3}(\cos t-\cos 2t)$

Section 8.7 Answers, pp. 285–286

8.7.1 (p. 285) $y=\frac{1}{2}e^{2t}-4e^{-t}+\frac{11}{2}e^{-2t}+2u(t-1)(e^{-(t-1)}-e^{-2(t-1)})$
8.7.2 (p. 285) $y=2e^{-2t}+5e^{-t}+\frac{5}{3}u(t-1)(e^{(t-1)}-e^{-2(t-1)})$
8.7.3 (p. 285) $y=\frac{1}{6}e^{2t}-\frac{2}{3}e^{-t}-\frac{1}{2}e^{-2t}+\frac{5}{2}u(t-1)\sinh 2(t-1)$
8.7.4 (p. 285) $y=\frac{1}{8}(8\cos t-5\sin t-\sin 3t)-2u(t-\pi/2)\cos t$
8.7.5 (p. 285) $y=1-\cos 2t+\frac{1}{2}\sin 2t+\frac{1}{2}u(t-3\pi)\sin 2t$
8.7.6 (p. 285) $y=4e^{t}+3e^{-t}-8+2u(t-2)\sinh(t-2)$
8.7.7 (p. 285) $y=\frac{1}{2}e^{t}-\frac{7}{2}e^{-t}+2+3u(t-6)(1-e^{-(t-6)})$
8.7.8 (p. 285) $y=e^{2t}+7\cos 2t-\sin 2t-\frac{1}{2}u(t-\pi/2)\sin 2t$

8.7.9 (p. 285) $y = \frac{1}{2}(1+e^{-2t}) + u(t-1)(e^{-(t-1)} - e^{-2(t-1)})$

8.7.10 (p. 285) $y = \frac{1}{4}e^{t} + \frac{1}{4}e^{-t}(2t-5) + 2u(t-2)(t-2)e^{-(t-2)}$

8.7.11 (p. 285) $y = \frac{1}{6}(2\sin t + 5\sin 2t) - \frac{1}{2}u(t-\pi/2)\sin 2t$

8.7.12 (p. 285) $y = e^{-t}(\sin t - \cos t) - e^{-(t-\pi)}\sin t - 3u(t-2\pi)e^{-(t-2\pi)}\sin t$

8.7.13 (p. 285) $y = e^{-2t}\left(\cos 3t + \frac{4}{3}\sin 3t\right) - \frac{1}{3}u(t-\pi/6)e^{-2(t-\pi/6)}\cos 3t - \frac{2}{3}u(t-\pi/3)e^{-2(t-\pi/3)}\sin 3t$

8.7.14 (p. 285) $y = \frac{7}{10}e^{2t} - \frac{6}{5}e^{-t/2} - \frac{1}{2} + \frac{1}{5}u(t-2)(e^{2(t-2)} - e^{-(t-2)/2})$

8.7.15 (p. 285) $y = \frac{1}{17}(12\cos t + 20\sin t) + \frac{1}{34}e^{t/2}(10\cos t - 11\sin t) - u(t-\pi/2)e^{(2t-\pi)/4}\cos t$
$+u(t-\pi)e^{(t-\pi)/2}\sin t$

8.7.16 (p. 285) $y = \frac{1}{3}(\cos t - \cos 2t - 3\sin t) - 2u(t-\pi/2)\cos t + 3u(t-\pi)\sin t$

8.7.17 (p. 285) $y = e^{t} - e^{-t}(1+2t) - 5u(t-1)\sinh(t-1) + 3u(t-2)\sinh(t-2)$

8.7.18 (p. 285) $y = \frac{1}{4}(e^{t} - e^{-t}(1+6t)) - u(t-1)e^{-(t-1)} + 2u(t-2)e^{-(t-2)})$

8.7.19 (p. 285) $y = \frac{5}{3}\sin t - \frac{1}{3}\sin 2t + \frac{1}{3}u(t-\pi)(\sin 2t + 2\sin t) + u(t-2\pi)\sin t$

8.7.20 (p. 285) $y = \frac{3}{4}\cos 2t - \frac{1}{2}\sin 2t + \frac{1}{4} + \frac{1}{4}u(t-\pi/2)(1+\cos 2t) + \frac{1}{2}u(t-\pi)\sin 2t + \frac{3}{2}u(t-3\pi/2)\sin 2t$

8.7.21 (p. 285) $y = \cos t - \sin t$ **8.7.22 (p. 285)** $y = \frac{1}{4}(8e^{3t} - 12e^{-2t})$

8.7.23 (p. 285) $y = 5(e^{-2t} - e^{-t})$ **8.7.24 (p. 285)** $y = e^{-2t}(1+6t)$

8.7.25 (p. 286) $y = \frac{1}{4}e^{-t/2}(4-19t)$

8.7.29 (p. 286) $y = (-1)^k m\omega_1 R e^{-c\tau/2m}\delta(t-\tau)$ **if** $\omega_1\tau - \phi = (2k+1)\pi/2$**(**$k =$ **integer)**

8.7.30 (p. 286) **(a)** $y = \dfrac{(e^{m+1}-1)(e^{t-m} - e^{-t})}{2(e-1)}$, $m \leqslant t < m+1$, **(**$m = 0, 1, \dots$**)**

(b) $y = (m+1)\sin t$, $2m\pi \leqslant t < 2(m+1)\pi$, **(**$m = 0, 1, \dots$**)**

(c) $y = e^{2(t-m)}\dfrac{e^{2m+2}-1}{e^2-1} - e^{(t-m)}\dfrac{e^{m+1}-1}{e-1}$, $m \leqslant t < m+1$ **(**$m = 0, 1, \dots$**)**

(d) $y = \begin{cases} 0, & 2m\pi \leqslant t < (2m+1)\pi, \\ -\sin t, & (2m+1)\pi \leqslant t < (2m+2)\pi, \end{cases}$ **(**$m = 0, 1, \dots$**)**

Section 9.1 Answers, pp. ??–??

9.1.2 (p. ??) $y = 2x^2 - 3x^3 + \frac{1}{x}$ **9.1.3 (p. ??)** $y = 2e^x + 3e^{-x} - e^{2x} + e^{-3x}$ **9.1.4 (p. ??)** $y_i = \dfrac{(x-x_0)^{i-1}}{(i-1)!}$, $1 \leqslant i \leqslant n$

9.1.5 (p. ??) **(b)** $y_1 = -\frac{1}{2}x^3 + x^2 + \frac{1}{2x}$, $y_2 = \frac{1}{3}x^2 - \frac{1}{3x}$, $y_3 = \frac{1}{4}x^3 - \frac{1}{3}x^2 + \frac{1}{12x}$
(c) $y = k_0y_1 + k_1y_2 + k_2y_3$

9.1.7 (p. ??) $2e^{-x^2}$ **9.1.8 (p. ??)** $\sqrt{2}K\cos x$ **9.1.9 (p. ??)** **(a)** $W(x) = 2e^{3x}$ **(d)** $y = e^x(c_1 + c_2x + c_3x^2)$

9.1.10 (p. ??) **(a)** 2 **(b)** $-e^{3x}$**(c)** 4 **(d)** $4/x^2$ **(e)** 1 **(f)** $2x$ **(g)** $2/x^2$**(h)** $e^x(x^2-2x+2)$
(i) $-240/x^5$ **(j)** $6e^{2x}(2x-1)$**(l)** $-128x$

9.1.24 (p. ??) **(a)** $y''' = 0$ **(b)** $xy''' - y'' - xy' + y = 0$ **(c)** $(2x-3)y''' - 2y'' - (2x-5)y' = 0$
(d) $(x^2-2x+2)y''' - x^2y'' + 2xy' - 2y = 0$ **(e)** $x^3y''' + x^2y'' - 2xy' + 2y = 0$
(f) $(3x-1)y''' - (12x-1)y'' + 9(x+1)y' - 9y = 0$

(g) $x^4y^{(4)} + 5x^3y''' - 3x^2y'' - 6xy' + 6y = 0$
(h) $x^4y^{(4)} + 3x^2y''' - x^2y'' + 2xy' - 2y = 0$
(i) $(2x-1)y^{(4)} - 4xy''' + (5-2x)y'' + 4xy' - 4y = 0$
(j) $xy^{(4)} - y''' - 4xy'' + 4y' = 0$

Section 9.2 Answers, pp. ??–??

