1. Sketch the region bounded by the graphs of $y = \tan x$, $y = 2 \sin x$ over $\left[ \frac{-\pi}{3}, \frac{\pi}{3} \right]$, and find the exact area of the region.

We will find the $x$-coordinates of the points of intersections by solving $\tan x = 2 \sin x$. This will give $\sin x = 2 \sin x \cos x$ from which, we get $x = 0$ or $x = \pm \frac{\pi}{3}$. Because of symmetry:

\[
A = 2 \int_{0}^{\frac{\pi}{3}} [2 \sin x - \tan x] \, dx
\]
\[
= 2 \left[ -2 \cos x - \ln |\sec x| \right]_{0}^{\frac{\pi}{3}} = 2 - 2 \ln 2.
\]

2. Find the exact volume of the solid that results when the area of the region enclosed by $y = 1 + \sec x$ and $y = 3$ is revolved about the line $y = 1$. Sketch the region.

We will find the $x$-coordinates of the points of intersections by solving $1 + \sec x = 3$. This will give $\cos x = \frac{1}{2}$ from which, we get $x = \pm \frac{\pi}{3}$. Because of symmetry:

\[
V = 2\pi \int_{0}^{\frac{\pi}{3}} [2^2 - \sec^2 x] \, dx = 2\pi \int_{0}^{\frac{\pi}{3}} [4 - \sec^2 x] \, dx
\]
\[
= 2\pi (4x - \tan x) \bigg|_{0}^{\frac{\pi}{3}} = 2\pi \left( \frac{4\pi}{3} - \sqrt{3} \right).
\]
3. Evaluate the integral \( \int_{\sqrt{\pi}/2}^{\sqrt{\pi}} x^3 \cos x^2 \, dx \).

We let \( z = x^2 \), then \( dz = 2x \, dx \). Therefore

\[
I = \frac{1}{2} \int_{\pi/2}^{\pi} z \cos z \, dz = \frac{1}{2} \left[ z \sin z + \cos z \right]_{\pi/2}^{\pi} = -\frac{1}{2} - \frac{\pi}{4},
\]

in (\( \ast \)), we used formula \#(21).

4. Use the cylindrical shell method to find the exact volume of the solid that results when the area of the region enclosed by \( y = x^3 \), \( y = 0 \), and \( x = 1 \) is revolved about the line \( y = 1 \). Sketch the region.

\[
V = 2\pi \int_0^1 (1 - y)(1 - y^{1/3}) \, dy = 2\pi \left[ (1 - y - y^{1/3} + y^{4/3}) \right]_0^1 = 5\pi/14.
\]

\[\text{Figure 3. Graph for Problem 4}\]

5. Find the average value \( f_{av} \) of \( f(x) = \cos^4 x \sin x \) over \([0,\pi]\). Sketch the graph of \( f \) and a rectangle whose area is the same as the area under the graph of \( f \).

\[
f_{av} = \frac{1}{\pi} \int_0^\pi \cos^4 x \sin x \, dx = \frac{1}{5\pi} \cos^5 x \bigg|_0^\pi = \frac{2}{5\pi}.
\]

\[\text{Figure 4. Graph for Problem 5}\]

6. Set up and evaluate the integral that gives the volume of the solid formed by revolving the region bounded by the graphs of \( y = xe^{-x}, \, y = 0 \) and \( x = 2 \) about the y-axis.
We will use the shell method:

\[ 2\pi \int_0^2 x^2 e^{-x} \, dx \]

We will use integration–by–parts in (\(\ast\)) and (\(\ast\ast\)) with:

\[ u = x^2, \quad dv = e^{-x} \, dx, \]
\[ du = 2x \, dx, \quad v = -e^{-x}. \]

and

\[ u = x, \quad dv = e^{-x} \, dx, \]
\[ du = dx, \quad v = -e^{-x}. \]

\[ 2\pi \int_0^2 x^2 e^{-x} \, dx = 2\pi \left[ -x^2 e^{-x} \bigg|_0^2 + 2 \int_0^2 xe^{-x} \, dx \right] \]

\[ = 2\pi \left[ -4e^{-2} + 2 \left( -xe^{-x} \bigg|_0^2 + \int_0^2 e^{-x} \, dx \right) \right] \]

\[ = 2\pi \left[ -4e^{-2} - 4e^{-2} - 2e^{-2} + 2 \right] = 2\pi \left[ 2 - 10e^{-2} \right]. \]

One could have used the formula (34) from the Handout.

7. Evaluate \( \int_0^1 \frac{x^3}{\sqrt{4 + x^2}} \, dx \) using an appropriate trigonometric substitution.

We let \( x = 2 \tan \theta \) so that \( dx = 2 \sec^2 \theta \, d\theta \). We get:

\[ I = \int_0^{\tan^{-1}(1/2)} \frac{8 \tan^3 \theta}{2 \sec \theta} \cdot 2 \sec^2 \theta \, d\theta \]

\[ = 8 \int_0^{\tan^{-1}(1/2)} \tan^3 \theta \sec \theta \, d\theta = \int_0^{\tan^{-1}(1/2)} (\sec^2 \theta - 1) \sec \theta \tan \theta \, d\theta \]

\[ = 8 \int_1^{\sqrt{5}/2} (z^2 - 1) \, dz = 8 \left( \frac{1}{3} z^3 - z \right) \bigg|_1^{\sqrt{5}/2} = \frac{16}{5} - \frac{7}{3} \sqrt{5}. \]
we could have used integration by parts with

\[ u = x^2, \quad dv = \frac{x}{\sqrt{x^2 + 4}} \, dx, \]
\[ du = 2x \, dx, \quad v = \sqrt{x^2 + 4}. \]

\[ I = x^2 \sqrt{x^2 + 4} \bigg|_{0}^{1} - 2 \int_{0}^{1} x \sqrt{x^2 + 4} \, dx = \sqrt{5} - \frac{2}{3} (x^2 + 4)^{3/2} \bigg|_{0}^{1} = \frac{16}{5} - \frac{7}{3} \sqrt{5}. \]

8. Evaluate \( \int_{0}^{\pi} e^{\cos x} \sin(2x) \, dx \). Hint: Double–angle formula.

First, in (*), we write \( \sin(2x) = 2 \sin x \cos x \), and in (***) let \( z = \cos x \) so that \( dz = -\sin x \, dx \).

\[ I \overset{(*)}{=} \int_{0}^{\pi} e^{\cos x} \sin(2x) \, dx = 2 \int_{0}^{\pi} e^{\cos x} \sin x \cos x \, dx \]
\[ \overset{(**)}{=} -2 \int_{1}^{-1} z \, e^z \, dz = 2 \int_{-1}^{1} z \, e^z \, dz \]
\[ \overset{(***)}{=} 2 \left[ ze^z \bigg|_{-1}^{1} - \int_{-1}^{1} e^z \, dz \right] = [ze^z - e^z] \bigg|_{-1}^{1} = 4. \]

We used integration–by–parts in (***) with, we could also use (34) from the Handout:

\[ u = z, \quad dv = e^z \, dz, \]
\[ du = dz, \quad v = e^z. \]