9.2.1 (p. ??) $y = e^x(c_1 + c_2x + c_3x^2)$ **9.2.2 (p. ??)** $y = c_1e^x + c_2e^{-x} + c_3\cos 3x + c_4\sin 3x$
9.2.3 (p. ??) $y = c_1e^x + c_2\cos 4x + c_3\sin 4x$ **9.2.4 (p. ??)** $y = c_1e^x + c_2e^{-x} + c_3e^{-3x/2}$
9.2.5 (p. ??) $y = c_1e^{-x} + e^{-2x}(c_1\cos x + c_2\sin x)$ **9.2.6 (p. ??)** $y = c_1e^x + e^{x/2}(c_2 + c_3x)$
9.2.7 (p. ??) $y = e^{-x/3}(c_1 + c_2x + c_3x^2)$ **9.2.8 (p. ??)** $y = c_1 + c_2x + c_3\cos x + c_4\sin x$
9.2.9 (p. ??) $y = c_1e^{2x} + c_2e^{-2x} + c_3\cos 2x + c_4\sin 2x$
9.2.10 (p. ??) $y = (c_1 + c_2x)\cos\sqrt{6}x + (c_3 + c_4x)\sin\sqrt{6}x$
9.2.11 (p. ??) $y = e^{3x/2}(c_1 + c_2x) + e^{-3x/2}(c_3 + c_4x)$
9.2.12 (p. ??) $y = c_1e^{-x/2} + c_2e^{-x/3} + c_3\cos x + c_4\sin x$
9.2.13 (p. ??) $y = c_1e^x + c_2e^{-2x} + c_3e^{-x/2} + c_4e^{-3x/2}$ **9.2.14 (p. ??)** $y = e^x(c_1 + c_2x + c_3\cos x + c_4\sin x)$
9.2.15 (p. ??) $y = \cos 2x - 2\sin 2x + e^{2x}$ **9.2.16 (p. ??)** $y = 2e^x + 3e^{-x} - 5e^{-3x}$
9.2.17 (p. ??) $y = 2e^x + 3xe^x - 4e^{-x}$
9.2.18 (p. ??) $y = 2e^{-x}\cos x - 3e^{-x}\sin x + 4e^{2x}$ **9.2.19 (p. ??)** $y = \frac{9}{5}e^{-5x/3} + e^x(1+2x)$
9.2.20 (p. ??) $y = e^{2x}(1 - 3x + 2x^2)$ **9.2.21 (p. ??)** $y = e^{3x}(2-x) + 4e^{-x/2}$
9.2.22 (p. ??) $y = e^{x/2}(1-2x) + 3e^{-x/2}$ **9.2.23 (p. ??)** $y = \frac{1}{8}(5e^{2x} + e^{-2x} + 10\cos 2x + 4\sin 2x)$
9.2.24 (p. ??) $y = -4e^x + e^{2x} - e^{4x} + 2e^{-x}$ **9.2.25 (p. ??)** $y = 2e^x - e^{-x}$
9.2.26 (p. ??) $y = e^{2x} + e^{-2x} + e^{-x}(3\cos x + \sin x)$ **9.2.27 (p. ??)** $y = 2e^{-x/2} + \cos 2x - \sin 2x$
9.2.28 (p. ??) **(a)** $\{e^x,\ xe^x,\ e^{2x}\}:\ 1$ **(b)** $\{\cos 2x,\ \sin 2x,\ e^{3x}\}:\ 26$
(c) $\{e^{-x}\cos x,\ e^{-x}\sin x,\ e^x\}:\ 5$ **(d)** $\{1,\ x,\ x^2,\ e^x\}\ 2e^x$
(e) $\{e^x,\ e^{-x},\ \cos x,\ \sin x\}8$ **(f)** $\{\cos x,\ \sin x,\ e^x\cos x,\ e^x\sin x\}:\ 5$
9.2.29 (p. ??) $\{e^{-3x}\cos 2x,\ e^{-3x}\sin 2x,\ e^{2x},\ xe^{2x},\ 1,\ x,\ x^2\}$
9.2.30 (p. ??) $\{e^x,\ xe^x,\ e^{x/2},\ xe^{x/2},\ x^2e^{x/2},\ \cos x,\ \sin x\}$
9.2.31 (p. ??) $\{\cos 3x,\ x\cos 3x,\ x^2\cos 3x,\ \sin 3x,\ x\sin 3x,\ x^2\sin 3x,\ 1,\ x\}$
9.2.32 (p. ??) $\{e^{2x},\ xe^{2x},\ x^2e^{2x},\ e^{-x},\ xe^{-x},\ 1\}$
9.2.33 (p. ??) $\{\cos x,\ \sin x,\ \cos 3x,\ x\cos 3x,\ \sin 3x,\ x\sin 3x,\ e^{2x}\}$
9.2.34 (p. ??) $\{e^{2x},\ xe^{2x},\ e^{-2x},\ xe^{-2x},\ \cos 2x,\ x\cos 2x,\ \sin 2x,\ x\sin 2x\}$
9.2.35 (p. ??) $\{e^{-x/2}\cos 2x,\ xe^{-x/2}\cos 2x,\ x^2e^{-x/2}\cos 2x,\ e^{-x/2}\sin 2x,\ xe^{-x/2}\sin 2x,$ $x^2e^{-x/2}\sin 2x\}$
9.2.36 (p. ??) $\{1,\ x,\ x^2,\ e^{2x},\ xe^{2x},\ \cos 2x,\ x\cos 2x,\ \sin 2x,\ x\sin 2x\}$
9.2.37 (p. ??) $\{\cos(x/2),\ x\cos(x/2),\ \sin(x/2),\ x\sin(x/2),\ \cos 2x/3\ x\cos(2x/3),$ $x^2\cos(2x/3),\ \sin(2x/3),\ x\sin(2x/3),\ x^2\sin(2x/3)\}$
9.2.38 (p. ??) $\{e^{-x},\ e^{3x},\ e^x\cos 2x,\ e^x\sin 2x\}$ **9.2.39 (p. ??)** **(b)** $e^{(a_1+a_2+\cdots+a_n)x}\prod_{1\leqslant i<j\leqslant n}(a_j - a_i)$

9.2.43 (p. ??) **(a)** $\left\{e^x,\ e^{-x/2}\cos\left(\frac{\sqrt{3}}{2}x\right),\ e^{-x/2}\sin\left(\frac{\sqrt{3}}{2}x\right)\right\}$ **(b)** $\left\{e^{-x},\ e^{x/2}\cos\left(\frac{\sqrt{3}}{2}x\right),\ e^{x/2}\sin\left(\frac{\sqrt{3}}{2}x\right)\right\}$
(c) $\{e^{2x}\cos 2x,\ e^{2x}\sin 2x,\ e^{-2x}\cos 2x,\ e^{-2x}\sin 2x\}$
(d) $\left\{e^x,\ e^{-x},\ e^{x/2}\cos\left(\frac{\sqrt{3}}{2}x\right),\ e^{x/2}\sin\left(\frac{\sqrt{3}}{2}x\right),\ e^{-x/2}\cos\left(\frac{\sqrt{3}}{2}x\right),\ e^{-x/2}\sin\left(\frac{\sqrt{3}}{2}x\right)\right\}$
(e) $\{\cos 2x,\ \sin 2x,\ e^{-\sqrt{3}x}\cos x,\ e^{-\sqrt{3}x}\sin x,\ e^{\sqrt{3}x}\cos x,\ e^{\sqrt{3}x}\sin x\}$
(f) $\left\{1,\ e^{2x},\ e^{3x/2}\cos\left(\frac{\sqrt{3}}{2}x\right),\ e^{3x/2}\sin\left(\frac{\sqrt{3}}{2}x\right),\ e^{x/2}\cos\left(\frac{\sqrt{3}}{2}x\right),\ e^{x/2}\sin\left(\frac{\sqrt{3}}{2}x\right)\right\}$

(g) $\left\{e^{-x},\ e^{x/2}\cos\left(\frac{\sqrt{3}}{2}x\right),\ e^{x/2}\sin\left(\frac{\sqrt{3}}{2}x\right),\ e^{-x/2}\cos\left(\frac{\sqrt{3}}{2}x\right),\ e^{-x/2}\sin\left(\frac{\sqrt{3}}{2}x\right)\right\}$

9.2.45 (p. ??) $y = c_1x^{r_1} + c_2x^{r_2} + c_3x^{r_3}$ (r_1, r_2, r_3 **distinct);** $y = c_1x^{r_1} + (c_2 + c_3 \ln x)x^{r_2}$ (r_1, r_2 **distinct**); $y = [c_1 + c_2 \ln x + c_3(\ln x)^2]x^{r_1}$**;** $y = c_1x^{r_1} + x^{\lambda}[c_2 \cos(\omega \ln x) + c_3 \sin(\omega \ln x)]$

Section 9.3 Answers, pp. ??–??

9.3.1 (p. ??) $y_p = e^{-x}(2 + x - x^2)$ **9.3.2 (p. ??)** $y_p = -\frac{e^{-3x}}{4}(3 - x + x^2)$ **9.3.3 (p. ??)** $y_p = e^x(1 + x - x^2)$

9.3.4 (p. ??) $y_p = e^{-2x}(1 - 5x + x^2)$. **9.3.5 (p. ??)** $y_p = -\frac{xe^x}{2}(1 - x + x^2 - x^3)$

9.3.6 (p. ??) $y_p = x^2e^x(1 + x)$ **9.3.7 (p. ??)** $y_p = \frac{xe^{-2x}}{2}(2 + x)$ **9.3.8 (p. ??)** $y_p = \frac{x^2e^x}{2}(2 + x)$

9.3.9 (p. ??) $y_p = \frac{x^2e^{2x}}{2}(1 + 2x)$ **9.3.10 (p. ??)** $y_p = x^2e^{3x}(2 + x - x^2)$ **9.3.11 (p. ??)** $y_p = x^2e^{4x}(2 + x)$

9.3.12 (p. ??) $y_p = \frac{x^3e^{x/2}}{48}(1 + x)$ **9.3.13 (p. ??)** $y_p = e^{-x}(1 - 2x + x^2)$ **9.3.14 (p. ??)** $y_p = e^{2x}(1 - x)$

9.3.15 (p. ??) $y_p = e^{-2x}(1 + x + x^2 - x^3)$ **9.3.16 (p. ??)** $y_p = \frac{e^x}{3}(1 - x)$ **9.3.17 (p. ??)** $y_p = e^x(1 + x)^2$

9.3.18 (p. ??) $y_p = xe^x(1 + x^3)$ **9.3.19 (p. ??)** $y_p = xe^x(2 + x)$ **9.3.20 (p. ??)** $y_p = \frac{xe^{2x}}{6}(1 - x^2)$

9.3.21 (p. ??) $y_p = 4xe^{-x/2}(1 + x)$ **9.3.22 (p. ??)** $y_p = \frac{xe^x}{6}(1 + x^2)$

9.3.23 (p. ??) $y_p = \frac{x^2e^{2x}}{6}(1 + x + x^2)$ **9.3.24 (p. ??)** $y_p = \frac{x^2e^{2x}}{6}(3 + x + x^2)$ **9.3.25 (p. ??)** $y_p = \frac{x^3e^x}{48}(2 + x)$

9.3.26 (p. ??) $y_p = \frac{x^3e^x}{6}(1 + x)$ **9.3.27 (p. ??)** $y_p = -\frac{x^3e^{-x}}{6}(1 - x + x^2)$ **9.3.28 (p. ??)** $y_p = \frac{x^3e^{2x}}{12}(2 + x - x^2)$

9.3.29 (p. ??) $y_p = e^{-x}[(1 + x)\cos x + (2 - x)\sin x]$ **9.3.30 (p. ??)** $y_p = e^{-x}[(1 - x)\cos 2x + (1 + x)\sin 2x]$

9.3.31 (p. ??) $y_p = e^{2x}[(1 + x - x^2)\cos x + (1 + 2x)\sin x]$

9.3.32 (p. ??) $y_p = \frac{e^x}{2}[(1+x)\cos 2x + (1-x+x^2)\sin 2x]$ **9.3.33 (p. ??)** $y_p = \frac{x}{13}(8\cos 2x + 14\sin 2x)$

9.3.34 (p. ??) $y_p = xe^x[(1+x)\cos x + (3+x)\sin x]$ **9.3.35 (p. ??)** $y_p = \frac{xe^{2x}}{2}[(3-x)\cos 2x + \sin 2x]$

9.3.36 (p. ??) $y_p = -\frac{xe^{3x}}{12}(x\cos 3x + \sin 3x)$ **9.3.37 (p. ??)** $y_p = -\frac{e^x}{10}(\cos x + 7\sin x)$

9.3.38 (p. ??) $y_p = \frac{e^x}{12}(\cos 2x - \sin 2x)$ **9.3.39 (p. ??)** $y_p = xe^{2x}\cos 2x$

9.3.40 (p. ??) $y_p = -\frac{e^{-x}}{2}[(1 + x)\cos x + (2 - x)\sin x]$ **9.3.41 (p. ??)** $y_p = \frac{xe^{-x}}{10}(\cos x + 2\sin x)$

9.3.42 (p. ??) $y_p = \frac{xe^x}{40}(3\cos 2x - \sin 2x)$ **9.3.43 (p. ??)** $y_p = \frac{xe^{-2x}}{8}[(1-x)\cos 3x + (1+x)\sin 3x]$

9.3.44 (p. ??) $y_p = -\frac{xe^x}{4}(1 + x)\sin 2x$ **9.3.45 (p. ??)** $y_p = \frac{x^2e^{-x}}{4}(\cos x - 2\sin x)$

9.3.46 (p. ??) $y_p = -\frac{x^2e^{2x}}{32}(\cos 2x - \sin 2x)$ **9.3.47 (p. ??)** $y_p = \frac{x^2e^{2x}}{8}(1 + x)\sin x$

9.3.48 (p. ??) $y_p = 2x^2e^x + xe^{2x} - \cos x$ **9.3.49 (p. ??)** $y_p = e^{2x} + xe^x + 2x\cos x$
9.3.50 (p. ??) $y_p = 2x + x^2 + 2xe^x - 3xe^{-x} + 4e^{3x}$
9.3.51 (p. ??) $y_p = xe^x(\cos 2x - 2\sin 2x) + 2xe^{2x} + 1$ **9.3.52 (p. ??)** $y_p = x^2e^{-2x}(1+2x) - \cos 2x + \sin 2x$
9.3.53 (p. ??) $y_p = 2x^2(1+x)e^{-x} + x\cos x - 2\sin x$ **9.3.54 (p. ??)** $y_p = 2xe^x + xe^{-x} + \cos x$
9.3.55 (p. ??) $y_p = \frac{xe^x}{6}(\cos x + \sin 2x)$ **9.3.56 (p. ??)** $y_p = \frac{x^2}{54}[(2+2x)e^x + 3e^{-2x}]$
9.3.57 (p. ??) $y_p = \frac{x}{8}\sinh x \sin x$ **9.3.58 (p. ??)** $y_p = x^3(1+x)e^{-x} + xe^{-2x}$
9.3.59 (p. ??) $y_p = xe^x(2x^2 + \cos x + \sin x)$ **9.3.60 (p. ??)** $y = e^{2x}(1+x) + c_1e^{-x} + e^x(c_2 + c_3x)$
9.3.61 (p. ??) $y = e^{3x}\left(1 - x - \frac{x^2}{2}\right) + c_1e^x + e^{-x}(c_2\cos x + c_3\sin x)$
9.3.62 (p. ??) $y = xe^{2x}(1+x)^2 + c_1e^x + c_2e^{2x} + c_3e^{3x}$
9.3.63 (p. ??) $y = x^2e^{-x}(1-x)^2 + c_1 + e^{-x}(c_2 + c_3x)$
9.3.64 (p. ??) $y = \frac{x^3e^x}{24}(4+x) + e^x(c_1 + c_2x + c_3x^2)$
9.3.65 (p. ??) $y = \frac{x^2e^{-x}}{16}(1 + 2x - x^2) + e^x(c_1 + c_2x) + e^{-x}(c_3 + c_4x)$
9.3.66 (p. ??) $y = e^{-2x}\left[\left(1 + \frac{x}{2}\right)\cos x + \left(\frac{3}{2} - 2x\right)\sin x\right] + c_1e^x + c_2e^{-x} + c_3e^{-2x}$
9.3.67 (p. ??) $y = -xe^x\sin 2x + c_1 + c_2e^x + e^x(c_3\cos x + c_4\sin x)$
9.3.68 (p. ??) $y = -\frac{x^2e^x}{16}(1+x)\cos 2x + e^x\left[(c_1 + c_2x)\cos 2x + (c_3 + c_4x)\sin 2x\right]$
9.3.69 (p. ??) $y = (x^2+2)e^x - e^{-2x} + e^{3x}$ **9.3.70 (p. ??)** $y = e^{-x}(1 + x + x^2) + (1-x)e^x$
9.3.71 (p. ??) $y = \left(\frac{x^2}{12} + 16\right)xe^{-x/2} - e^x$ **9.3.72 (p. ??)** $y = (2-x)(x^2+1)e^{-x} + \cos x - \sin x$
9.3.73 (p. ??) $y = (2-x)\cos x - (1-7x)\sin x + e^{-2x}$ **9.3.74 (p. ??)** $2 + e^x\left[(1+x)\cos x - \sin x - 1\right]$

Section 9.4 Answers, pp. ??–??

9.4.1 (p. ??) $y_p = 2x^3$ **9.4.2 (p. ??)** $y_p = \frac{8}{105}x^{7/2}e^{-x^2}$ **9.4.3 (p. ??)** $y_p = x\ln|x|$
9.4.4 (p. ??) $y_p = -\frac{2(x^2+2)}{x}$ **9.4.5 (p. ??)** $y_p = -\frac{xe^{-3x}}{64}$ **9.4.6 (p. ??)** $y_p = -\frac{2x^2}{3}$
9.4.7 (p. ??) $y_p = -\frac{e^{-x}(x+1)}{x}$ **9.4.8 (p. ??)** $y_p = 2x^2\ln|x|$ **9.4.9 (p. ??)** $y_p = x^2 + 1$
9.4.10 (p. ??) $y_p = \frac{2x^2+6}{3}$ **9.4.11 (p. ??)** $y_p = \frac{x^2\ln|x|}{3}$ **9.4.12 (p. ??)** $y_p = -x^2 - 2$
9.4.13 (p. ??) $\frac{1}{4}x^3\ln|x| - \frac{25}{48}x^3$ **9.4.14 (p. ??)** $y_p = \frac{x^{5/2}}{4}$ **9.4.15 (p. ??)** $y_p = \frac{x(12 - x^2)}{6}$
9.4.16 (p. ??) $y_p = \frac{x^4\ln|x|}{6}$ **9.4.17 (p. ??)** $y_p = \frac{x^3e^x}{2}$ **9.4.18 (p. ??)** $y_p = x^2\ln|x|$
9.4.19 (p. ??) $y_p = \frac{xe^x}{2}$ **9.4.20 (p. ??)** $y_p = \frac{3xe^x}{2}$ **9.4.21 (p. ??)** $y_p = -x^3$
9.4.22 (p. ??) $y = -x(\ln x)^2 + 3x + x^3 - 2x\ln x$ **9.4.23 (p. ??)** $y = \frac{x^3}{2}(\ln|x|)^2 + x^2 - x^3 + 2x^3\ln|x|$
9.4.24 (p. ??) $y = -\frac{1}{2}(3x+1)xe^x - 3e^x - e^{2x} + 4xe^{-x}$ **9.4.25 (p. ??)** $y = \frac{3}{2}x^4(\ln x)^2 + 3x - x^4 + 2x^4\ln x$
9.4.26 (p. ??) $y = -\frac{x^4+12}{6} + 3x - x^2 + 2e^x$ **9.4.27 (p. ??)** $y = \left(\frac{x^2}{3} - \frac{x}{2}\right)\ln|x| + 4x - 2x^2$
9.4.28 (p. ??) $y = -\frac{xe^x(1+3x)}{2} + \frac{x+1}{2} - \frac{e^x}{4} + \frac{e^{3x}}{2}$ **9.4.29 (p. ??)** $y = -8x + 2x^2 - 2x^3 + 2e^x -$

e^{-x}

9.4.30 (p. ??) $y = 3x^2 \ln x - 7x^2$ **9.4.31 (p. ??)** $y = \frac{3(4x^2+9)}{2} + \frac{x}{2} - \frac{e^x}{2} + \frac{e^{-x}}{2} + \frac{e^{2x}}{4}$

9.4.32 (p. ??) $y = x\ln x + x - \sqrt{x} + \frac{1}{x} + \frac{1}{\sqrt{x}}$. **9.4.33 (p. ??)** $y = x^3 \ln|x| + x - 2x^3 + \frac{1}{x} - \frac{1}{x^2}$

9.4.35 (p. ??) $y_p = \int_{x_0}^{x} \frac{e^{(x-t)} - 3e^{-(x-t)} + 2e^{-2(x-t)}}{6} F(t)\,dt$ **9.4.36 (p. ??)** $y_p = \int_{x_0}^{x} \frac{(x-t)^2(2x+t)}{6xt^3} F(t)\,dt$

9.4.37 (p. ??) $y_p = \int_{x_0}^{x} \frac{xe^{(x-t)} - x^2 + x(t-1)}{t^4} F(t)\,dt$ **9.4.38 (p. ??)** $y_p = \int_{x_0}^{x} \frac{x^2 - t(t-2) - 2te^{(x-t)}}{2x(t-1)^2} F(t)\,dt$

9.4.39 (p. ??) $y_p = \int_{x_0}^{x} \frac{e^{2(x-t)} - 2e^{(x-t)} + 2e^{-(x-t)} - e^{-2(x-t)}}{12} F(t)\,dt$

9.4.40 (p. ??) $y_p = \int_{x_0}^{x} \frac{(x-t)^3}{6x} F(t)\,dt$

9.4.41 (p. ??) $y_p = \int_{x_0}^{x} \frac{(x+t)(x-t)^3}{12x^2t^3} F(t)\,dt$

9.4.42 (p. ??) $y_p = \int_{x_0}^{x} \frac{e^{2(x-t)}(1+2t) + e^{-2(x-t)}(1-2t) - 4x^2 + 4t^2 - 2}{32t^2} F(t)\,dt$

Section 10.1 Answers, pp. 297–298

10.1.1 (p. 297) $\begin{aligned} Q_1' &= 2 - \frac{1}{10}Q_1 + \frac{1}{25}Q_2 \\ Q_2' &= 6 + \frac{3}{50}Q_1 - \frac{1}{20}Q_2. \end{aligned}$ **10.1.2 (p. 297)** $\begin{aligned} Q_1' &= 12 - \frac{5}{100+2t}Q_1 + \frac{1}{100+3t}Q_2 \\ Q_2' &= 5 + \frac{1}{50+t}Q_1 - \frac{4}{100+3t}Q_2. \end{aligned}$

10.1.3 (p. 297) $m_1y_1'' = -(c_1+c_2)y_1' + c_2y_2' - (k_1+k_2)y_1 + k_2y_2 + F_1$

$m_2y_2'' = (c_2-c_3)y_1' - (c_2+c_3)y_2' + c_3y_3' + (k_2-k_3)y_1 - (k_2+k_3)y_2 + k_3y_3 + F_2$

$m_3y_3'' = c_3y_1' + c_3y_2' - c_3y_3' + k_3y_1 + k_3y_2 - k_3y_3 + F_3$

10.1.4 (p. 297) $x'' = -\frac{\alpha}{m}x' + \frac{gR^2x}{(x^2+y^2+z^2)^{3/2}}$ $y'' = -\frac{\alpha}{m}y' + \frac{gR^2y}{(x^2+y^2+z^2)^{3/2}}$

$z'' = -\frac{\alpha}{m}z' + \frac{gR^2z}{(x^2+y^2+z^2)^{3/2}}$

10.1.5 (p. 297) **(a)** $\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ x_3' &= f(t,x_1,y_1,y_2) \\ y_1' &= y_2 \\ y_2' &= g(t,y_1,y_2) \end{aligned}$ **(b)** $\begin{aligned} u_1' &= f(t,u_1,v_1,v_2,w_2) \\ ;v_1' &= v_2 \\ v_2' &= g(t,u_1,v_1,v_2,w_1) \\ w_1' &= w_2 \\ w_2' &= h(t,u_1,v_1,v_2,w_1,w_2) \end{aligned}$

(c) $\begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ y_3' &= f(t,y_1,y_2,y_3) \end{aligned}$ **(d)** $\begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ y_3' &= y_4 \\ y_4' &= f(t,y_1) \end{aligned}$

(e) $\begin{aligned} x_1' &= x_2 \\ x_2' &= f(t,x_1,y_1) \\ y_1' &= y_2 \\ y_2' &= g(t,x_1,y_1) \end{aligned}$

10.1.6 (p. 298)
$$\begin{aligned} x' &= x_1 \\ y' &= y_1 \\ z' &= z_1 \end{aligned} \qquad \begin{aligned} x_1' &= -\frac{gR^2x}{(x^2+y^2+z^2)^{3/2}} \\ y_1' &= -\frac{gR^2y}{(x^2+y^2+z^2)^{3/2}} \\ z_1' &= -\frac{gR^2z}{(x^2+y^2+z^2)^{3/2}} \end{aligned}$$

Section 10.2 Answers, pp. 302–305

10.2.1 (p. 302) **(a)** $\mathbf{y}' = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}\mathbf{y}$ **(b)** $\mathbf{y}' = \begin{bmatrix} -2 & -2 \\ -5 & 1 \end{bmatrix}\mathbf{y}$
(c) $\mathbf{y}' = \begin{bmatrix} -4 & -10 \\ 3 & 7 \end{bmatrix}\mathbf{y}$ **(d)** $\mathbf{y}' = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\mathbf{y}$

10.2.2 (p. 302) **(a)** $\mathbf{y}' = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 1 & 6 \\ 0 & 0 & -2 \end{bmatrix}\mathbf{y}$ **(b)** $\mathbf{y}' = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix}\mathbf{y}$
(c) $\mathbf{y}' = \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}\mathbf{y}$ **(d)** $\mathbf{y}' = \begin{bmatrix} 3 & -1 & -1 \\ -2 & 3 & 2 \\ 4 & -1 & -2 \end{bmatrix}\mathbf{y}$

10.2.3 (p. 303) **(a)** $\mathbf{y}' = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}\mathbf{y},\ \mathbf{y}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ **(b)** $\mathbf{y}' = \begin{bmatrix} 5 & 3 \\ -1 & 1 \end{bmatrix}\mathbf{y},\ \mathbf{y}(0) = \begin{bmatrix} 9 \\ -5 \end{bmatrix}$

10.2.4 (p. 303) **(a)** $\mathbf{y}' = \begin{bmatrix} 6 & 4 & 4 \\ -7 & -2 & -1 \\ 7 & 4 & 3 \end{bmatrix}\mathbf{y},\ \mathbf{y}(0) = \begin{bmatrix} 3 \\ -6 \\ 4 \end{bmatrix}$
(b) $\mathbf{y}' = \begin{bmatrix} 8 & 7 & 7 \\ -5 & -6 & -9 \\ 5 & 7 & 10 \end{bmatrix}\mathbf{y},\ \mathbf{y}(0) = \begin{bmatrix} 2 \\ -4 \\ 3 \end{bmatrix}$

10.2.5 (p. 303) **(a)** $\mathbf{y}' = \begin{bmatrix} -3 & 2 \\ -5 & 3 \end{bmatrix} + \begin{bmatrix} 3-2t \\ 6-3t \end{bmatrix}$ **(b)** $\mathbf{y}' = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix}\mathbf{y} + \begin{bmatrix} -5e^t \\ e^t \end{bmatrix}$

10.2.10 (p. 305) **(a)** $\frac{d}{dt}Y^2 = Y'Y + YY'$
(b) $\frac{d}{dt}Y^n = Y'Y^{n-1} + YY'Y^{n-2} + Y^2Y'Y^{n-3} + \cdots + Y^{n-1}Y' = \sum_{r=0}^{n-1} Y^rY'Y^{n-r-1}$

10.2.13 (p. 305) $B = (P' + PA)P^{-1}$.

Section 10.3 Answers, pp. 310–315

10.3.2 (p. 310) $\mathbf{y}' = \begin{bmatrix} 0 & 1 \\ -\frac{P_2(x)}{P_0(x)} & -\frac{P_1(x)}{P_0(x)} \end{bmatrix}\mathbf{y}$ **10.3.3 (p. 311)** $\mathbf{y}' = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -\frac{P_n(x)}{P_0(x)} & -\frac{P_{n-1}(x)}{P_0(x)} & \cdots & -\frac{P_1(x)}{P_0(x)} \end{bmatrix}\mathbf{y}$

10.3.7 (p. 313) **(b)** $\mathbf{y} = \begin{bmatrix} 3e^{6t} - 6e^{-2t} \\ 3e^{6t} + 6e^{-2t} \end{bmatrix}$ **(c)** $\mathbf{y} = \frac{1}{2}\begin{bmatrix} e^{6t} + e^{-2t} & e^{6t} - e^{-2t} \\ e^{6t} - e^{-2t} & e^{6t} + e^{-2t} \end{bmatrix}\mathbf{k}$

10.3.8 (p. 313) **(b)** $\mathbf{y} = \begin{bmatrix} 6e^{-4t} + 4e^{3t} \\ 6e^{-4t} - 10e^{3t} \end{bmatrix}$ **(c)** $\mathbf{y} = \frac{1}{7}\begin{bmatrix} 5e^{-4t} + 2e^{3t} & 2e^{-4t} - 2e^{3t} \\ 5e^{-4t} - 5e^{3t} & 2e^{-4t} + 5e^{3t} \end{bmatrix}\mathbf{k}$

10.3.9 (p. 313) **(b)** $\mathbf{y} = \begin{bmatrix} -15e^{2t} - 4e^t \\ 9e^{2t} + 2e^t \end{bmatrix}$ **(c)** $\mathbf{y} = \begin{bmatrix} -5e^{2t} + 6e^t & -10e^{2t} + 10e^t \\ 3e^{2t} - 3e^t & 6e^{2t} - 5e^t \end{bmatrix}\mathbf{k}$

10.3.10 (p. 313) **(b)** $\mathbf{y} = \begin{bmatrix} 5e^{3t} - 3e^{t} \\ 5e^{3t} + 3e^{t} \end{bmatrix}$ **(c)** $\mathbf{y} = \frac{1}{2}\begin{bmatrix} e^{3t} + e^{t} & e^{3t} - e^{t} \\ e^{3t} - e^{t} & e^{3t} + e^{t} \end{bmatrix}\mathbf{k}$

10.3.11 (p. 313) **(b)** $\mathbf{y} = \begin{bmatrix} e^{2t} - 2e^{3t} + 3e^{-t} \\ 2e^{3t} - 9e^{-t} \\ e^{2t} - 2e^{3t} + 21e^{-t} \end{bmatrix}$ **(c)** $\mathbf{y} = \frac{1}{6}\begin{bmatrix} 4e^{2t} + 3e^{3t} - e^{-t} & 6e^{2t} - 6e^{3t} & 2e^{2t} - 3e^{3t} + e^{-t} \\ -3e^{3t} + 3e^{-t} & 6e^{3t} & 3e^{3t} - 3e^{-t} \\ 4e^{2t} + 3e^{3t} - 7e^{-t} & 6e^{2t} - 6e^{3t} & 2e^{2t} - 3e^{3t} + 7e^{-t} \end{bmatrix}$

10.3.12 (p. 314) **(b)** $\mathbf{y} = \frac{1}{3}\begin{bmatrix} -e^{-2t} + e^{4t} \\ -10e^{-2t} + e^{4t} \\ 11e^{-2t} + e^{4t} \end{bmatrix}$ **(c)** $\mathbf{y} = \frac{1}{3}\begin{bmatrix} 2e^{-2t} + e^{4t} & -e^{-2t} + e^{4t} & -e^{-2t} + e^{4t} \\ -e^{-2t} + e^{4t} & 2e^{-2t} + e^{4t} & -e^{-2t} + e^{4t} \\ -e^{-2t} + e^{4t} & -e^{-2t} + e^{4t} & 2e^{-2t} + e^{4t} \end{bmatrix}\mathbf{k}$

10.3.13 (p. 314) **(b)** $\mathbf{y} = \begin{bmatrix} 3e^{t} + 3e^{-t} - e^{-2t} \\ 3e^{t} + 2e^{-2t} \\ -e^{-2t} \end{bmatrix}$ **(c)** $\mathbf{y} = \begin{bmatrix} e^{-t} & e^{t} - e^{-t} & 2e^{t} - 3e^{-t} + e^{-2t} \\ 0 & e^{t} & 2e^{t} - 2e^{-2t} \\ 0 & 0 & e^{-2t} \end{bmatrix}\mathbf{k}$

10.3.14 (p. 314) YZ^{-1} **and** ZY^{-1}

Section 10.4 Answers, pp. 325–328

10.4.1 (p. 325) $\mathbf{y} = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{3t} + c_2\begin{bmatrix} 1 \\ -1 \end{bmatrix}e^{-t}$ **10.4.2 (p. 325)** $\mathbf{y} = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{-t/2} + c_2\begin{bmatrix} -1 \\ 1 \end{bmatrix}e^{-2t}$

10.4.3 (p. 325) $\mathbf{y} = c_1\begin{bmatrix} -3 \\ 1 \end{bmatrix}e^{-t} + c_2\begin{bmatrix} -1 \\ 2 \end{bmatrix}e^{-2t}$ **10.4.4 (p. 325)** $\mathbf{y} = c_1\begin{bmatrix} 2 \\ 1 \end{bmatrix}e^{-3t} + c_2\begin{bmatrix} -2 \\ 1 \end{bmatrix}e^{t}$

10.4.5 (p. 325) $\mathbf{y} = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{-2t} + c_1\begin{bmatrix} -4 \\ 1 \end{bmatrix}e^{3t}$ **10.4.6 (p. 325)** $\mathbf{y} = c_1\begin{bmatrix} 3 \\ 2 \end{bmatrix}e^{2t} + c_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{t}$

10.4.7 (p. 325) $\mathbf{y} = c_1\begin{bmatrix} -3 \\ 1 \end{bmatrix}e^{-5t} + c_2\begin{bmatrix} -1 \\ 1 \end{bmatrix}e^{-3t}$

10.4.8 (p. 325) $\mathbf{y} = c_1\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}e^{-3t} + c_2\begin{bmatrix} -1 \\ -4 \\ 1 \end{bmatrix}e^{-t} + c_3\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}e^{2t}$

10.4.9 (p. 326) $\mathbf{y} = c_1\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}e^{-16t} + c_2\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}e^{2t} + c_3\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}e^{2t}$

10.4.10 (p. 326) $\mathbf{y} = c_1\begin{bmatrix} -2 \\ -4 \\ 3 \end{bmatrix}e^{t} + c_2\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}e^{-2t} + c_3\begin{bmatrix} -7 \\ -5 \\ 4 \end{bmatrix}e^{2t}$

10.4.11 (p. 326) $\mathbf{y} = c_1\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}e^{-2t} + c_2\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}e^{-3t} + c_3\begin{bmatrix} -2 \\ -6 \\ 3 \end{bmatrix}e^{-5t}$

10.4.12 (p. 326) $\mathbf{y} = c_1\begin{bmatrix} 11 \\ 7 \\ 1 \end{bmatrix}e^{3t} + c_2\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}e^{-2t} + c_3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}e^{-t}$

10.4.13 (p. 326) $\mathbf{y} = c_1\begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix}e^{-4t} + c_2\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}e^{6t} + c_3\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}e^{4t}$

10.4.14 (p. 326) $\mathbf{y} = c_1\begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}e^{-5t} + c_2\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}e^{5t} + c_3\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}e^{5t}$

10.4.15 (p. 326) $\mathbf{y} = c_1\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + c_2\begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}e^{6t} + c_3\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}e^{6t}$

10.4.16 (p. 326) $\mathbf{y} = -\begin{bmatrix} 2 \\ 6 \end{bmatrix}e^{5t} + \begin{bmatrix} 4 \\ 2 \end{bmatrix}e^{-5t}$ **10.4.17 (p. 326)** $\mathbf{y} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}e^{t/2} + \begin{bmatrix} -2 \\ 1 \end{bmatrix}e^{t}$

10.4.18 (p. 326) $\mathbf{y} = \begin{bmatrix} 7 \\ 7 \end{bmatrix}e^{9t} - \begin{bmatrix} 2 \\ 4 \end{bmatrix}e^{-3t}$ **10.4.19 (p. 326)** $\mathbf{y} = \begin{bmatrix} 3 \\ 9 \end{bmatrix}e^{5t} - \begin{bmatrix} 4 \\ 2 \end{bmatrix}e^{-5t}$

10.4.20 (p. 326) $\mathbf{y} = \begin{bmatrix} 5 \\ 5 \\ 0 \end{bmatrix}e^{t/2} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}e^{t/2} + \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}e^{-t/2}$ **10.4.21 (p. 326)** $\mathbf{y} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}e^{t} + \begin{bmatrix} -2 \\ -2 \\ 2 \end{bmatrix}e^{-t}$

10.4.22 (p. 326) $\mathbf{y} = \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix}e^{t} - \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix}e^{-2t} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}e^{3t}$

10.4.23 (p. 326) $\mathbf{y} = -\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}e^{t} + \begin{bmatrix} 4 \\ 2 \\ 4 \end{bmatrix}e^{-t} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}e^{2t}$

10.4.24 (p. 326) $\mathbf{y} = \begin{bmatrix} -2 \\ -2 \\ 2 \end{bmatrix}e^{2t} - \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}e^{-2t} + \begin{bmatrix} 4 \\ 12 \\ 4 \end{bmatrix}e^{4t}$

10.4.25 (p. 327) $\mathbf{y} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}e^{-6t} + \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix}e^{2t} + \begin{bmatrix} 7 \\ -7 \\ -7 \end{bmatrix}e^{4t}$

10.4.26 (p. 327) $\mathbf{y} = \begin{bmatrix} 1 \\ 4 \\ 4 \end{bmatrix}e^{-t} + \begin{bmatrix} 6 \\ 6 \\ -2 \end{bmatrix}e^{2t}$ **10.4.27 (p. 327)** $\mathbf{y} = \begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ -9 \\ 6 \end{bmatrix}e^{4t} + \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}e^{2t}$

10.4.29 (p. 327) Half lines of $L_1 : y_2 = y_1$ and $L_2 : y_2 = -y_1$ are trajectories other trajectories

are asymptotically tangent to L_1 as $t \to -\infty$ and asymptotically tangent to L_2 as $t \to \infty$.

10.4.30 (p. 327) Half lines of $L_1 : y_2 = -2y_1$ and $L_2 : y_2 = -y_1/3$ are trajectories

other trajectories are asymptotically parallel to L_1 as $t \to -\infty$ and asymptotically tangent to L_2 as $t \to \infty$.

10.4.31 (p. 327) Half lines of $L_1 : y_2 = y_1/3$ and $L_2 : y_2 = -y_1$ are trajectories other trajectories

are asymptotically tangent to L_1 as $t \to -\infty$ and asymptotically parallel to L_2 as $t \to \infty$.

10.4.32 (p. 327) Half lines of $L_1 : y_2 = y_1/2$ and $L_2 : y_2 = -y_1$ are trajectories other trajectories

are asymptotically tangent to L_1 as $t \to -\infty$ and asymptotically tangent to L_2 as $t \to \infty$.

10.4.33 (p. 327) Half lines of $L_1 : y_2 = -y_1/4$ and $L_2 : y_2 = -y_1$ are trajectories other trajectories

are asymptotically tangent to L_1 as $t \to -\infty$ and asymptotically parallel to L_2 as $t \to \infty$.

10.4.34 (p. 327) Half lines of $L_1 : y_2 = -y_1$ and $L_2 : y_2 = 3y_1$ are trajectories other trajectories are asymptotically parallel to L_1 as $t \to -\infty$ and asymptotically tangent to L_2 as $t \to \infty$.

10.4.36 (p. 328) Points on $L_2 : y_2 = y_1$ are trajectories of constant solutions. The trajectories of nonconstant solutions are half-lines on either side of L_1, parallel to $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, traversed toward L_1.

10.4.37 (p. 328) Points on $L_1 : y_2 = -y_1/3$ are trajectories of constant solutions. The trajectories of nonconstant solutions are half-lines on either side of L_1, parallel to $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$, traversed away from L_1.

10.4.38 (p. 328) Points on $L_1 : y_2 = y_1/3$ are trajectories of constant solutions. The trajectories of nonconstant solutions are half-lines on either side of L_1, parallel to $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\begin{bmatrix} \\ 1 \end{bmatrix} -1$, traversed away from L_1.

10.4.39 (p. 328) Points on $L_1 : y_2 = y_1/2$ are trajectories of constant solutions. The trajectories of nonconstant solutions are half-lines on either side of L_1, parallel to $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, L_1.

10.4.40 (p. 328) Points on $L_2 : y_2 = -y_1$ are trajectories of constant solutions. The trajectories of nonconstant solutions are half-lines on either side of L_2, parallel to $\begin{bmatrix} -4 \\ 1 \end{bmatrix}$, traversed toward L_1.

10.4.41 (p. 328) Points on $L_1 : y_2 = 3y_1$ are trajectories of constant solutions. The trajectories of nonconstant solutions are half-lines on either side of L_1, parallel to $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, traversed away from L_1.

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10.5.1 (p. 342) $\mathbf{y} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{5t} + c_2 \left(\begin{bmatrix} -1 \\ 0 \end{bmatrix} e^{5t} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} te^{5t} \right)$.

10.5.2 (p. 342) $\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + c_2 \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-t} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{-t} \right)$

10.5.3 (p. 342) $\mathbf{y} = c_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{-9t} + c_2 \left(\begin{bmatrix} -1 \\ 0 \end{bmatrix} e^{-9t} + \begin{bmatrix} -2 \\ 1 \end{bmatrix} te^{-9t} \right)$

10.5.4 (p. 342) $\mathbf{y} = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{2t} + c_2 \left(\begin{bmatrix} -1 \\ 0 \end{bmatrix} e^{2t} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} te^{2t} \right)$

10.5.5 (p. 342) $c_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_2 \left(\begin{bmatrix} -1 \\ 0 \end{bmatrix} \frac{e^{-2t}}{3} + \begin{bmatrix} -2 \\ 1 \end{bmatrix} te^{-2t} \right)$

10.5.6 (p. 342) $\mathbf{y} = c_1 \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{-4t} + c_2 \left(\begin{bmatrix} -1 \\ 0 \end{bmatrix} \frac{e^{-4t}}{2} + \begin{bmatrix} 3 \\ 2 \end{bmatrix} te^{-4t} \right)$

10.5.7 (p. 342) $\mathbf{y} = c_1 \begin{bmatrix} 4 \\ 3 \end{bmatrix} e^{-t} + c_2 \left(\begin{bmatrix} -1 \\ 0 \end{bmatrix} \frac{e^{-t}}{3} + \begin{bmatrix} 4 \\ 3 \end{bmatrix} te^{-t} \right)$

10.5.8 (p. 342) $\mathbf{y}=c_1\begin{bmatrix}-1\\-1\\2\end{bmatrix}+c_2\begin{bmatrix}1\\1\\2\end{bmatrix}e^{4t}+c_3\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\frac{e^{4t}}{2}+\begin{bmatrix}1\\1\\2\end{bmatrix}te^{4t}\right)$

10.5.9 (p. 342) $\mathbf{y}=c_1\begin{bmatrix}-1\\1\\1\end{bmatrix}e^{t}+c_2\begin{bmatrix}1\\-1\\1\end{bmatrix}e^{-t}+c_3\left(\begin{bmatrix}0\\3\\0\end{bmatrix}e^{-t}+\begin{bmatrix}1\\-1\\1\end{bmatrix}te^{-t}\right).$

10.5.10 (p. 342) $\mathbf{y}=c_1\begin{bmatrix}0\\1\\1\end{bmatrix}e^{2t}+c_2\begin{bmatrix}1\\0\\1\end{bmatrix}e^{-2t}+c_3\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\frac{e^{-2t}}{2}+\begin{bmatrix}1\\0\\1\end{bmatrix}te^{-2t}\right)$

10.5.11 (p. 342) $\mathbf{y}=c_1\begin{bmatrix}-2\\-3\\1\end{bmatrix}e^{2t}+c_2\begin{bmatrix}0\\-1\\1\end{bmatrix}e^{4t}+c_3\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\frac{e^{4t}}{2}+\begin{bmatrix}0\\-1\\1\end{bmatrix}te^{4t}\right)$

10.5.12 (p. 342) $\mathbf{y}=c_1\begin{bmatrix}-1\\-1\\1\end{bmatrix}e^{-2t}+c_2\begin{bmatrix}1\\1\\1\end{bmatrix}e^{4t}+c_3\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\frac{e^{4t}}{2}+\begin{bmatrix}1\\1\\1\end{bmatrix}te^{4t}\right).$

10.5.13 (p. 342) $\mathbf{y}=\begin{bmatrix}6\\2\end{bmatrix}e^{-7t}-\begin{bmatrix}8\\4\end{bmatrix}te^{-7t}$ **10.5.14 (p. 342)** $\mathbf{y}=\begin{bmatrix}5\\8\end{bmatrix}e^{3t}-\begin{bmatrix}12\\16\end{bmatrix}te^{3t}$

10.5.15 (p. 342) $\mathbf{y}=\begin{bmatrix}2\\3\end{bmatrix}e^{-5t}-\begin{bmatrix}8\\4\end{bmatrix}te^{-5t}$ **10.5.16 (p. 342)** $\mathbf{y}=\begin{bmatrix}3\\1\end{bmatrix}e^{5t}-\begin{bmatrix}12\\6\end{bmatrix}te^{5t}$

10.5.17 (p. 342) $\mathbf{y}=\begin{bmatrix}0\\2\end{bmatrix}e^{-4t}+\begin{bmatrix}6\\6\end{bmatrix}te^{-4t}$

10.5.18 (p. 342) $\mathbf{y}=\begin{bmatrix}4\\8\\-6\end{bmatrix}e^{t}+\begin{bmatrix}2\\-3\\-1\end{bmatrix}e^{-2t}+\begin{bmatrix}-1\\1\\0\end{bmatrix}te^{-2t}$

10.5.19 (p. 343) $\mathbf{y}=\begin{bmatrix}3\\3\\6\end{bmatrix}e^{2t}-\begin{bmatrix}9\\5\\6\end{bmatrix}+\begin{bmatrix}2\\2\\0\end{bmatrix}t$

10.5.20 (p. 343) $\mathbf{y}=-\begin{bmatrix}2\\0\\2\end{bmatrix}e^{-3t}+\begin{bmatrix}-4\\9\\1\end{bmatrix}e^{t}-\begin{bmatrix}0\\4\\4\end{bmatrix}te^{t}$

10.5.21 (p. 343) $\mathbf{y}=\begin{bmatrix}-2\\2\\2\end{bmatrix}e^{4t}+\begin{bmatrix}0\\-1\\1\end{bmatrix}e^{2t}+\begin{bmatrix}3\\-3\\3\end{bmatrix}te^{2t}$

10.5.22 (p. 343) $\mathbf{y}=-\begin{bmatrix}1\\1\\0\end{bmatrix}e^{-4t}+\begin{bmatrix}-3\\2\\-3\end{bmatrix}e^{8t}+\begin{bmatrix}8\\0\\-8\end{bmatrix}te^{8t}$

10.5.23 (p. 343) $\mathbf{y}=\begin{bmatrix}3\\6\\3\end{bmatrix}e^{4t}-\begin{bmatrix}3\\4\\1\end{bmatrix}+\begin{bmatrix}8\\4\\4\end{bmatrix}t$

10.5.24 (p. 343) $\mathbf{y}=c_1\begin{bmatrix}0\\1\\1\end{bmatrix}e^{6t}+c_2\left(\begin{bmatrix}-1\\1\\0\end{bmatrix}\frac{e^{6t}}{4}+\begin{bmatrix}0\\1\\1\end{bmatrix}te^{6t}\right)$

$$+c_3\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\frac{e^{6t}}{8}+\begin{bmatrix}-1\\1\\0\end{bmatrix}\frac{te^{6t}}{4}+\begin{bmatrix}0\\1\\1\end{bmatrix}\frac{t^2e^{6t}}{2}\right)$$

10.5.25 (p. 343) $\mathbf{y} = c_1 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} e^{3t} + c_2 \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \frac{e^{3t}}{2} + \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} te^{3t} \right)$

$+c_3 \left(\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \frac{e^{3t}}{36} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \frac{te^{3t}}{2} + \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \frac{t^2e^{3t}}{2} \right)$

10.5.26 (p. 343) $\mathbf{y} = c_1 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^{-2t} + \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} te^{-2t} \right)$

$+c_3 \left(\begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix} \frac{e^{-2t}}{4} + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} te^{-2t} + \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \frac{t^2e^{-2t}}{2} \right)$

10.5.27 (p. 343) $\mathbf{y} = c_1 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} e^{2t} + c_2 \left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \frac{e^{2t}}{2} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} te^{2t} \right)$

$+c_3 \left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \frac{e^{2t}}{8} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \frac{te^{2t}}{2} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \frac{t^2e^{2t}}{2} \right)$

10.5.28 (p. 343) $\mathbf{y} = c_1 \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} e^{-6t} + c_2 \left(-\begin{bmatrix} 6 \\ 1 \\ 0 \end{bmatrix} \frac{e^{-6t}}{6} + \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} te^{-6t} \right)$

$+c_3 \left(-\begin{bmatrix} 12 \\ 1 \\ 0 \end{bmatrix} \frac{e^{-6t}}{36} - \begin{bmatrix} 6 \\ 1 \\ 0 \end{bmatrix} \frac{te^{-6t}}{6} + \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} \frac{t^2e^{-6t}}{2} \right).$

10.5.29 (p. 343) $\mathbf{y} = c_1 \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} 6 \\ 1 \\ 0 \end{bmatrix} e^{-3t} + c_3 \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{-3t} + \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} te^{-3t} \right)$

10.5.30 (p. 343) $\mathbf{y} = c_1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{-3t} + c_3 \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{-3t} + \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} te^{-3t} \right)$

10.5.31 (p. 343) $\mathbf{y} = c_1 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} -3 \\ 2 \\ 0 \end{bmatrix} e^{-t} + c_3 \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \frac{e^{-t}}{2} + \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} te^{-t} \right)$

10.5.32 (p. 343) $\mathbf{y} = c_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{-2t} + c_3 \left(\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} e^{-2t} + \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} te^{-2t} \right)$

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10.6.1 (p. 354) $\mathbf{y} = c_1e^{2t} \begin{bmatrix} 3\cos t + \sin t \\ 5\cos t \end{bmatrix} + c_2e^{2t} \begin{bmatrix} 3\sin t - \cos t \\ 5\sin t \end{bmatrix}.$

10.6.2 (p. 354) $\mathbf{y} = c_1e^{-t}\begin{bmatrix} 5\cos 2t + \sin 2t \\ 13\cos 2t \end{bmatrix} + c_2e^{-t}\begin{bmatrix} 5\sin 2t - \cos 2t \\ 13\sin 2t \end{bmatrix}$.

10.6.3 (p. 354) $\mathbf{y} = c_1e^{3t}\begin{bmatrix} \cos 2t + \sin 2t \\ 2\cos 2t \end{bmatrix} + c_2e^{3t}\begin{bmatrix} \sin 2t - \cos 2t \\ 2\sin 2t \end{bmatrix}$.

10.6.4 (p. 354) $\mathbf{y} = c_1e^{2t}\begin{bmatrix} \cos 3t - \sin 3t \\ \cos 3t \end{bmatrix} + c_2e^{2t}\begin{bmatrix} \sin 3t + \cos 3t \\ \sin 3t \end{bmatrix}$.

10.6.5 (p. 354) $\mathbf{y} = c_1\begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}e^{-2t} + c_2e^{4t}\begin{bmatrix} \cos 2t - \sin 2t \\ \cos 2t + \sin 2t \\ 2\cos 2t \end{bmatrix} + c_3e^{4t}\begin{bmatrix} \sin 2t + \cos 2t \\ \sin 2t - \cos 2t \\ 2\sin 2t \end{bmatrix}$.

10.6.6 (p. 354) $\mathbf{y} = c_1\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}e^{-t} + c_2e^{-2t}\begin{bmatrix} \cos 2t - \sin 2t \\ -\cos 2t - \sin 2t \\ 2\cos 2t \end{bmatrix} + c_3e^{-2t}\begin{bmatrix} \sin 2t + \cos 2t \\ -\sin 2t + \cos 2t \\ 2\sin 2t \end{bmatrix}$

10.6.7 (p. 354) $\mathbf{y} = c_1\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}e^{2t} + c_2e^{t}\begin{bmatrix} -\sin t \\ \sin t \\ \cos t \end{bmatrix} + c_3e^{t}\begin{bmatrix} \cos t \\ -\cos t \\ \sin t \end{bmatrix}$

10.6.8 (p. 354) $\mathbf{y} = c_1\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}e^{t} + c_2e^{-t}\begin{bmatrix} -\sin 2t - \cos 2t \\ 2\cos 2t \\ 2\cos 2t \end{bmatrix} + c_3e^{-t}\begin{bmatrix} \cos 2t - \sin 2t \\ 2\sin 2t \\ 2\sin 2t \end{bmatrix}$

10.6.9 (p. 354) $\mathbf{y} = c_1e^{3t}\begin{bmatrix} \cos 6t - 3\sin 6t \\ 5\cos 6t \end{bmatrix} + c_2e^{3t}\begin{bmatrix} \sin 6t + 3\cos 6t \\ 5\sin 6t \end{bmatrix}$

10.6.10 (p. 354) $\mathbf{y} = c_1e^{2t}\begin{bmatrix} \cos t - 3\sin t \\ 2\cos t \end{bmatrix} + c_2e^{2t}\begin{bmatrix} \sin t + 3\cos t \\ 2\sin t \end{bmatrix}$

10.6.11 (p. 354) $\mathbf{y} = c_1e^{2t}\begin{bmatrix} 3\sin 3t - \cos 3t \\ 5\cos 3t \end{bmatrix} + c_2e^{2t}\begin{bmatrix} -3\cos 3t - \sin 3t \\ 5\sin 3t \end{bmatrix}$

10.6.12 (p. 354) $\mathbf{y} = c_1e^{2t}\begin{bmatrix} \sin 4t - 8\cos 4t \\ 5\cos 4t \end{bmatrix} + c_2e^{2t}\begin{bmatrix} -\cos 4t - 8\sin 4t \\ 5\sin 4t \end{bmatrix}$

10.6.13 (p. 354) $\mathbf{y} = c_1\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}e^{-2t} + c_2e^{t}\begin{bmatrix} \sin t \\ -\cos t \\ \cos t \end{bmatrix} + c_3e^{t}\begin{bmatrix} -\cos t \\ -\sin t \\ \sin t \end{bmatrix}$

10.6.14 (p. 354) $\mathbf{y} = c_1\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}e^{-2t} + c_2e^{2t}\begin{bmatrix} -\cos 3t - \sin 3t \\ -\sin 3t \\ \cos 3t \end{bmatrix} + c_3e^{2t}\begin{bmatrix} -\sin 3t + \cos 3t \\ \cos 3t \\ \sin 3t \end{bmatrix}$

10.6.15 (p. 354) $\mathbf{y} = c_1\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}e^{3t} + c_2e^{6t}\begin{bmatrix} -\sin 3t \\ \sin 3t \\ \cos 3t \end{bmatrix} + c_3e^{6t}\begin{bmatrix} \cos 3t \\ -\cos 3t \\ \sin 3t \end{bmatrix}$

10.6.16 (p. 354) $\mathbf{y} = c_1\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}e^{t} + c_2e^{t}\begin{bmatrix} 2\cos t - 2\sin t \\ \cos t - \sin t \\ 2\cos t \end{bmatrix} + c_3e^{t}\begin{bmatrix} 2\sin t + 2\cos t \\ \cos t + \sin t \\ 2\sin t \end{bmatrix}$

10.6.17 (p. 354) $\mathbf{y} = e^{t}\begin{bmatrix} 5\cos 3t + \sin 3t \\ 2\cos 3t + 3\sin 3t \end{bmatrix}$ **10.6.18 (p. 355)** $\mathbf{y} = e^{4t}\begin{bmatrix} 5\cos 6t + 5\sin 6t \\ \cos 6t - 3\sin 6t \end{bmatrix}$

10.6.19 (p. 355) $\mathbf{y} = e^{t}\begin{bmatrix} 17\cos 3t - \sin 3t \\ 7\cos 3t + 3\sin 3t \end{bmatrix}$ **10.6.20 (p. 355)** $\mathbf{y} = e^{t/2}\begin{bmatrix} \cos(t/2) + \sin(t/2) \\ -\cos(t/2) + 2\sin(t/2) \end{bmatrix}$

10.6.21 (p. 355) $\mathbf{y} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}e^{t} + e^{4t}\begin{bmatrix} 3\cos t + \sin t \\ \cos t - 3\sin t \\ 4\cos t - 2\sin t \end{bmatrix}$

10.6.22 (p. 355) $\mathbf{y} = \begin{bmatrix} 4 \\ 4 \\ 2 \end{bmatrix} e^{8t} + e^{2t} \begin{bmatrix} 4\cos 2t + 8\sin 2t \\ -6\sin 2t + 2\cos 2t \\ 3\cos 2t + \sin 2t \end{bmatrix}$

10.6.23 (p. 355) $\mathbf{y} = \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix} e^{-4t} + e^{4t} \begin{bmatrix} 15\cos 6t + 10\sin 6t \\ 14\cos 6t - 8\sin 6t \\ 7\cos 6t - 4\sin 6t \end{bmatrix}$

10.6.24 (p. 355) $\mathbf{y} = \begin{bmatrix} 6 \\ -3 \\ 3 \end{bmatrix} e^{8t} + \begin{bmatrix} 10\cos 4t - 4\sin 4t \\ 17\cos 4t - \sin 4t \\ 3\cos 4t - 7\sin 4t \end{bmatrix}$

10.6.29 (p. 356) $\mathbf{U} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \mathbf{V} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

10.6.30 (p. 356) $\mathbf{U} \approx \begin{bmatrix} .5257 \\ .8507 \end{bmatrix}, \mathbf{V} \approx \begin{bmatrix} -.8507 \\ .5257 \end{bmatrix}$

10.6.31 (p. 356) $\mathbf{U} \approx \begin{bmatrix} .8507 \\ .5257 \end{bmatrix}$, $\mathbf{V} \approx \begin{bmatrix} -.5257 \\ .8507 \end{bmatrix}$

10.6.32 (p. 356) $\mathbf{U} \approx \begin{bmatrix} -.9732 \\ .2298 \end{bmatrix}, \mathbf{V} \approx \begin{bmatrix} .2298 \\ .9732 \end{bmatrix}$

10.6.33 (p. 356) $\mathbf{U} \approx \begin{bmatrix} .5257 \\ .8507 \end{bmatrix}, \mathbf{V} \approx \begin{bmatrix} -.8507 \\ .5257 \end{bmatrix}$

10.6.34 (p. 356) $\mathbf{U} \approx \begin{bmatrix} -.5257 \\ .8507 \end{bmatrix}, \mathbf{V} \approx \begin{bmatrix} .8507 \\ .5257 \end{bmatrix}$

10.6.35 (p. 356) $\mathbf{U} \approx \begin{bmatrix} -.8817 \\ .4719 \end{bmatrix}, \mathbf{V} \approx \begin{bmatrix} .4719 \\ .8817 \end{bmatrix}$

10.6.36 (p. 356) $\mathbf{U} \approx \begin{bmatrix} .8817 \\ .4719 \end{bmatrix}, \mathbf{V} \approx \begin{bmatrix} -.4719 \\ .8817 \end{bmatrix}$

10.6.37 (p. 356) $\mathbf{U} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \mathbf{V} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ **10.6.38 (p. 356)** $\mathbf{U} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \mathbf{V} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

10.6.39 (p. 356) $\mathbf{U} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{V} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ **10.6.40 (p. 356)** $\mathbf{U} \approx \begin{bmatrix} .5257 \\ .8507 \end{bmatrix}, \mathbf{V} \approx \begin{bmatrix} -.8507 \\ .5257 \end{bmatrix}$

Section 10.7 Answers, pp. 365–367

10.7.1 (p. 365) $\begin{bmatrix} 5e^{4t} + e^{-3t}(2+8t) \\ -e^{4t} - e^{-3t}(1-4t) \end{bmatrix}$ **10.7.2 (p. 365)** $\begin{bmatrix} 13e^{3t} + 3e^{-3t} \\ -e^{3t} - 11e^{-3t} \end{bmatrix}$ **10.7.3 (p. 365)** $\frac{1}{9} \begin{bmatrix} 7-6t \\ -11+3t \end{bmatrix}$

10.7.4 (p. 365) $\begin{bmatrix} 5-3e^{t} \\ -6+5e^{t} \end{bmatrix}$

10.7.5 (p. 365) $\begin{bmatrix} e^{-5t}(3+6t) + e^{-3t}(3-2t) \\ -e^{-5t}(3+2t) - e^{-3t}(1-2t) \end{bmatrix}$ **10.7.6 (p. 365)** $\begin{bmatrix} t \\ 0 \end{bmatrix}$ **10.7.7 (p. 365)** $-\frac{1}{6} \begin{bmatrix} 2-6t \\ 7+6t \\ 1-12t \end{bmatrix}$

10.7.8 (p. 365) $-\frac{1}{6} \begin{bmatrix} 3e^{t}+4 \\ 6e^{t}-4 \\ 10 \end{bmatrix}$

10.7.9 (p. 365) $\frac{1}{18} \begin{bmatrix} e^{t}(1+12t) - e^{-5t}(1+6t) \\ -2e^{t}(1-6t) - e^{-5t}(1-12t) \\ e^{t}(1+12t) - e^{-5t}(1+6t) \end{bmatrix}$ **10.7.10 (p. 365)** $\frac{1}{3} \begin{bmatrix} 2e^{t} \\ e^{t} \\ 2e^{t} \end{bmatrix}$ **10.7.11 (p. 365)**

$\begin{bmatrix} t\sin t \\ 0 \end{bmatrix}$ **10.7.12 (p. 365)** $-\begin{bmatrix} t^2 \\ 2t \end{bmatrix}$

10.7.13 (p. 365) $(t-1)\,(\ln|t-1|+t)\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ **10.7.14 (p. 365)** $\dfrac{1}{9}\begin{bmatrix} 5e^{2t}-e^{-3t} \\ e^{3t}-5e^{-2t} \end{bmatrix}$ **10.7.15 (p. 365)** $\dfrac{1}{4t}\begin{bmatrix} 2t^3\ln|t|+t^3(t+2) \\ 2\ln|t|+3t-2 \end{bmatrix}$

10.7.16 (p. 365) $\dfrac{1}{2}\begin{bmatrix} te^{-t}(t+2)+(t^3-2) \\ te^{t}(t-2)+(t^3+2) \end{bmatrix}$ **10.7.17 (p. 365)** $-\begin{bmatrix} t \\ t \\ t \end{bmatrix}$ **10.7.18 (p. 366)** $\dfrac{1}{4}\begin{bmatrix} -3e^t \\ 1 \\ e^{-t} \end{bmatrix}$

10.7.19 (p. 366) $\begin{bmatrix} 2t^2+t \\ t \\ -t \end{bmatrix}$ **10.7.20 (p. 366)** $\dfrac{e^t}{4t}\begin{bmatrix} 2t+1 \\ 2t-1 \\ 2t+1 \end{bmatrix}$

10.7.22 (p. 366) **(a)** $\mathbf{y}' = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -P_n(t)/P_0(t) & -P_{n-1}/P_0(t) & \cdots & -P_1(t)/P_0(t) \end{bmatrix}\mathbf{y} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ F(t)/P_0(t) \end{bmatrix}.$

(b) $\begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{bmatrix}$

A BRIEF TABLE OF INTEGRALS

$$\int u^{\alpha}\, du = \frac{u^{\alpha+1}}{\alpha+1} + c, \quad \alpha \neq -1$$

$$\int \frac{du}{u} = \ln|u| + c$$

$$\int \cos u\, du = \sin u + c$$

$$\int \sin u\, du = -\cos u + c$$

$$\int \tan u\, du = -\ln|\cos u| + c$$

$$\int \cot u\, du = \ln|\sin u| + c$$

$$\int \sec^2 u\, du = \tan u + c$$

$$\int \csc^2 u\, du = -\cot u + c$$

$$\int \sec u\, du = \ln|\sec u + \tan u| + c$$

$$\int \cos^2 u\, du = \frac{u}{2} + \frac{1}{4}\sin 2u + c$$

$$\int \sin^2 u\, du = \frac{u}{2} - \frac{1}{4}\sin 2u + c$$

$$\int \frac{du}{1+u^2}\, du = \tan^{-1} u + c$$

$$\int \frac{du}{\sqrt{1-u^2}}\, du = \sin^{-1} u + c$$

$$\int \frac{1}{u^2-1}\, du = \frac{1}{2}\ln\left|\frac{u-1}{u+1}\right| + c$$

$$\int \cosh u\, du = \sinh u + c$$

$$\int \sinh u\, du = \cosh u + c$$

$$\int u\, dv = uv - \int v\, du$$

$$\int u\cos u\, du = u\sin u + \cos u + c$$

$$\int u\sin u\, du = -u\cos u + \sin u + c$$

$$\int ue^{u}\, du = ue^{u} - e^{u} + c$$

$$\int e^{\lambda u} \cos \omega u \, du = \frac{e^{\lambda u}(\lambda \cos \omega u + \omega \sin \omega u)}{\lambda^2 + \omega^2} + c$$

$$\int e^{\lambda u} \sin \omega u \, du = \frac{e^{\lambda u}(\lambda \sin \omega u - \omega \cos \omega u)}{\lambda^2 + \omega^2} + c$$

$$\int \ln |u| \, du = u \ln |u| - u + c$$

$$\int u \ln |u| \, du = \frac{u^2 \ln |u|}{2} - \frac{u^2}{4} + c$$

$$\int \cos \omega_1 u \cos \omega_2 u \, du = \frac{\sin(\omega_1 + \omega_2)u}{2(\omega_1 + \omega_2)} + \frac{\sin(\omega_1 - \omega_2)u}{2(\omega_1 - \omega_2)} + c \quad (\omega_1 \neq \pm\omega_2)$$

$$\int \sin \omega_1 u \sin \omega_2 u \, du = -\frac{\sin(\omega_1 + \omega_2)u}{2(\omega_1 + \omega_2)} + \frac{\sin(\omega_1 - \omega_2)u}{2(\omega_1 - \omega_2)} + c \quad (\omega_1 \neq \pm\omega_2)$$

$$\int \sin \omega_1 u \cos \omega_2 u \, du = -\frac{\cos(\omega_1 + \omega_2)u}{2(\omega_1 + \omega_2)} - \frac{\cos(\omega_1 - \omega_2)u}{2(\omega_1 - \omega_2)} + c \quad (\omega_1 \neq \pm\omega_2)$$

APPENDIX B

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